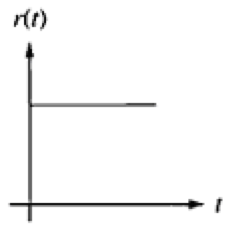
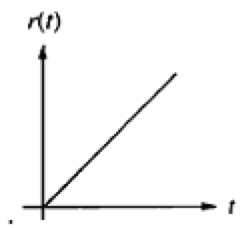
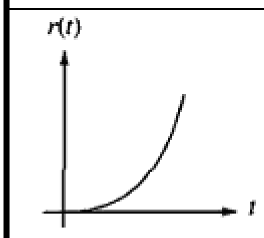
Lecture # 5

Transient Response

In analysis of control systems, it is usual to evaluate the output responses with respect to time, which is referred to as the time response. In the analysis control systems, a standard reference input signal is applied to a system, and the performance of the system is evaluated by studying the system response in the time domain. The standard signals used in control system analysis are given in Table (1).

Table 1

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Most of the control systems response in time domain is studied by applying a standard unit step input, and such response is then referred to as a unit step response. All dynamic systems exhibit transient response on the application of input signal. In electrical circuits the closing and opening of a switch at some particular time sets the state of the circuit, so that the circuit currents and voltages are time dependent and the behavior of the circuit is determined as time-response (transient response) due to operation of switch. In circuits analysis we have analyzed electrical circuits with representing them by differential equations, which on solution for dependent variable provided a complete response (behavior) consisting of two parts; the complementary (transient) and particular solution (steady-state) that is:

$$c(t) = c_t(t) + c_{SS}(t) \qquad 1$$

Where $c_t(t)$ is the transient component and $c_{SS}(t)$ is the steady-state component of the total response (time response) of the system. In control systems, transient response is defined as the part of the time response that goes to zero as time becomes very large. Thus $c_t(t)$ has the property for a stable system if:

$$\lim_{t \to \infty} c_t(t) = 0 \qquad 2$$

Transient response is responsible for stability and must be closely controlled, besides it is a significant part of the dynamic behavior of the system, and is also responsible for deviation between the output response and the input (desired response), The steady-state response of a control system is equally important, and is the part of the total response which remain after the transient component has decayed. Steady-state response indicates where the system output ends up when time becomes large. Thus, the steady-state response can still vary in a fixed pattern with time, such as a step function, the amplitude of which can remain constant with time or a ramp function that increases with time.

First-Order System

A first-order system is characterized by a first-order characteristic equation with a single pole irrespective of the number of zeros in a transfer function. Many systems are approximately first-order. A first-order system without zeros can be described generally by the transfer function:

$$G(s) = \frac{C(s)}{R(s)} = \frac{K}{s+a} \qquad 3$$

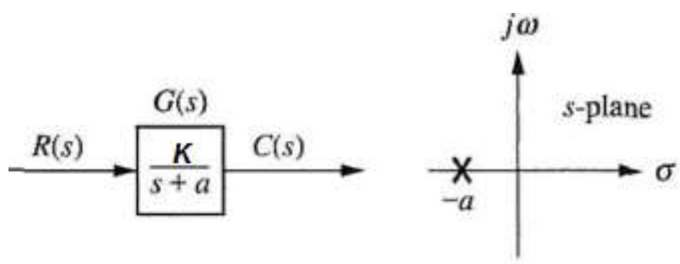


Figure 1: Simple First Order System and its Pole Location

If the input is a unit step function $r(t) = u(t)$, then the Laplace transform of the unit step function is: $1/s$, thus: $R(s) = 1/s$. The Laplace transform of the step response is $C(s)$, which is then given as:

$$C(s) = \frac{K}{(s+a)} R(s) = \frac{K}{s(s+a)}$$

Decomposing into partial fractions:

$$C(s) = \frac{K}{s(s+a)} \equiv \frac{k_1}{s} + \frac{k_2}{(s+a)}$$

Or $$C(s) = \frac{K}{s(s+a)} \equiv \frac{K/a}{s} - \frac{K/a}{(s+a)}$$

Taking inverse Laplace transform of the above expression, we have:

$$c(t) = \frac{K}{a}(1 - e^{-at}) \qquad 4$$

The step response is illustrated in Figure (2) the output has an initial value $c(0) = 0$, which approaches $c(t) = 1$ as its final value for $t \to \infty$ following a trend; $1 - e^{-at}$. Since the value of the unit-step input is equal to 1 and the final value of the output is also equal to 1, the error between input and output as $t \to \infty$ is equal to zero.

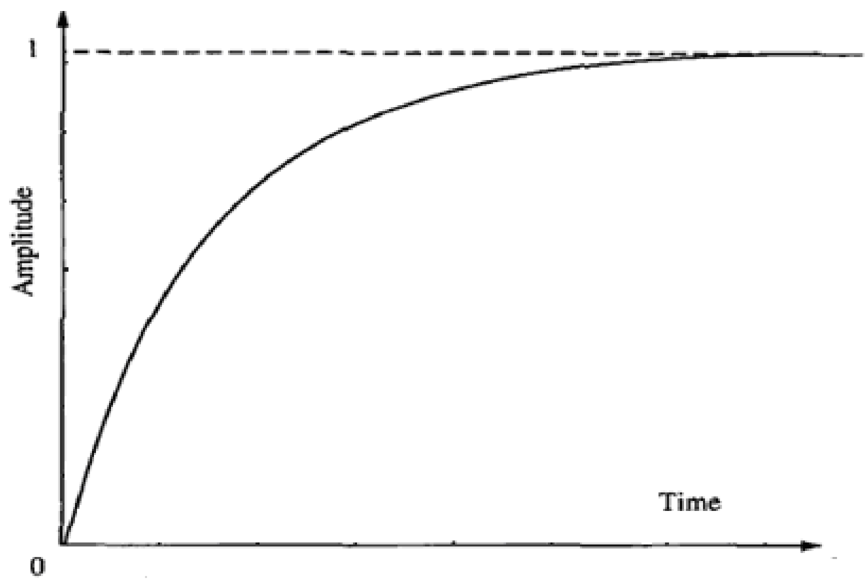


Figure 2: Unit-Step Response of a First Order System

Example 1: Obtain the unit-step and unit-ramp response of an *RC* low-pass filter with value of *R* = 1 and *C* = ½ *F*.

