Lecture # 6

Stability

Stability is the most important property of control systems, which are generally designed to be stable. A system is stable if every Bounded Input (BI) produces a Bounded Output (BO). This stability criterion is refers the system as BIBO stable. A system on the other hand is unstable if any bounded input produces an unbounded output. In a stable system, the transients due to disturbance dies out with time and the system revert back to its normal state often through oscillations. Under the BIBO definitions, thus:

1. A system is stable if every bounded input produces a bounded output.
2. A system is unstable if any bounded input produces an unbounded output.

Stability from System Response

Stability can be determined from system response of the system represented by linear differential equations. The solution of the differential equation is a two-part solution giving the total response of the system, comprising of homogeneous or complementary part and particular solution part. A simple test of stability would be to consider the complementary or the transient-part by making $t \rightarrow \infty$, for which: $\lim_{t \to 0} c_t(t) = 0$ is regarded as stable system. The stability criterion relies on a description of the natural response, which thus implies that only the forced response remains as the natural response approaches zero. Using the natural response criterion, the followings are the definitions of stability, instability, and marginal stability applicable to real time or LTI systems.

1. An LTI system is stable if the natural response approaches zero as time approaches infinity, otherwise the system is unstable.
2. An LTI system is marginally stable if the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity.

From the perspective of the time-response plot of a physical system, instability is displayed by transients that grow without limit and, consequently, the total response does not approach a steady-state value for any forcing function.

Stability from Closed-Loop Poles

Stability of a system can also be determined from the poles of its closed-loop transfer function. This necessitates obtaining the solution of the characteristic equation. The characteristic equation is obtained by equating the characteristic polynomial (denominator of a transfer function) to zero. Consider a general form of a closed-loop transfer function, which can be expressed as:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{P(s)}{Q(s)} \qquad 1$$

The denominator of closed-loop transfer function as Eq (1), which when equated to zero gives the characteristic equation that is:

$$Q(s) = 1 + G(s)H(s) = 0 \qquad 2$$

The solution of the characteristic equation will yield roots that are referred to as closed loop poles. Thus if all the poles of characteristic equation have negative real parts, or in other words all the poles of the system lie in the LHP, the system is said to be stable. In case any of the system poles or any root of the characteristic equation have positive real part will lie in the RHP, then the system is said to be unstable. The problem, however, can become difficult for equations higher than second or third-order. Stability of LTI systems can be determined from the location of the closed-loop poles as follows:

1. If the poles are only in the LHP, the system is stable.
2. If any poles are in the RHP, the system is unstable.
3. If the poles are on the $j\omega$-axis and in the LHP, the system is marginally stable as long as the poles on the $j\omega$-axis are of unit multiplicity; it is unstable if there are any multiple $j\omega$ poles.

If any of the coefficient(s) of characteristic equation is zero or negative in the presence of at least one positive coefficient, there is a root or roots that are imaginary or that have positive real parts. However, it is also not necessary that if all the coefficients of the characteristic equation have positive sign, all the roots will have negative real parts. For example, consider the characteristic polynomial $Q(s)$ of a closed-loop transfer function of a system:

$$Q(s) = s^3 + s^2 + 3s + 10$$

It can be noted from this polynomial that the system apparently seems to be stable, since all of the coefficients are positive integers. However, the factors of this polynomial are:

$$Q(s) = (s+2)(s^2 - s + 5)$$

The quadratic factor in the above expression reveals that there are two roots with positive real parts and are therefore in the RHP. The system is thus unstable.

Point to Remember: Consider an under-damped system (most frequently encountered), the general expression of the natural or transient response is given as:

$$c(t) = Ke^{-\sigma t}\cos(\omega t + \phi) \qquad 3$$

For its natural response to diminish, the system poles must be in the LHP. A negative real part: $-\sigma$ (lying in LHP), according to Eq (3) is indication of stable system is stable. Thus, if the poles of a system are in the RHP, hence has a positive real part ($+\sigma$), the system is unstable.

Causes of System Instability

The main causes of instability are the system gain and the nature of feedback. Increasing the gain will tend to drag the system's closed loop poles to lie in the RHP. Incorporating positive feedback tends to make the system unstable due to changes in output continuously added to the reference input thus making the system get out of control.

