Lecture # 7

Steady-State Response

Design and analysis of control systems mainly focuses on the following three important specifications that can be conveniently obtained from transfer function:

1. Transient response.
2. Stability.
3. Steady-state response.

The above three specifications orient around each other for evaluating the overall performance of control systems. One of the factors of prime importance is stability. Control systems are designed to be stable. A system is stable if the transients disappear leaving behind the steady-state component of the total-response. We are already familiar with the total-response of a control system that consists of two components; the natural-response and the forced-response. The natural-response gives us information about the transient behavior and the forced-response yields information regarding the steady-state behavior. The transient behavior or performance is based on specifications; rise time, time to peak, percent overshoot and the settling time. The steady-state design of control systems specify how close the system's output is in terms of accuracy and is based on minimizing the steady-state error, whereas the transient design of control system specify how quickly the system responds to changes and is based on time-response.

Steady-State Error

Once the desired final steady-state is reached, the operation constraints of the system should be such that this value is maintained. However, in real systems there is a deviation from the desired steady-state value, depending on the system, giving a constant steady-state error. The definition of system error has not been unified in the literature. In control system studies an error $e(t)$ refers to the difference between the sensed output $c(t)$ and the reference signal $r(t)$ which can be expressed as:

$$e(t) = e(\infty) = r(t) - c(t) \qquad 1$$

Steady-state error is referred to the difference between the final steady-state value of the output with respect to the reference input as illustrated in Figure (1). In Figure (1), the output 1 has zero steady-state error, and output 2 has a finite steady-state error, $e(\infty)$ = input – output 2. Mathematically:

$$e_{SS}(t) = e(\infty) = \lim_{t\to\infty} e(t) = \lim_{t\to\infty}[r(t) - c(t)] \qquad 2$$

Control systems are subject to steady-state errors due to different causes besides the applied input, and in order to reduce the steady-state error, remedial actions are required. Practically, because of friction and other features, such as imperfections and the natural behavior of the system, the steady-state of the output response does not exactly agree with the reference. In addition to the input, a disturbance $d(t)$ may be acting on the system from some source, and the output due to disturbance acting alone may also be considered an error. In a design problem, one of the objectives is to keep the steady-state error to a minimum, or below a certain tolerable value. The accuracy requirement for control systems depends to a great extent on the control objectives of the system.

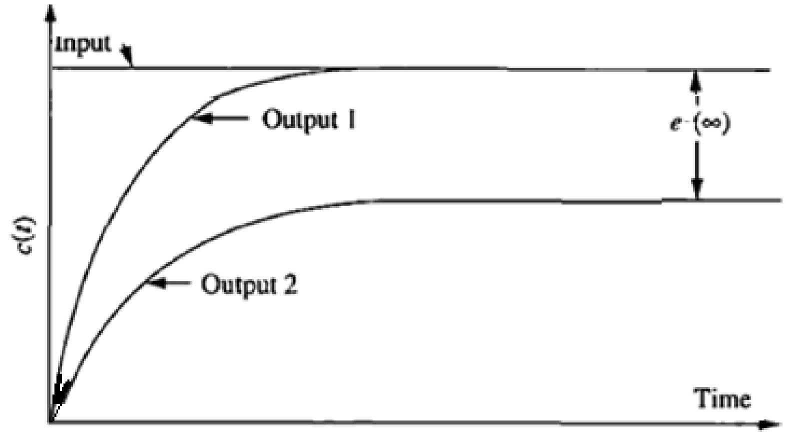

Figure 1: Illustration of Error

In order to measure and predict accuracy in a control system, a standard measure of performance in terms of steady-state error is widely used. The steady-state error is a concept that assumes the followings:

1. The system under test is excited with some standard input (having proper mathematical form), depending on the requirements. In control system analysis and synthesis the standard inputs for most purpose are step, ramp and parabolic. Step inputs represent constant position and are thus useful in determining the ability of the control system to position itself with respect to a stationary target. Ramp inputs represent constant-velocity inputs to a position control system by their linearly increasing amplitude. These waveforms can be used to test a system's ability to follow a linearly increasing input or, equivalently, to track a constant velocity target in a radio guidance systems. Finally, parabolas, whose second derivatives are constant, represent constant acceleration inputs to position control systems and can be used to represent accelerating targets. The curves shown in Figure (2) represent acceleration, velocity, and position respectively for unit-step, unit-ramp and parabolic quantities.