Solution: An *RC* LP filter is a series combination of *R* and *C* in which case the input is applies across the combination and output voltage is taken across *C*. Thus the transfer function of the *RC* low-pass filter is:

$$G(s) = \frac{C(s)}{R(s)} = \frac{1/RC}{(s + 1/RC)} = \frac{2}{s+2}$$

For unit-step response, the input is $r(t) = u(t) = 1$ for $t \geq 0$, the Laplace transform of the input function is: $R(s) = 1/s$. Therefore:

$$C(s) = R(s)G(s) = \frac{2}{s(s+2)}$$

Decomposing into partial fractions, we have:

$$C(s) = \frac{2}{s(s+2)} = \frac{k_1}{s} + \frac{k_2}{s+2}$$

For which: $k_1 = 1$ and $k_2 = -1$

Therefore: $C(s) = \frac{1}{s} - \frac{1}{s+2}$

Taking the inverse Laplace transform, we have the unit-step response:

$$c(t) = 1 - e^{-2t}$$

To obtain a unit-ramp response, the input function will be a unit-ramp, thus $r(t) = t$ for which the Laplace transform is: $R(s) = 1/s^2$. Therefore:

$$C(s) = R(s)G(s) = \frac{2}{s^2(s+2)}$$

Decomposing into partial fractions, we have:

$$C(s) = \frac{2}{s^2(s+2)} = \frac{k_1}{s^2} + \frac{k_2}{s} + \frac{k_3}{s+2}$$

For which: $k_1 = 1,\ k_2 = -\frac{1}{2}$ and $k_3 = \frac{1}{2}$

Therefore: $C(s) = \frac{1}{s^2} - \frac{1/2}{s} + \frac{1/2}{s+2}$

Taking the inverse Laplace transform yields the ramp response:

$$c(t) = t - \frac{1}{2} + \frac{1}{2}e^{-2t}$$

Second-Order Systems

Second-order system is characterized by second-order characteristic equation with two poles, irrespective of the number of zeros in a transfer function. Physical second-order system models contain two independent energy-storage devices which exchange stored energy, and may contain additional dissipative devices. Engineers often use second-order system models in the preliminary stages of design in order to establish the parameters of the energy-storage and dissipation devices required to achieve a satisfactory response.

The general form of the closed-loop transfer function of a second-order system as a function of ω_n and ζ is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 5$$

The quantities ω_n (undamped natural frequency, which describes the oscillatory nature) and ζ (damping coefficient or damping ratio, which describes the damping) are regarded as the two most important parameters defining the characteristics of a second-order system. The damping ratio ζ, will thus determine how much the system oscillates as the response decays toward steady-state (final value). The undamped natural frequency ω_n, on the other hand will determine how fast the system oscillates during any transient response containing a decaying sinusoid with a frequency of ω_d, where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

Example 2: A second-order system is represented by a closed-loop transfer function: $T(s) = \frac{10}{s^2 + 6s + 10}$. Find its natural frequency and damping ratio.

Solution: Comparing with the general form as given in Eq (5), we have:

$$\omega_n = \sqrt{10} = 3.16 \text{ rad/sec}$$

$$2\zeta\omega_n = 6 \text{ and } \zeta = \frac{6}{2\times 3.16} = 0.95$$

Step Response of Second Order System

The unit-step response can likewise be found by multiplying the transfer function of the given system by $1/s$, followed by a partial-fraction expansion and the inverse Laplace transform. The type of transient or natural response, however, depends on the characteristics of poles, which may be real and equal, real and unequal, complex conjugate or purely imaginary. The types of transient response depending on the nature of the roots and categorizing the system in terms of its damping type, depending on value of ζ. These are summarized as follows:

Overdamped response ($\zeta > 1$): When the two poles are real but different. The system attains a steady-state value gradually and without oscillations. The natural response is of the type:

$$c(t) = Ae^{-at} + Be^{-bt}$$

The overdamped system is a slow acting and does not oscillate about the final position. This may be necessary in some systems, for example, an elevator (lift).

Critically damped response ($\zeta = 1$): When the two poles are real and equal, the system attains a steady-state value gradually without oscillations but the rise time is less than that of over-damped system. The natural response of the type:

$$c(t) = Ae^{-at} + Bte^{-at}$$

Underdamped response (0 < ζ < 1): When the two complex conjugate (*a* and *b*), the system transient response is associated with overshoot and the system settles to a steady-state value through oscillations. The natural response of the type:

$$c(t) = Ae^{-at} + Be^{-bt}$$

The natural response is damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian or angular frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles.

Undamped response (ζ = 0): When the two poles are purely imaginary conjugate. The natural response of the type:

$$c(t) = A\cos(\omega_n t - \phi)$$

Where: ω_n is the undamped natural frequency of the system. Natural response is undamped sinusoid with frequency equal to the imaginary part of the poles. Figure (3) shows the response of each category of the system.