System Gain: Consider a unity feedback system shown in Figure (1). The transfer function with; $H(s) = 1$ and $G(s) = \frac{3}{s(s+1)(s+2)}$ has a closed-loop transfer function:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{3}{s^3+3s^2+2s+3} \quad 4$$

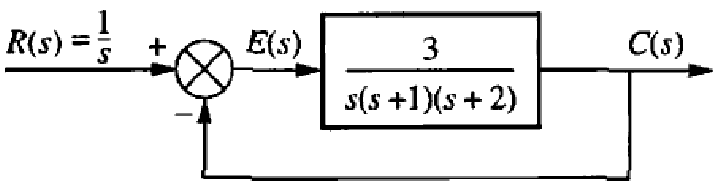
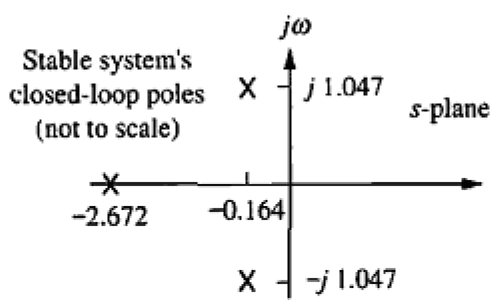


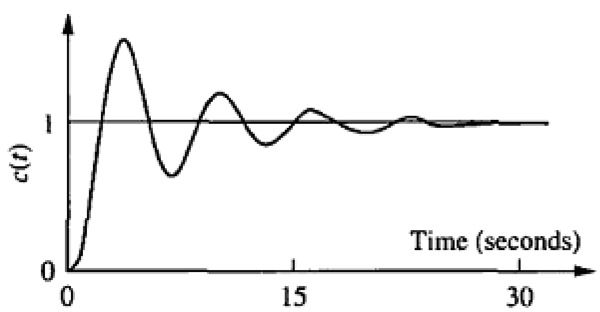
Figure 1

From the closed-loop transfer function of Eq (8.4), the characteristic equation is:

$$Q(s) = s^3 + 3s^2 + 2s + 3 = 0$$

The roots or system closed-loop poles are: -2.67 and $-0.164 \pm j1.847$. These poles are located in LHP as shown in Figure (2a) and therefore the system is stable. Thus the transient response to a unit step consists of damped oscillations, which settle after a certain finite time as illustrated in Figure (2b).





(a) **(b)**

Figure 2: (a) Closed-loop Poles Location (b) Unit-Step Response

Let us now consider the same system as shown in Figure (1) in which the gain is increased from 3 to 7, the transfer function of which become:

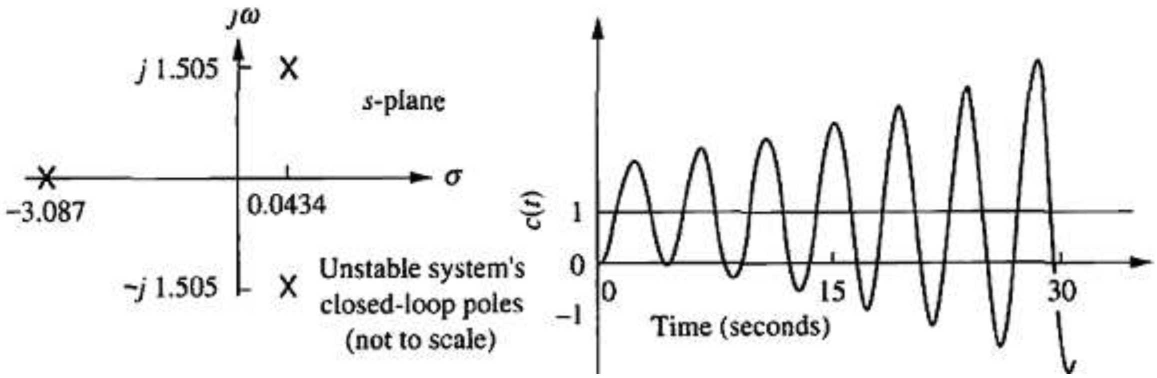
$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{7}{s^3+3s^2+2s+7} \quad 5$$

The characteristic equation from the closed-loop transfer function of Eq (5) is:

$Q(s) = s^3 + 3s^2 + 2s + 7 = 0$

The closed-loop poles are: –3.086 and 0.04 ± j1.505. Thus it can be seen that the complex poles are located in RHP as shown in Figure (3a) and therefore the system is unstable.