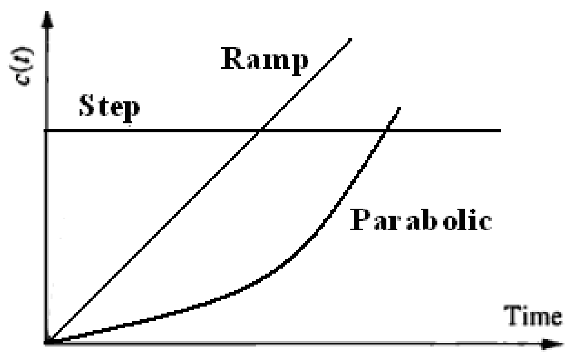

Figure 2

2. When the system comes to a steady-state, the difference between the input and the output, referred to as the error is measured.

Test Inputs

Test inputs are standard inputs for which the mathematical form is known. In the case of a generally non-standard input, it may however be necessary to split the input into pieces. Each piece of the input can then be compared with some standard input and dealt with during the analysis separately. For example refer to Figure (3), in which input quantity is split into different pieces in such a way that it can be compared with one or more of the standard inputs. In Figure (3), 0a is regarded as a ramp function, ab and bcd are the step function and de can be approximated as a parabolic function.

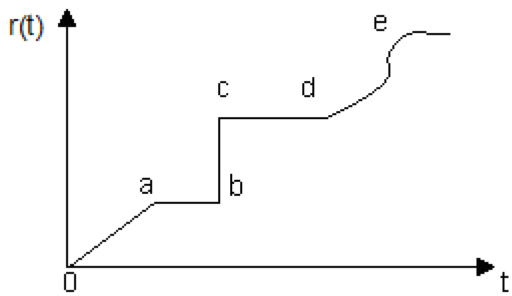

Figure 3

Steady-State Error in terms of *R*(*s*) and *T*(*s*)

Most system responses in a way in that the actual output is less than the desired value, so that there is always an error, how large or small will depend on the system itself and the type of input. Since the error is the difference between the input and the output of a system, we assume a closed-loop transfer function, $T(s)$, and form the error, $E(s)$, by taking the difference between the input and the output. Consider a unity feedback system as shown in Figure (4a). The negative feedback system is considered because it is stable and that the steady-state error is only determined for stable system. Here we are interested in the steady-state error $e(\infty)$.

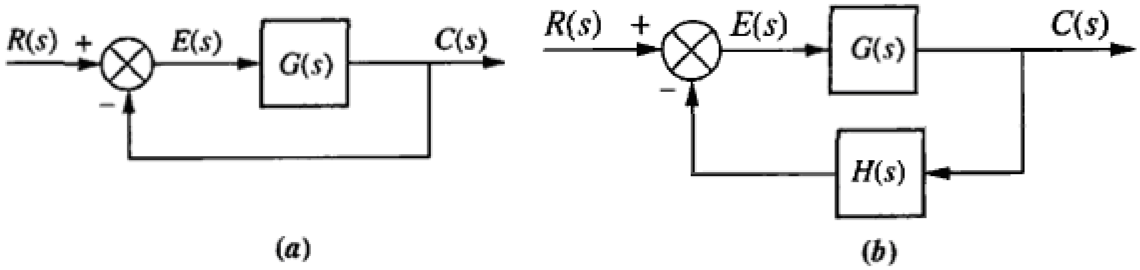

Figure 4: (a) Unity Feedback System (b) Non-unity Feedback System

For unity feedback systems as shown in Figure (4a), $E(s)$ is:

$$E(s) = R(s) - C(s) \qquad 3$$

Also $C(s) = G(s)E(s)$

Or $E(s) = R(s) - G(s)E(s)$ 4

Rearranging Eq (4):

$$E(s) = \frac{R(s)}{1 + G(s)} \quad 5$$

Using the final value theorem, we have:

$$e(\infty) = \lim_{S \to 0} sE(s) = \lim_{S \to 0} \frac{sR(s)}{1 + G(s)} \quad 6$$