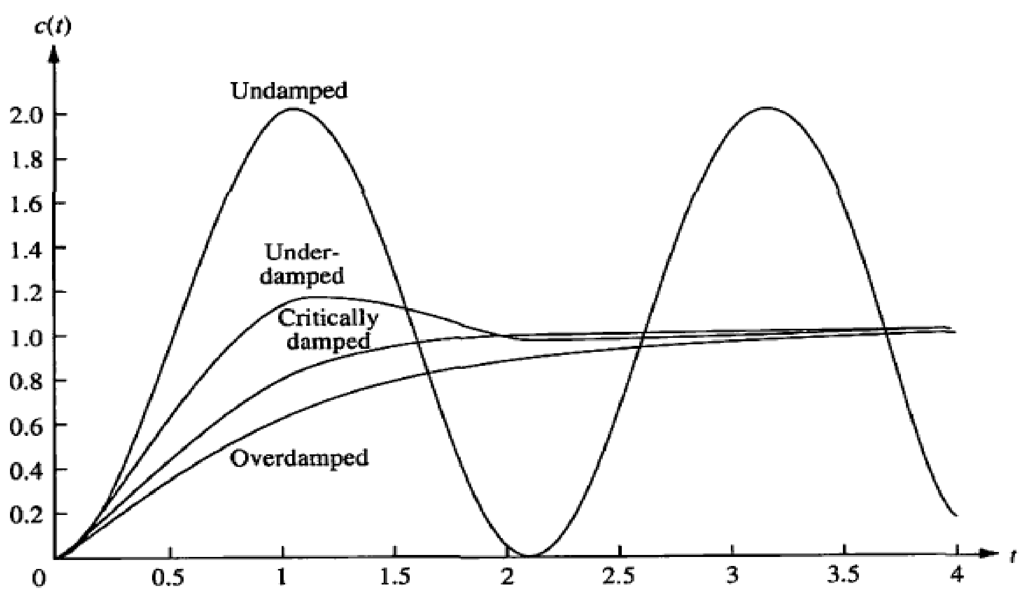


Figure 3: Response of a Second-order System

Since majority of systems we come across are under-damped, so in this lecture emphasis will be given to the step response of under-damped system only. The roots of the characteristic equation have for an under-damped system have a general form: $-\sigma \pm j\omega_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$. Thus for a unit step response:

$$C(s) = R(s)\frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

Or $$C(s) = R(s)\frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

For the unit-step response, the input has a Laplace transform of 1/*s*. Thus the unit-step response can be expressed as:

$$C(s) = \frac{\omega_n^{\,2}}{s(s^2 + 2\zeta\omega_n s + \omega_n^{\,2})} \qquad 6$$

In order to obtain the response, Eq (6) is decomposed into partial fractions as follows:

$$C(s) = \frac{\omega_n^{\,2}}{s(s^2 + 2\zeta\omega_n s + \omega_n^{\,2})} \equiv \frac{k_1}{s} + \frac{k_2 s + k_3}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

The constants k_1, k_2 and k_3 are: 1, -1 and $-2\zeta\omega_n$ respectively. Therefore:

$$C(s) = \frac{1}{s} + \frac{-s - 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

Or $$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2} + \zeta^2\omega_n^{\,2} - \zeta^2\omega_n^{\,2}}$$

Or $$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^{\,2}(1 - \zeta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^{\,2}(1 - \zeta^2)}$$

Multiplying and dividing the second term in the numerator of the above expression by: $\sqrt{1-\zeta^2}$, we have:

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\,\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^{\,2}(1 - \zeta^2)}$$

Or $$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^{\,2}(1 - \zeta^2)} - \frac{\zeta}{\sqrt{1-\zeta^2}}\,\frac{\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^{\,2}(1 - \zeta^2)}$$

Taking the inverse Laplace transform of both sides, we have:

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_n\sqrt{1-\zeta^2}\,t - \frac{\zeta}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t$$

Or $$c(t)=1-e^{-\zeta\omega_n t}\left[\cos\omega_n\sqrt{1-\zeta^2}\,t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right] \quad 7$$

Or $$c(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\sqrt{1-\zeta^2}\cos\omega_n\sqrt{1-\zeta^2}\,t+\zeta\sin\omega_n\sqrt{1-\zeta^2}\,t\right]$$

Supposing that: $\sqrt{1-\zeta^2}=\sin\beta$ and $\zeta=\cos\beta$. Then:

$$c(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\sin\beta\cos\omega_n\sqrt{1-\zeta^2}\,t+\cos\beta\sin\omega_n\sqrt{1-\zeta^2}\,t\right]$$

Or $$c(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t+\beta) \quad 8$$

Where: $\beta=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$. Alternatively we can suppose that: $\sqrt{1-\zeta^2}=\cos\alpha$ and $\zeta=\sin\alpha$. Then the unit step response is in the form:

$$c(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}\,t-\alpha) \quad 9$$

Where: $\alpha=\tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$ and $\alpha=\frac{\pi}{2}-\beta$. The unit-step response can also be obtained by factorizing the denominator of the closed-loop transfer function into complex factors as follows:

$$T(s)=\frac{\omega_n^{\ 2}}{s^2+2\zeta\omega_n s+\omega_n^{\ 2}}=\frac{\omega_n^{\ 2}}{(s+\zeta\omega_n+j\omega_n\sqrt{1-\zeta^2})(s+\zeta\omega_n-j\omega_n\sqrt{1-\zeta^2})}$$

For a unit-step response: $R(s)=1/s$, so that the above expression can be written as:

Or $$C(s)=\frac{\omega_n^{\ 2}}{s(s+\zeta\omega_n+j\omega_n\sqrt{1-\zeta^2})(s+\zeta\omega_n-j\omega_n\sqrt{1-\zeta^2})}$$

Or $$C(s)\equiv\frac{k_1}{s}+\frac{k_2}{s+\zeta\omega_n+j\omega_n\sqrt{1-\zeta^2}}+\frac{k_3}{s+\zeta\omega_n-j\omega_n\sqrt{1-\zeta^2}}$$

For making the mathematics simpler and for convenience, let us suppose that:

$$\zeta\omega_n+j\omega_n\sqrt{1-\zeta^2}=a$$

And $$\zeta\omega_n-j\omega_n\sqrt{1-\zeta^2}=b$$

So that: $C(s) \equiv \frac{k_1}{s} + \frac{k_2}{s+a} + \frac{k_3}{s+b}$

$$k_1 = 1$$

$$k_2]_{s=-a} = \frac{\omega_n^{\ 2}}{-a(b-a)} = \frac{\omega_n^{\ 2}}{-2\omega_n^{\ 2}\sqrt{1-\zeta^2}(\sqrt{1-\zeta^2} - j\zeta)} = \frac{-1}{2\sqrt{1-\zeta^2}} e^{j\alpha}$$