Solving Eq (5) for a unit-step gives the natural response in the form of rising oscillations, not settling to zero as t approaches infinity as illustrated in Figure (3b).



(a) **(b)**

Figure 3: (a) Closed-loop Poles Location (b) Unit Step Response

Increased gain causing instability can be easily compensated by using attenuator in cascade with the plant.

Nature of Feedback: Let us consider the system shown in Figure (1) with a positive feedback instead of negative feedback. The transfer function is then given as:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)} = \frac{3}{s^3 + 3s^2 + 2s - 3} \qquad 6$$

The characteristic equation of transfer function of Eq (6) is:

$Q(s) = s^3 + 3s^2 + 2s – 3 = 0$

The solution gives closed-loop poles, which are: 0.6717 and –1.835 ± j1.046. Thus it can be seen that one of the poles; 0.6717 is located in RHP and therefore the system is unstable.

Poles that result in instability can be compensated by adding zeros to a system's transfer function, appropriately to cancel the unstable poles. This is accomplished by adding a subsystem in cascade with the given system, whose transfer function contains the zero that will cancel with the unstable pole.

Routh-Hurwitz Criterion

There are several techniques in control systems that can provide information on stability. A simple way is to know about the closed-loop poles and locating them in s-plane. For locating the closed-loop poles in s-plane it is essential to solve the system's characteristic equation. However, solving polynomial equations of second order is within the scope of our knowledge that we have obtained in our matriculation. Moreover, most of the calculators facilitate to solve polynomial equation up to third order. Higher order polynomial equations are difficult to solve. Other method is to find the total response of a system to an input. However, the total response requires a transfer function of the system and the mathematical form of the input which is sometimes difficult to determine, often complicated by the system's transfer function, especially for systems higher than third-order.

The Routh-Hurwitz stability method of stability is based on characteristic polynomial of the system without solving the characteristic equation. The Routh-Hurwitz criterion states that the number of roots of the characteristic polynomial that are in the RHP is equal to the number of sign changes in the first column of the Routh table.

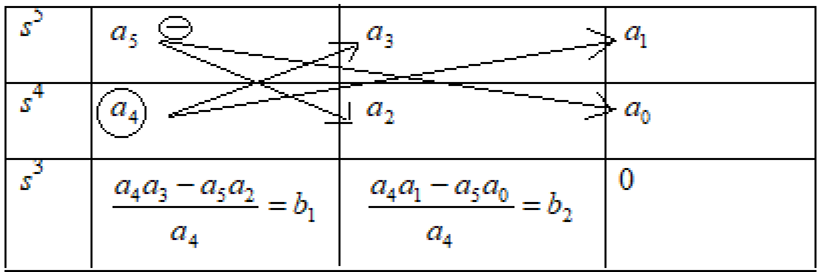
Forming the Routh Table

The method of stability using Routh-Hurwitz criterion requires two steps: First, formation of a data table called a Routh table from the characteristic polynomial and second, to find out, from the Routh table as to how many closed-loop system poles are in the LHP. First, arrange the coefficients of the n^{th}-order (say n, odd) polynomial $Q(s)$ into two rows. The first row consists of the odd number power of s coefficients that is first, third, fifth, and so on, and the second row then consists of the even power of s coefficients that is second, fourth, sixth and so on, all counting from the highest-order term (descending order). It must be remembered that in continuation from the highest to lowest power of s of the polynomial, if any power of s (even or odd) is missing, its coefficient is placed as zero in the array. In order to understand the technique of forming the complete table a fifth-order characteristic polynomial [n = 5, (odd)] of a system transfer function is considered as:

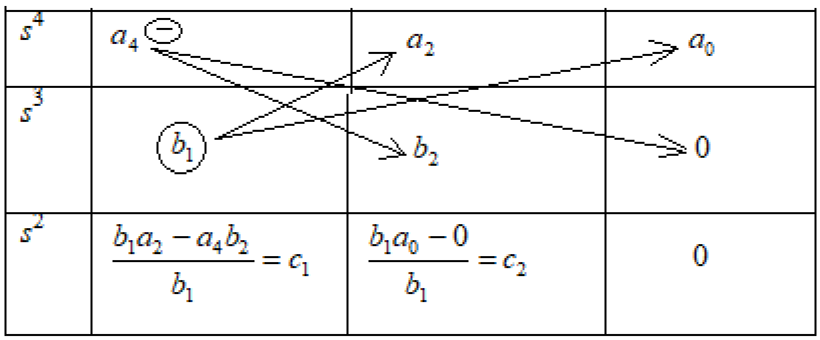
$$Q(s) = a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \qquad 7$$

s^5	a_5	a_3	a_1
s^4	a_4	a_2	a_0
s^3			
s^2			
s^1			
s^0			

The other rubrics in the above table are empty and have to be filled through evaluation using the elements which are present in the s^5 and s^4 rows. In order to fill the rubrics of s^3 row, the base number is a_4 and will act as a divisor. The technique to obtain the elements of s^3 row is illustrated in the following table. The circled quantity is the base number. The multiplication trend is positive upward and negative downward.



The rubric for the element which does not exist is filled with 0. Similarly to obtain the elements of s^2 row, the base number will be b_1. The technique is illustrated below:



Likewise, similar procedure is followed for creating elements of s^1 and s^0 rows. The completed table is shown below.

s^5	a_5	a_3	a_1
s^4	a_4	a_2	a_0
s^3	$\frac{a_4 a_3 - a_5 a_2}{a_4} = b_1$	$\frac{a_4 a_1 - a_5 a_0}{a_4} = b_2$	0
s^2	$\frac{b_1 a_2 - a_4 b_2}{b_1} = c_1$	$\frac{b_1 a_0 - 0}{b_1} = c_2$	0
s^1	$\frac{c_1 b_2 - b_1 c_2}{c_1} = d_1$	0	0
s^0	$\frac{d_1 c_2 - 0}{d_1} = c_2 = e_1$	0	0

Example 1: Using Routh-Hurwitz stability criterion, characterize whether the system with characteristic polynomial: $Q(s) = s^5 + 3s^4 + 6s^3 + 4s^2 + 2s + 9$, is stable.

Solution: The Routh's table is formed as follows:

s^5	1	6	2
s^4	3	4	9
s^3	$\frac{3\times 6-1\times 4}{3}=4.66$	$\frac{3\times 2-1\times 9}{3}=-1$	0
s^2	$\frac{4.66\times 4-3\times -1}{4.66}=4.64$	$\frac{4.66\times 9-3\times 0}{4.66}=9$	0
s^1	$\frac{4.64\times -1-(4.66\times 9)}{4.64}=-10.03$	0	0
s^0	9	0	0

Since there are two changes in sign in the first column (one from +4.64 to –10.03 and then from –10.03 to +9), therefore according to Routh-Hurwitz stability criterion, the system has two poles in the RHP. The system is therefore unstable.

Special Cases

Case 1: Zero in the first column: When the first element of a row in the Routh table turns out to be zero, this would lead to division by zero, resulting in the next element of the row to be undefined. To avoid this, a quantity, epsilon (ε) is introduced to replace the zero in the first column. The quantity ε approaches zero to limit when approached from positive side, so that $\varepsilon = 0^+$ (not exactly zero but just a little above, because of limit) and for interpretation, we must then assume a positive sign for ε.

Example 2: A system has transfer function: $T(s)=\frac{1}{s^4+s^3+2s^2+2s+5}$. Find whether the system is stable or not.

Solution: The characteristic polynomial is: $Q(s) = s^4 + s^3 + 2s^2 + 2s + 5$.

s^4	1	2	5
s^3	1	2	0
s^2	$0 = \varepsilon = 0^+$	5	0
s^1	$\frac{2\varepsilon-5}{\varepsilon}$	0	0
s^0	5	0	0

Since ε is positive and just above zero, therefore in fourth row of the first column, the element is negative, meaning a change of sign. The system is therefore unstable.