Now consider a non-unity feedback system as shown in Figure (4b). The closed-loop transfer function is:

$$T(s) = \frac{G(s)}{1 + H(s)G(s)}$$

To find $E(s)$, the error between the input, $R(s)$, and the output, $C(s)$, we refer to Eq (3) that is $E(s) = R(s) - C(s)$. Since:

$$C(s) = R(s)T(s) \quad 7$$

Substituting the value of $C(s)$ from Eq (7) into Eq (3), we have:

$$E(s) = R(s)[1 - T(s)] \quad 8$$

Eq (8) gives the error in terms of closed-loop transfer function from which we can find $e(t)$ by taking the inverse Laplace transform and then allowing t to approach infinity, giving $e(\infty)$. Application of the final value theorem on the other hand, allows us to find the final value of $e(t)$ without taking the inverse Laplace transform of $E(s)$. That is:

$$e(\infty) = \lim_{S \to 0} sE(s) = \lim_{S \to 0} sR(s)[1 - T(s)] \quad 9$$

For finite steady-state errors, the final value theorem is valid if and only if $T(s)$ has poles in the LHP and, at most, one pole at the origin. However, correct results that yield steady-state errors that are infinite can be obtained if $T(s)$ has more than one pole at the origin. If $T(s)$ has poles in the RHP or on the imaginary axis other than at the origin, the final value theorem is invalid. Note that by using Eq (9), numerical results can also be obtained for unstable systems. These results, however, are meaningless.

Example 1: Consider a unity feedback system shown in Figure (5). Find the steady-state error for a unit-step input.

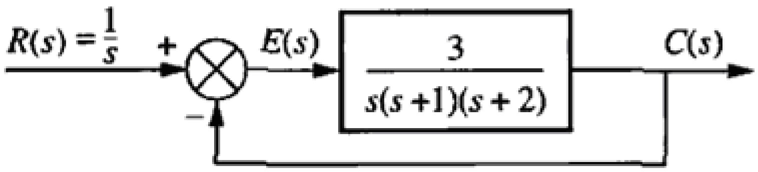

Figure 5

Solution: The closed-loop transfer function of the given system is:

$$T(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{3}{s^3+3s^2+2s+3}$$

Using: $E(s) = R(s)\,[1 - T(s)]$

We have: $\frac{1}{s}\left[1-\frac{3}{s^3+3s^2+2s+3}\right]$

Using: $e(\infty) = \lim_{s\to 0}[sE(s)] = \lim_{s\to 0} s \times \frac{1}{s}\left[1-\frac{3}{s^3+3s^2+2s+3}\right]$

$$e(\infty) = \lim_{s\to 0}\left[1-\frac{3}{s^3+3s^2+2s+3}\right] = 0$$

This means a stable system generates no error. In contrast consider the same system of Figure (5) with positive feedback so as to make it unstable, we have:

$$T(s) = \frac{G(s)}{1-G(s)H(s)} = \frac{3}{s^3+3s^2+2s-3}$$

Using: $E(s) = R(s)\,[1 - T(s)]$

We have: $\frac{1}{s}\left[1-\frac{3}{s^3+3s^2+2s-3}\right]$

Using: $e(\infty) = \lim_{s\to 0}[sE(s)] = \lim_{s\to 0} s \times \frac{1}{s}\left[1-\frac{3}{s^3+3s^2+2s-3}\right]$

$$e(\infty) = \lim_{s\to 0}\left[1-\frac{3}{s^3+3s^2+2s-3}\right] = 2$$

This means an unstable system produces a large error.

Steady-State Error in Terms of *G*(*s*)

Majority of control systems can be configured as a unity feedback system with a forward transfer function, $G(s)$, so that steady-state error can also be formulated by considering the forward transfer function. Consider the feedback control system shown in Figure (4a). In this case $E(s)$ is actually the difference between the input, $R(s)$, and the output, $C(s)$ and therefore the error between $R(s)$ and $C(s)$. Knowing the transfer function of a unity feedback system as shown in Figure (4a): $\frac{G(s)}{1+G(s)}$, so that:

$$C(s) = R(s)\frac{G(s)}{1+G(s)} \qquad 10$$

Substituting Eq (10) in Eq (3) and applying the final value theorem to the resultant expression, we will have:

$$e(\infty) = \lim_{S\to 0} sE(s) = \lim_{S\to 0} \frac{sR(s)}{1+G(s)} \qquad 11$$