Similarly: $k_3]_{s=-b} = \frac{\omega_n^{\ 2}}{-b(a-b)} = \frac{-1}{2\sqrt{1-\zeta^2}} e^{-j\alpha}$

Where: $\alpha = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$

Therefore: $C(s) \equiv \frac{1}{s} + \frac{[-1/2\sqrt{1-\zeta^2}]e^{j\alpha}}{s+a} + \frac{[-1/2\sqrt{1-\zeta^2}]e^{-j\alpha}}{s+b}$

Or $C(s) \equiv \frac{1}{s} - \frac{[1/2\sqrt{1-\zeta^2}]e^{j\alpha}}{s+\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}} - \frac{[1/2\sqrt{1-\zeta^2}]e^{-j\alpha}}{s+\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}}$

Rearranging and taking the inverse Laplace transform of both sides and using the trigonometric relationship with complex exponent, we have:

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2}\, t - \alpha) \ \text{(as before)}$$

The unit-step response of the second-order underdamped system is shown in Figure (4). As illustrated in Figure (4), the unit-step response of the second-order system starts at zero and goes above the final value (for a unit-step response, final value is 1). The response increases in the magnitude well-beyond the final value is termed as overshoot and in control system studies it is usually determined in terms of percentage of the final value, referred to as percentage overshoot. Overshoot is important and is closely related to damping ratio and is independent of the undamped natural frequency. In a second-order system, percent overshoot depends entirely upon damping ratio. As the damping ratio increases, the percent overshoot decreases. When the damping ratio attains a value around 0.8, the overshoot becomes almost unobservable.

Unit-step input is important and is often used as test input to determine how well a system is performing. In addition, the shape of the step response; how fast it occurs, how much it oscillates, etc. help the designer to predict the ability of the system's response to other inputs.

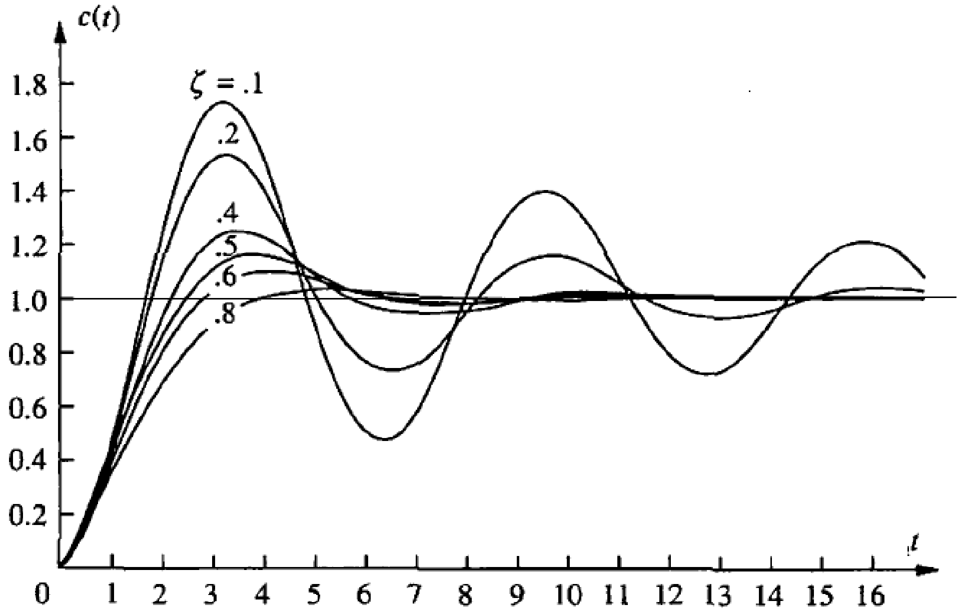


Figure 4: Unit Step Response for Different Values of Damping Ratio

The Damping Line

When the complex poles of a second order under-damped system are plotted in s-plane, they are symmetrically placed above and below the real axis as shown in Figure (5). The length of the line connecting the origin to the pole: $p_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$ is equal to ω_n from Pythagoras theorem: $\sqrt{\zeta^2\omega_n^{\ 2} + \omega_n^{\ 2}(1-\zeta^2)}$ .The line making an angle: $\beta = \cos^{-1}\zeta$ at the origin with the negative real axis or angle $\phi = 180^o - \cos^{-1}\zeta$ at the origin with the positive real axis that passes through the pole p_1 is referred to as the constant damping line; the line on which the value of damping ratio ζ is constant.

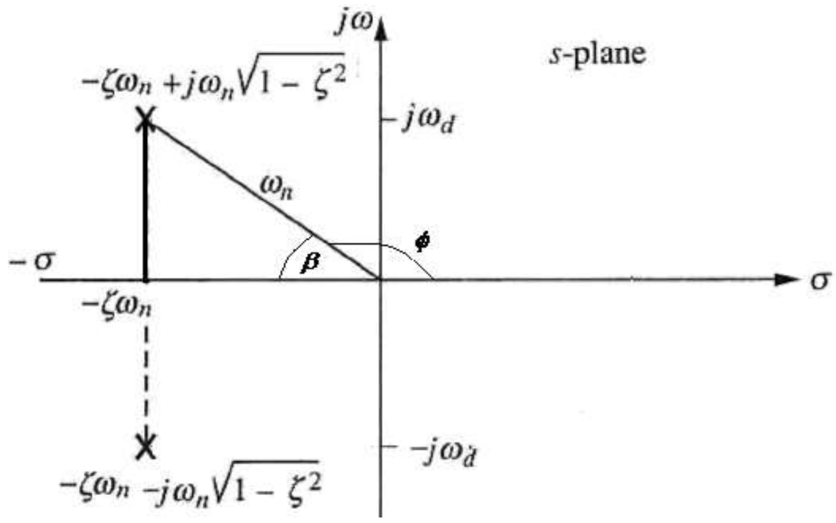


Figure 5: Complex Pole Location and Constant Damping Line

The angle β will lie in the second quadrant; therefore the damping line will make and angle of ϕ with origin with respect to the real-axis given by: $\phi = 180^o - \cos^{-1}\zeta$. It must be remembered that two second-order systems having the same value of ζ but different values of ω_n will have the same overshoot and same oscillatory pattern. Such systems are then said to have the same relative stability.