Case 2: Entire row of zeros: A situation may arise such that an entire row in a Routh table will consist of zero elements, referred to as 'all-zero row'. An 'all zero-row' usually appear in the Routh table when a purely even polynomial is a factor of the original polynomial. This situation is of an advantage in root locus technique when determining the points of imaginary-axis crossing of the root locus, which will be discussed later.

The Routh array can be completed by replacing the 'all-zero row' with the coefficients obtained by differentiating the auxiliary equation formed from the coefficients just preceding the 'all-zero row'. This is illustrated by the following example.

Example 3: Determine the stability of the system with characteristic equation:

$$Q(s) = s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128$$

Solution: The Routh table for the given system is as follow:

s^8	1	10	48	128	128
s^7	3	24	96	192	0
s^6	**2**	**16**	**64**	**128**	0
s^5	**0 → 12**	**0 → 64**	**0 → 128**	0	0
s^4	5.33	42.67	128	0	0
s^3	– 32.06	– 160.18	0	0	0
s^2	16.04	128	0	0	0
s^1	95.66	0	0	0	0
s^0	128	0	0	0	0

The s^5 'all-zero row' allow us to form an auxiliary equation from the coefficients of the just preceding row (s^6) that is of even power. The auxiliary equation formed from the coefficients (blue) of the s^6 row in the Routh table is: $2s^6 + 16s^4 + 64s^2 + 128 = A(s)$. Next taking the derivative of $A(s)$ with respect to s, we have:

$$\frac{dA(s)}{ds} = 12s^5 + 64s^3 + 128s$$

The 'all-zero row' elements are then replaced by the coefficients of the differentiated auxiliary equation. Thus the coefficients to replace the zero elements of s^5 row are: 12, 64 and 128 to permit us to complete the Routh table. The remaining table is completed in usual way. Since there are two changes in sign in the first column of the Routh table, the system has two poles in the RHP. The system is therefore unstable.

Range of Gain for Stability

Changes in gain affect stability. Thus simple gain adjustments from the controller can affect performance. When the system gain K is varied, the system may become unstable at certain values of K. Thus in control system design, it is necessary to find the range of values of K for which the system can become unstable. There is also a value of K for which the system is marginally stable or will oscillate. The basic element of control system is a plant cascaded with a simple gain controller K in a unity feedback configuration as shown in Figure (4). By adjusting the value of K, optimum performance can be obtained.

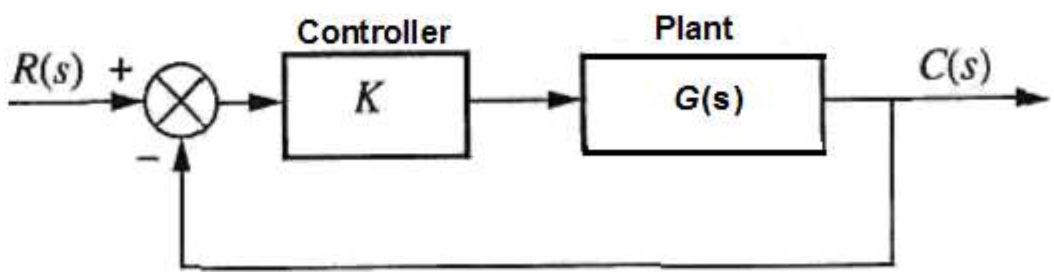


Figure 4: Plant with a Simple Gain Controller

The Routh-Hurwitz criterion, besides locating the system poles can also provide information about the changes in the gain of a feedback control system that result in differences in transient response due to changes in closed loop pole locations. From Routh table, we can find the range of values of controller gain K for which the system can be stable. Thus a simple gain adjustment within this range can provide an optimum performance without leading to instability. Furthermore, the system poles on the $j\omega$-axis can be obtained for a value of gain K that leads to marginal stability or oscillations. This is illustrated in the following example.

Example 4: Consider a feedback control system shown in Figure (5). Find the range of values of K for which the system should remain stable and also a value of K for which the system will be marginally stable and will oscillate.