The steady-state error for the standard inputs can be obtained by knowing their Laplace transform. That is for a unit step input; $R(s) = 1/s$. Eq (11) is therefore expressed as:

$$e(\infty)_{Step} = \frac{1}{1+\lim_{S\to 0} G(s)} \qquad 12$$

For a standard ramp input $R(s) = 1/s^2$, we have:

$$e(\infty)_{Ramp} = \lim_{S\to 0} \frac{1}{s[1+G(s)]} \qquad 13$$

For a standard parabolic input $R(s) = 1/s^3$, we have:

$$e(\infty)_{Parabolic} = \lim_{S\to 0} \frac{1}{s^2[1+G(s)]} \qquad 14$$

Eqs (12), (13) and (14), however, is only applicable to a unity feedback system. In order to extend the same formula for finding the steady-state error to non-unity feedback systems, it will be essential to transform the non-unity feedback system to equivalent unity feedback system.

Example 2: For a system shown in Figure (6), find the steady-state error due to all the three standard inputs (unit step, unit ramp and unit parabolic).

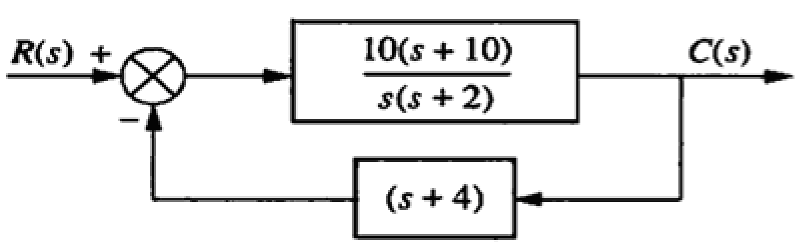

Figure 6

Solution: The given system must be transformed into equivalent unity feedback system. To transform to unity feedback system, 1 is added and subtracted from the feedback block function. That is: (s + 4) + 1 – 1. The function is then: (s + 3) +1, resulting in two blocks with functions (s + 3) and +1 in parallel forming inner and outer loop respectively. The inner loop is transformed into a single block having a closed-loop transfer function: $\frac{10(s+10)}{11s^2+132s+300}$. The equivalent unity feedback system of the given system is shown in Figure (7).

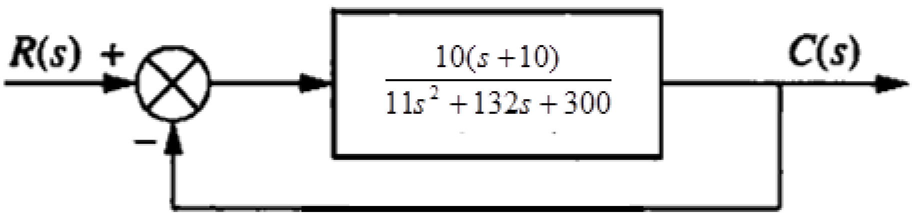

Figure 7

Form Figure (7), $G(s)=\frac{10(s+10)}{11s^2+132s+300}$. Therefore:

$$e(\infty)=\lim_{S\to 0} sE(s)=\lim_{S\to 0}\frac{sR(s)}{1+G(s)}$$

For the unit-step input; $R(s) = 1/s$. Therefore:

$$e(\infty)=\lim_{S\to 0}\frac{s(1/s)}{1+\frac{10(s+10)}{11s^2+132s+300}}=\frac{3}{4}$$

For the unit-ramp input; $R(s) = 1/s^2$. Therefore:

$$e(\infty)=\lim_{S\to 0}\frac{s(1/s^2)}{1+\frac{10(s+10)}{11s^2+132s+300}}=\infty$$

For the unit-parabolic input; $R(s) = 1/s^3$. Therefore:

$$e(\infty)=\lim_{S\to 0}\frac{s(1/s^3)}{1+\frac{10(s+10)}{11s^2+132s+300}}=\infty$$

Static Error Constants and System Type

It can be noted that for a system, the steady-state error due to step, ramp and parabolic inputs may be finite or un-defined. This means that systems only accept a specific input depending on its characteristics to give a finite error. In Eqs (12), (13) and (14), the three terms in the denominator that are taken to the limit determine the steady-state error. These limits are therefore called static error constants. They are:

Static position error constant (K_p): where $K_p = \lim_{S\to 0} G(s)$

Static velocity error constant (K_v): where $K_v = \lim_{S\to 0} sG(s)\ \sec^{-1}$