The damping ratio is constant along the radial line drawn at the origin, whereas the natural frequency ω_n changes. On the other hand the damping ratio varies from 0 to 1 along the arc of radius ω_n drawn at the origin from $+j\omega$-axis to $-\sigma$-axis. Thus the $+j\omega$-axis is referred to as a zero damping line. A system pole lying on the $j\omega$-axis will be that of an undamped system and system poles lying on the negative real axis will be that of a critical damped system. Pole location is an important element for predicting responses of all kinds of inputs, and is a function of damping ratio as illustrated in Figure (5). Changing the damping ratio or the natural frequency changes the location of the closed-loop pole. For example changing the natural frequency by keeping the damping ratio same will shift the closed-loop pole up along the damping line leftward or rightward, respectively for increasing and decreasing the value of the natural frequency of the system.

Inter-conversion of Standard Inputs

The standard inputs are inter-related through mathematical operations of integration and differentiated as can be seen in Table (2)

Table 2

Function	Integral	Derivative
Unit impulse δ(t)	$\int_0^t \delta(t)dt = u(t)$ (unit step)	$\frac{d}{dt}\delta(t) = \delta^{/}(t)$ (unit doublet)
Unit step u(t)	$\int_0^t u(t)dt = t$ (ramp)	$\frac{d}{dt}u(t) = \delta(t)$ (unit impulse)
Ramp (t)	$\int_0^t tdt = \frac{t^2}{2}$ (parabolic)	$\frac{d}{dt}(t) = 1 = u(t)$ (unit step)
Parabolic $\left(\frac{t^2}{2}\right)$	$\int_0^t \frac{t^2}{2}dt = \frac{t^3}{6}$ (jerk function)	$\frac{d}{dt}\left(\frac{t^2}{2}\right) = t$ (ramp)

Important point to remember: In classical time domain control systems are represented by differential equation whereas in modern time domain they are represented in state-space. However, the most important is the transfer function which is in s-domain. All three representations are convertible from one to the other.

Transient-Response Specifications of Second-Order Systems

The unit-step response of a second-order system consists of two parts; natural response and forces response. The type of natural responses depends of the characteristics of poles, which may be real and equal, real and unequal, complex conjugate or purely imaginary. Most of the real time systems we come across in our daily life have underdamped transient response, which for a second-order system is characterized by damped oscillations before reaching a steady-state under the application of a step input. It is meaningful to mention that the damping ratio remains the same regardless of the time scale of the response.

The transient response of a second-order system is governed by time-reponse specifications, which are rise time T_r, peak time T_p, percent overshoot $\%OS$, and settling time T_S. These time-response specifications are discussed as follows:

Rise Time (T_r): Rise time refers to the time required for the response to go from 0.1 of its final value to 0.9 of its final value.

$$T_r = \frac{\pi - \beta(rads)}{\omega_d} = \frac{\pi - \cos^{-1}\zeta}{\omega_d} \qquad 10$$

Peak Time (T_p): Peak time T_p refers to the time required for the response to reach maximum peak.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \text{ (for } n = 1) \qquad 11$$

Percent Overshoot ($\%OS$): The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value. Usually in the analysis of transient response, the first overshoot is important since its magnitude is the highest and it determines the ability of a system to sustain these magnitudes.

$$\%OS = \frac{c(t)_{\max} - c(t)_{final}}{c(t)_{final}} \times 100 \qquad 12$$

Or

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 \qquad 13$$

It can be seen from Eq (13) that the percent overshoot depends only on the damping ratio ζ. Thus any system's closed-loop pole placed on the constant damping line will exhibit the same percent overshoot, irrespective of the value of the un-damped natural frequency ω_n. Since percent overshoot is only a function of ζ, radial lines are thus lines of constant percent overshoot.

Settling Time (T_s): Settling time is the time it takes for the amplitude of the oscillatory term of the step response to reach 0.02, or:

$$T_S = \frac{n}{\zeta\omega_n} \qquad 14$$

Thus for a tolerance band of ±2%, which will be used for calculating the settling time. This yield:

$$T_S = \frac{4}{\zeta\omega_n} \quad (15)$$

Example 3: Find the transient-response specifications of a second-order system for a unit-step input, whose transfer function is: $\frac{16}{s^2+3s+16}$.

Solution: By comparing the given transfer function with the general transfer function of the second-order system, we have: $\omega_n = 4$ rad/sec and $\zeta = 0.375$. Thus:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\times\sqrt{1-0.375^2} = 3.7 \text{ rad/s}$$

The rise time is:

$$T_r = \frac{\pi-\beta(rads)}{\omega_d} = \frac{\pi-\cos^{-1}\zeta}{\omega_d} = \frac{3.14-1.18}{3.7} = 0.53\,\text{sec}$$

The peak time T_p is:

$$T_p = \frac{\pi}{\omega_d} = \frac{3.14}{3.7} = 0.85\,\text{sec.}$$

The percentage overshoot (*%OS*) is evaluated as:

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}}\times 100$$

Or $$\%OS = e^{0.375\pi/\sqrt{1-0.375^2}}\times 100 = 28\%$$

The settling time is:

$$T_S = \frac{4}{\zeta\omega_n} = \frac{4}{0.375\times 4} = 2.66\,\text{sec}$$

Summarizing the results:

ω_n	ζ	T_r	T_p	$\%OS$	T_S
4 rad/sec	0.375	0.53 sec	0.84 sec	28%	2.66 sec

Example 4: Evaluate the transient-response specifications for a system whose state-space representation matrices are:

$$A = \begin{bmatrix} 0 & 1 \\ -100 & -15 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 100 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$ and D = [0]

Solution: From the matrices; ***A***, ***B***, ***C*** and ***D***, the transfer function is:

$$T(s) = B[sI - A]^{-1}C + D$$

Substituting the given matrices, we have the transfer function as:

$$T(s) = \frac{100}{s^2 + 15s + 100}.$$

By comparing the given transfer function with the general transfer function of the second-order system, we have: $\omega_n = 10$ rad/sec and $\zeta = 0.75$. Therefore:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 10 \times \sqrt{1 - 0.75^2} = 6.6 \text{ rad/sec}$$

The rise time is:

$$T_r = \frac{\pi - \beta(rads)}{\omega_d} = \frac{\pi - \cos^{-1}\zeta}{\omega_d} = \frac{3.14 - 0.72}{6.6} = 0.36 \sec$$

The peak time T_p is:

$$T_p = \frac{\pi}{\omega_d} = \frac{3.14}{6.6} = 0.47 \text{ sec.}$$

The percentage overshoot (*%OS*) is evaluated as:

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

Or $$\%OS = e^{0.75\pi/\sqrt{1-0.75^2}} \times 100 = 2.8\%$$

The settling time is:

$$T_S = \frac{4}{\zeta\omega_n} = \frac{4}{0.75 \times 10} = 0.53 \sec$$

Summarizing the results:

ω_n	ζ	T_r	T_p	$\%OS$	T_S
10 rad/sec	0.75	0.36 sec	0.47 sec	2.8%	0.53 sec

Example 5: A mechanical system is shown in Figure (6). Find the transient-response specifications; T_r, T_p, $\%OS$ and T_s. The input is the torque $T(t)$ and the output is angular displacement $\theta_2(t)$.

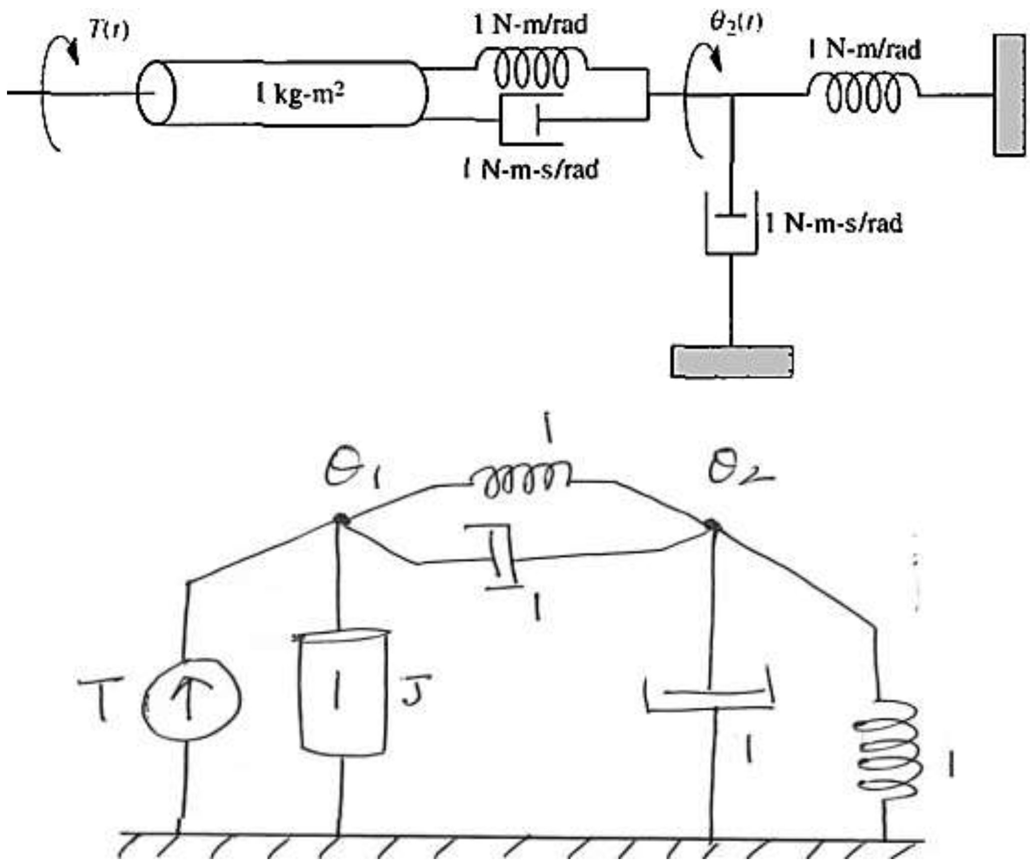


Figure 6

Solution: The equations of motion can be built-up either by drawing the equivalent mechanical network or simply by inspection. Referring to Figure (6), the torque $T(t)$ causes an angular displacement $\theta_1(t)$ to the point of application on inertia of 1 kg-m^2, which transforms to $\theta_2(t)$ after the parallel combination of spring and damper due to dissipation and storage of energy. Thus the equations of motion are:

$$T(t) = \frac{d^2\theta_1(t)}{dt^2} + \frac{d\theta_1(t)}{dt} + \theta_1(t) - \frac{d\theta_2(t)}{dt} - \theta_2(t)$$

And $$0 = 2\frac{d\theta_2(t)}{dt} + 2\theta_2(t) - \frac{d\theta_1(t)}{dt} - \theta_1(t)$$

In terms of s-domain:

$$T(s) = (s^2 + s + 1)\theta_1(s) - (s+1)\theta_2(s) \qquad 16$$

And $$0 = -(s+1)\theta_1(s) + 2(s+1)\theta_2 \qquad 17$$

Eliminating $\theta_1(t)$ between Eq (16) and (17), we have the transfer function:

$$G(s)=\frac{\theta_2(s)}{T(s)}=\frac{1}{2s^2+s+1}$$

Or
$$\frac{\theta_2(s)}{T(s)}=\frac{1/2}{\left(s^2+\frac{1}{2}s+\frac{1}{2}\right)}$$