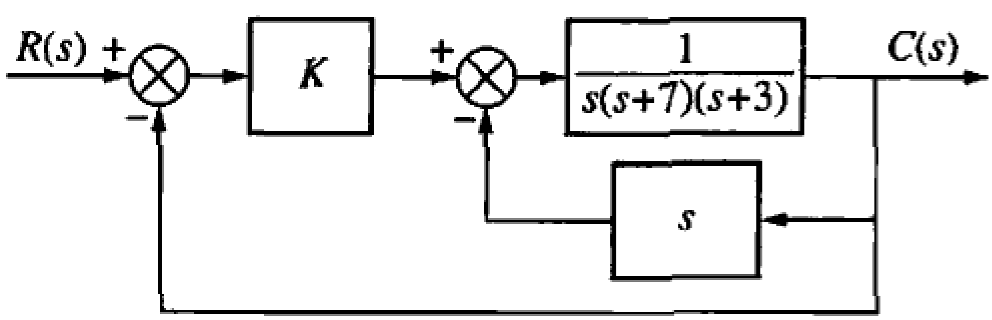


Figure 5

Solution: First we obtain the overall closed-loop transfer function through step-by-step simplification. The transfer function of the inner loop is obtained by considering $H(s) = s$ and $G_1(s) = \dfrac{1}{s(s+7)(s+3)}$. The transfer function of the inner loop is then:

$$G_2(s) = \frac{1}{s(s+7)(s+3)+s} = \frac{1}{s^2+10s^2+22s}$$

The resultant system in unity feedback configuration is shown in Figure (6)

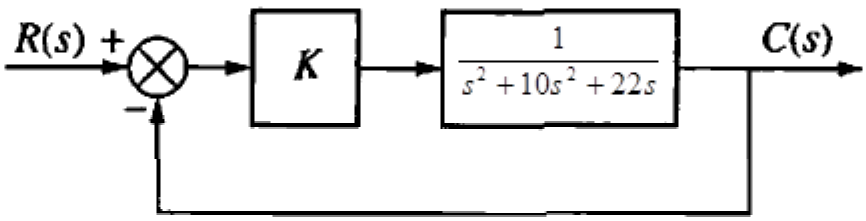


Figure 6

The closed-loop transfer function is:

$$T(s) = \frac{K}{s^3+10s^2+22s+K}$$

The characteristic equation of the system from the overall closed-loop transfer function is:

$$Q(s) = s^3 + 10s^2 + 22s + K$$

The Routh table is:

s^3	1	22
s^2	10	K
s^1	$\dfrac{220-K}{10}$	0
s^0	K	0

Apparently there are no changes in sign in the first column of the Routh table above. It must also be noted from the above Routh table that the first column containing numerical values have a positive sign. Thus in order to form a stable system, the consistency of positive sign must be maintained throughout in the first column. This all depends on the value of K we choose in the first column. Examination of the first column of the above Routh table shows that there are two rows (s^1 and s^0 row), which contain K. In the s^0 row, the value of K must be greater than zero that is $K > 0$ for making the element of first column and s^0 row positive. However, this condition is restricted by the presence of K expression in the first column and s^1 row. In order to maintain a positive sign in the first column and s^1 row, the entire expression must also be greater than zero that is:

$$\frac{220 - K}{10} > 0$$

Or $\quad 220 - K > 0$

Or $\quad K < 220$

Thus the range of values of K for which the above system will be stable must be:

$$0 < K < 220$$

That is the value of K must be set within 0 and 220. To find the condition for the value of K for which the system will be marginally stable, we refer to the s^1 row. This row must contain zero in the first column and therefore, the expression containing K must then be equated to zero. That is:

$$\frac{220 - K}{10} = 0$$

Thus K = 220 will make the system as marginally stable. The poles of the system under the condition for the system to be marginally stable can be obtained by considering the auxiliary equation for zero condition in the following row. The auxiliary equation is: $10s^2 + K = 0$. From which: $s = \pm j\sqrt{\frac{K}{10}} = \pm j\ 4.69$ for K = 220 (gain at marginal stability). It must be remembered that the poles of a marginally stable system lie on the $j\omega$-axis in the s-plane, which leads to oscillations. Since $j\omega = j4.69$. Therefore, the frequency of oscillation of the system will be 4.69 rad/sec.