Static acceleration error constant (K_a): where $K_a = \lim_{S\to 0} s^2 G(s) \sec^{-2}$

The steady-state error performance specifications of a system are generally based on static error constants. These quantities, depending upon the form of $G(s)$, can assume values of zero, finite (constant), or infinity. Since the static error constant appears in the denominator of the steady-state error formulas; Eqs (12), (13) and (14), the value of the steady-state error decreases as the static error constant increases. The steady-state errors for the standard test inputs can also be defined in terms of static error constants by using Eq (12) as follows:

$$e(\infty)_{Step} = \frac{1}{1 + \lim_{S\to 0} G(s)}$$

$$e(\infty)_{Step} = \frac{1}{1 + K_p} \qquad 15$$

For a unit-ramp function:

$$e(\infty)_{Ramp} = \lim_{S\to 0} \frac{1}{s[1 + G(s)]}$$

$$e(\infty)_{Ramp} = \frac{1}{\lim_{S\to 0}(s) + \lim_{S\to 0}[sG(s)]}$$

$$e(\infty)_{Ramp} = \frac{1}{K_v} \qquad 16$$

For a unit-parabolic function:

$$e(\infty)_{Parabolic} = \lim_{S\to 0} \frac{1}{s^2[1 + G(s)]}$$

$$e(\infty)_{Parabolic} = \frac{1}{\lim_{S\to 0}(s^2) + \lim_{S\to 0}[s^2 G(s)]}$$

$$e(\infty)_{Parabolic} = \frac{1}{K_a} \qquad 17$$

System Types

The values of the static error constants, again, depend upon the form of $G(s)$, especially on the number of pure integrations; $1/s$, $1/s^2$ etc in the forward path. Since steady-state errors are therefore dependent upon the number of integrations in the forward path, this system attribute identifies the system-type as Type-0, Type-1 and Type-2 system etc. Control systems are usually analyzed on Type-basis. With no integration in thye forward path, the system is called Type-0, with a single integration in the forward path, the system is Type-1, and with two integrations in its forward path, the system is Type-2. Higher types are beyond the scope of this course and will not be considered. The various types exhibit the following steady-state properties:

Type-0: In Type-0 systems a constant actuating signal results in a constant value for the controlled variable. This system gives a finite error constant with a step input.

Type-1: In Type-1 systems a constant actuating signal results in a constant rate of change (constant velocity) of the controlled variable. This system-type gives a finite error with a ramp input.

Type-2: In Type-2 systems a constant actuating signal results in a constant second derivative (constant acceleration) of the controlled variable. This system-type gives a finite error with a parabolic input.

Table (1) gives a summary of steady-state error and system-types.

Table 1

Input	Steady-state error formula	Type 0		Type 1		Type 2	
		Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

Example 3: For a system shown in Figure (8), categorize the system-type on the basis of static error constants and evaluate the steady-state error for an input of $50u(t)$, $50tu(t)$ and $50t^2u(t)$.

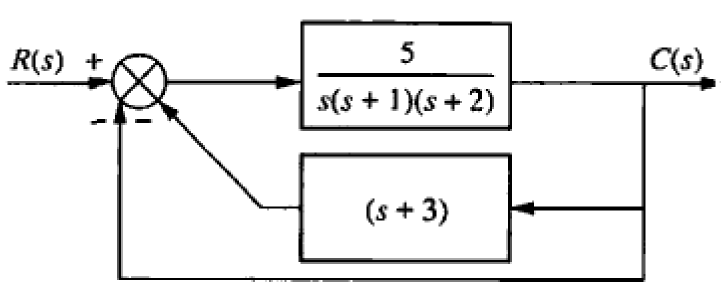

Figure 8

Solution: The given system of Figure (8) can be reduced to an equivalent negative feedback canonical form as shown in Figure (9), which gives:

$$G(s) = \frac{5}{s^3 + 3s^2 + 7s + 15}.$$

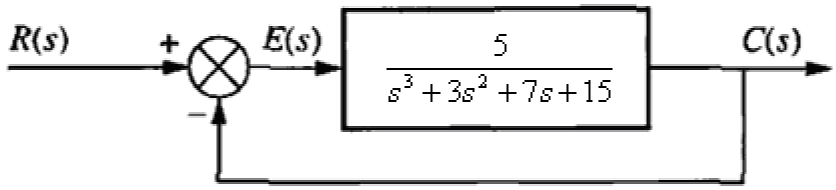

Figure 9

The static position error constant is:

$$K_p = \lim_{S \to 0} G(s)$$

$$K_p = \lim_{S \to 0} \frac{5}{s^3 + 3s^2 + 7s + 15} = 0.33$$

For a unit-step input, the steady-state error is: $e(\infty)_{Step} = \frac{1}{1+K_p}$. Let us suppose that the amplitude of each input is represented by *A*, then the steady-state error for the step input of the form: *A* = 50 is:

$$e(\infty)_{step} = \frac{A}{1+0.33} = \frac{50}{1.33} = 37.5$$

The static error constant due to velocity is:

$$K_v = \lim_{s \to 0}[sG(s)] = \lim_{S \to 0}\left[s\left(\frac{5}{s^3 + 3s^2 + 7s + 15}\right)\right] = 0$$

Likewise for ramp input of the form: *A* = 50, the steady-state error is:

$$e(\infty)_{ramp} = \frac{A}{K_V} = \frac{50}{0} = \infty$$

The static error due to constant acceleration is:

$$K_a = \lim_{S \to 0} s^2 G(s)$$

$$K_a = \lim_{S \to 0}\left[s^2\left(\frac{5}{s^3 + 3s^2 + 7s + 15}\right)\right] = 0$$

The steady-state error for a parabolic input of the form: A = 50 is:

$$e(\infty)_{parabolic} = \frac{A}{K_a} = \frac{100}{0} = \infty$$

The system has a finite static error constant due to position and yields a finite error with a step input. The system is therefore classified as Type-0 according to Table (1).

Example 4: A system is represented by a block diagram as shown in Figure (10). Determine the value of K so that for an input of 100$tu(t)$ there will be an error of 0.01 in the steady-state.

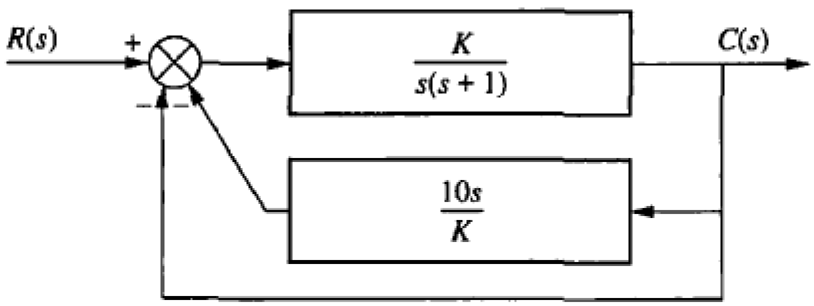

Figure 10

Solution: As shown in Figure (10), the inner loop consists of $H(s) = \frac{10s}{K}$ and a forward block: $G_1(s) = \frac{K}{s(s+1)}$. The transfer function of the inner loop is then:

$$G(s) = \frac{K}{s(s+11)}$$

The resultant system will then be a unity feedback system with the forward block $G(s)$. From the transfer function of the forward block $G(s)$, the system-type is identified as Type-1 because of single integration in its forward path. The static error constant of the given system is:

$$K_v = \lim_{s\to 0} \frac{K}{(s+11)} = \frac{K}{11}$$

For ramp input of the form: 100$tu(t)$, the steady-state error is:

$$e(\infty)_{ramp} = \frac{100}{K_V} = \frac{1100}{K}$$

To maintain the steady-state error of 0.01, the value of K will therefore be:

$$\frac{1100}{K} = 0.01 \quad \text{or} \quad K = 110000$$

Sources of Steady-State Error

Most of the steady-state errors in control systems arise from nonlinear sources, such as backlash in gears in the case of mechanical systems or in electrical system a dead band due to which motor that will not move unless the input voltage exceeds a threshold. The steady-state errors arise from the configuration of the system itself and the type of applied input. Moreover, the use of sub-standard devices to fabricate system is also source of error. Furthermore, disturbances acting on the system introduce errors in the system.

Larger the value of K, smaller is the value of e_{SS} that should be to yield a smaller value of c_{SS}. Thus with a pure gain systems, there will always be a steady-state error for a step input which, however, diminishes as the value of K increases.