Comparing the transfer function with the general transfer function of the second-order system, we have: $\omega_n=0.707\,\text{rad/sec}$ and $\zeta=0.35$. Therefore:

$$\omega_d=\omega_n\sqrt{1-\zeta^2}=0.707\times\sqrt{1-0.35^2}=0.66\,\text{rad/sec}$$

The rise time is:

$$T_r=\frac{\pi-\beta(rads)}{\omega_d}=\frac{\pi-\cos^{-1}\zeta}{\omega_d}=\frac{3.14-1.21}{0.66}=2.92\,\text{sec}$$

The peak time T_p is:

$$T_p=\frac{\pi}{\omega_d}=\frac{3.14}{0.66}=4.75\,\text{sec.}$$

The percentage overshoot (*%OS*) is evaluated as:

$$\%OS=e^{-\zeta\pi/\sqrt{1-\zeta^2}}\times 100$$

Or
$$\%OS=e^{0.35\pi/\sqrt{1-0.35^2}}\times 100=31\%$$

The settling time is:

$$T_S=\frac{4}{\zeta\omega_n}=\frac{4}{0.35\times 0.707}=16.2\,\text{sec}$$

Summarizing the results:

ω_n	ζ	T_r	T_p	%OS	T_S
0.707 rad/sec	0.35	2.92 sec	4.75 sec	31%	16.2 sec

Example 6: An electrical system has the form of *RLC* series circuit with R = 1Ω, L = 1H and C = ½ *F*. Find the transient-response specifications; T_r, T_p, *%OS* and T_s. The input is the voltage $v(t)$ and the output is the voltage $v_C(t)$ across the capacitor.

Solution: Applying KVL, the loop equation of the system is:

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt$$

Since the output is the voltage across the capacitor and the current through the capacitor is $i(t)$, therefore: $i(t) = C\frac{dv_C(t)}{dt}$. The loop equation of the system is:

$$v(t) = LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t)$$

Putting the values of R, L and C and taking the Laplace transform we have:

$$V(s) = \left(\frac{1}{2}s^2 + \frac{1}{2}s + 1\right)V_C(s)$$

The transfer function is: $T(s) = \frac{V_C(s)}{V(s)} = \frac{2}{s^2 + s + 2}$

By comparing the given transfer function with the general transfer function of the second-order system, we have: $\omega_n = 1.414$ rad/sec and $\zeta = 0.35$. Therefore:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 1.414 \times \sqrt{1-0.35^2} = 1.32 \text{ rad/sec}$$

The rise time is:

$$T_r = \frac{\pi - \beta(rads)}{\omega_d} = \frac{\pi - \cos^{-1}\zeta}{\omega_d} = \frac{3.14 - 1.21}{1.32} = 1.46 \sec$$

The peak time T_p is:

$$T_p = \frac{\pi}{\omega_d} = \frac{3.14}{1.32} = 2.38 \sec.$$

The percentage overshoot (%OS) is evaluated as:

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

Or $$\%OS = e^{0.35\pi/\sqrt{1-0.35^2}} \times 100 = 31\%$$

The settling time is:

$$T_S = \frac{4}{\zeta\omega_n} = \frac{4}{0.35 \times 1.414} = 8.1 \sec$$

Summarizing the results:

ω_n	ζ	T_r	T_p	%OS	T_S
1.414 rad/sec	0.35	1.46 sec	2.38 sec	31%	8.1 sec

Effect of Pole Shifting on System Response

Shifting of poles in the *s*-plane horizontally (right or left), vertically (up and down) or radially provide valuable insight into the system transient response.

Shifting pole vertically: Altering the imaginary component of the system's complex pole: $-\sigma + j\omega_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$ with real component unchanged, the movement of the complex pole will be along the vertical line as shown in Figure (7). Both the damping ratio ζ and ω_n will change and since the product $\zeta\omega_n$ is constant (real pole unchanged) the corresponding increase in frequency will be accompanied by corresponding decrease in damping ratio. This will affect the transient-performance parameters; rise time T_r, peak time T_p, %OS and settling time T_S.

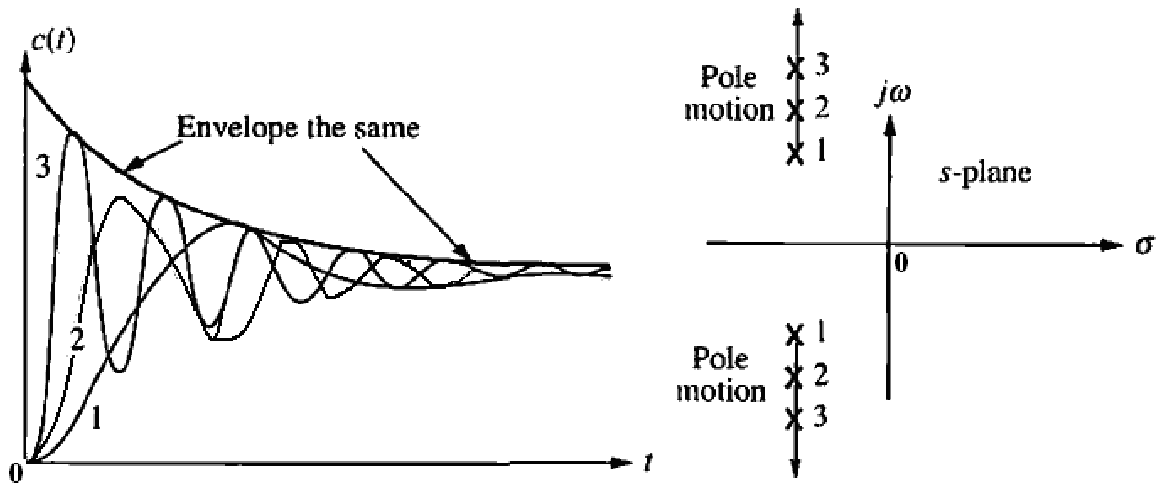


Figure 7: Effect of Shifting the Pole Vertically

Shifting pole horizontally: When a pole: $-\sigma + j\omega_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$ is shifted horizontally (left or right), the real component of the pole $\sigma = \zeta\omega_n$ is altered while the imaginary component ω_d is unchanged as shown in Figure (8).

Shifting of pole to the left would change the real component (both ζ and ω_n). In this case both the damping ratio ζ will and the natural frequency ω_n will increase. The transient response damps out more rapidly. The rise time will increase slightly, and the transient response thus becomes steeper. The peak time will of course be the same so long as the imaginary part remains the same, however, the percentage overshoot will be reduce as the real pole is shifted further towards left. The settling time will reduce when the pole is shifted towards left, meaning that the transient due to disturbances or otherwise will die out quickly and the system will tends to be more stable.

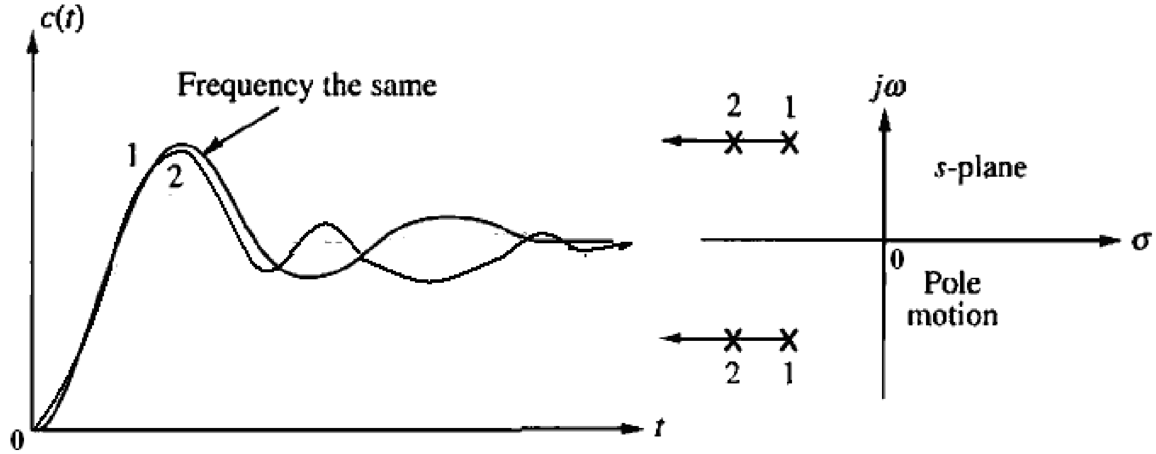


Figure 8: Effect of Pole Shifting Horizontally

Shifting pole radially: Shifting the pole: $-\sigma + j\omega_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$ along a constant radial line (constant damping line) yields the responses as illustrated in Figure (9). In this case the percent overshoot remains the same because the damping ratio ζ is constant along this line. When the complex pole is shifted up along the radial line, it tends to move towards the left in the *s*-plane. Since ω_n is inversely proportional to the rise time, peak time and settling time, therefore the rise time, peak time and settling time all decreases as ω_n increases. Thus the transient response speeds up as the complex pole is shifted upward radially making the system more stable.

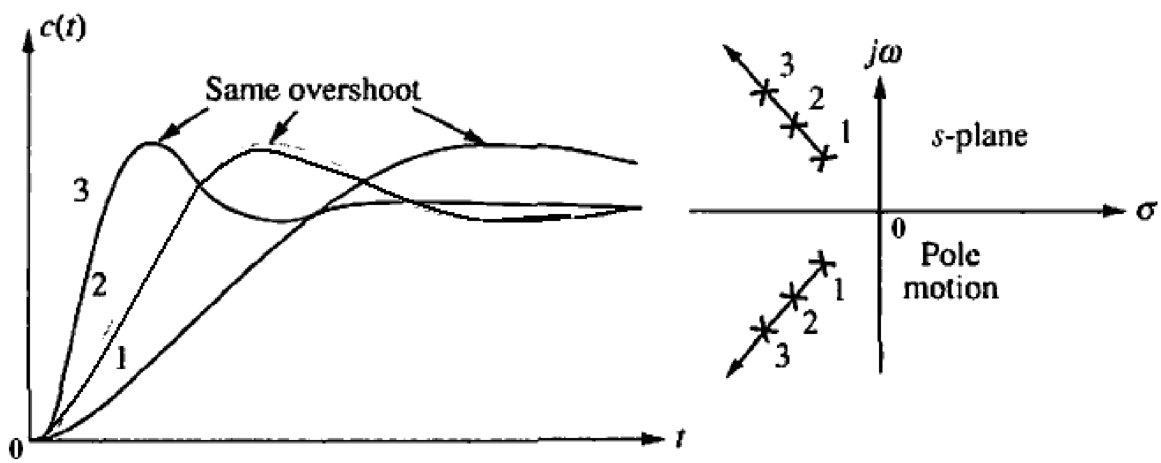


Figure 9: Effect of Pole Shifting Radially

On the other hand, when the pole is shifted downwards along the radial line, it tends to move to the right in the *s*-plane, thereby increasing the rise time, peak time and settling time, whereas the percent overshoot remains unchanged. The transient response will be slow. Thus the pole location and the transient-response specifications of the second-order under-damped response are related.

The effect of the transient-response specifications on the pole shifting is summarized in Table (3).

Table 3

Pole Shifting	Pole Component	Rise Time	Peak Time	Percent Overshoot	Settling Time	Behavior
Up	Imaginary component	Decrease	Decrease	Increases	Unaffected	Envelope same
Left	Real component	Increase	Unaffected	Decreases	Decreases	Frequency same
Radial Upward	Both components	Decreases	Decreases	Unaffected	Decreases	Rapid response