Lecture # 3

Control System Representation in Modern Time-Domain

For the analysis of control systems, mathematical modeling is the first step. Modeling can be time domain (differential equations) or frequency domain (s-domain). In fact, the representation of control systems by differential equations is referred to as classical time domain. The primary disadvantage of the classical time domain approach is, however, limited to LTI systems. or systems that can be approximated as such. A major advantage of *s*-domain techniques is that they rapidly provide stability and transient-response information. A modern time domain representation is also frequently used known as state-space representation, applicable to both LTI and non-linear systems.

State Space

The state-space representation is used for the same class of systems modeled by classical methods. In state-space, the state of a system refers to the past, present, and future conditions. State-space approach defines set of state variables and state equations to model dynamic systems. Some important terminologies used in state-space approach are as follows:

System variable: Variable that responds to an input or places initial conditions in a system is referred to as system variable. Usually quantities that are associated with energy storage devices, for example in electric circuits; the inductor current and the capacitor voltage.

State variables: (phase variables): The state variables of a system are defined as $x_1(t)$, $x_2(t),\ldots,x_n(t)$, that determine the state of the system at any time $t > t_0$. At any initial time $t = t_0$, the state variables $x_1(t_0)$, $x_2(t_0)$, • • •, $x_n(t_0)$ define the initial states of the system. In physical systems, the state variables are those which incorporate change of state and are associated with the energy-storage devices. Thus in an electrical circuit, the inductor current and the capacitor voltage are the state variables since the voltage and current for these devices exists in derivative form, exhibiting a rate of change.

Column and Row Vector: A single column matrix is a column vector whereas a single row matrix is a row vector.

State equations: State equations are a set of *n* number of first-order differential equations with *n* state variables.

Output equation: The output equation expresses the output variables of a system as linear combinations of the state variables and the inputs.

The state and output equations can be written in matrix form if the system is linear. The state represents the state of energy-storage devices in terms of their currents and voltages for electrical system and displacement and velocity for mechanical systems. The state equation can generally be written as:

$$\frac{dx(t)}{dt} = \boldsymbol{A}x(t) + \boldsymbol{B}r(t) \qquad 1$$

Where $x(t)$ are the state variables and $r(t)$ is the input variable. One should not confuse the state variables with the outputs of a system. An output of a system is a variable that can be measured, but a state variable does not always satisfy this requirement. In the output equation, the output quantity is defined in terms of state variables and can generally be written as:

$$c(t) = \boldsymbol{C}x(t) + \boldsymbol{D}r(t) \qquad 2$$

Where:

$x(t)$ is the state variable

$\frac{dx(t)}{dt}$ is the time-derivative of the state variable

$c(t)$ is the output variable

And $r(t)$ is the input variable

In Eq (1) and Eq (2), the quantities ***A***, ***B, C*** and ***D*** are space vectors and are expressed in matrix form. Thus:

A = System or evolution matrix (n x n)

B = Input or control matrix (n x m)

C = Output or observation matrix (m x p)

D = Direct transmittance matrix (p x q)

State-Space Approach

Our scope for state-space representation will only be confined to electrical systems. The modeling of a system using state-space method takes the following approach:

1. Select all possible system variables and refer them as state variables (those quantities, which produce a change of state (input) and those which exhibit state change (inductor current and capacitor voltage).
2. Label each voltage drop with polarity across each passive element and label current with direction in each branch. Remember that for voltage drop across each element, the direction of current must be from positive to negative terminal of voltage drop.
3. Relate state variables with state derivatives through the use of KVL and KCL for every possible loop and nodes.
4. For an n^{th}-order system, obtain n simultaneous, first-order differential equations in terms of the state variables. These simultaneous differential equations are then referred to as state equations.
5. Algebraically combine the state variables with the system's input and find all of the other system variables for $t > t_0$. This algebraic equation is referred to as the output equation.
6. The state equations and the output equations are represented in matrix form. This mathematical representation of the system is referred to as a state-space representation.

Let us consider an *RLC* series circuit with a voltage $v(t)$ applied across the combination and that the output is taken across the inductor as; inductor voltage $v_L(t)$. Let us assume a current $i(t)$ flowing in the circuit. The circuit is a single loop and there are no nodes. The equation of the circuit through the use of KVL is:

$$v(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)dt \qquad 3$$

Remember is state equation there must not be any integral term. But we cannot directly take the derivative of both sides in order to get rid of integral will not make sense, since the time-derivative of the input $v(t)$ is not defined. However, we can replace the current by the time-derivative of the charge $q(t)$. Thus Eq (3) can be expressed in terms of charge $q(t)$ as:

$$v(t) = L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) \qquad 4$$

But in Eq (4) a second derivative appears, which is also not allowed. Thus we can simply replace the integral term directly bu the derivative of current (charge). Eq (3) can then be expressed as:

$$v(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\frac{dq(t)}{dt} \qquad 5$$

Therefore the state variable in this example are $q(t)$ and $i(t)$, since their derivative exists and that:

$$\frac{dq(t)}{dt} = i(t) \qquad 6$$

Using Eq (5) can be expressed as:

$$\frac{di(t)}{dt} = -\frac{1}{LC}q(t) - \frac{R}{L}i(t) + \frac{1}{L}v(t) \qquad 7$$

Thus the state equations are Eq (6) and Eq (7) can be arranged as one of the state equations in matrix form with state derivatives on one side and all the state variables on the other side.

$$\begin{bmatrix} \frac{dq(t)}{dt} \\ \frac{di(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} q(t) \\ i(t) \end{bmatrix} + v(t) \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \qquad 8$$

The output is $v_L(t)$, which is: $L\frac{di(t)}{dt}$. Therefore substituting the value of $\frac{di(t)}{dt}$ from Eq (7) we have:

$$v_L(t) = L\frac{di(t)}{dt} = -\frac{1}{C}q(t) - Ri(t) + v(t)$$

Or

$$v_L(t) = \begin{bmatrix} -\frac{1}{C} & -R \end{bmatrix} \begin{bmatrix} q(t) \\ i(t) \end{bmatrix} + v(t)[1] \qquad 9$$

Thus: $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} -\frac{1}{C} & -R \end{bmatrix}$ and $\boldsymbol{D} = [1]$

Let us now consider an *RLC* parallel combination across a current source $i(t)$ as shown in Figure (1) in which all the branch currents are labeled and there is a single node.

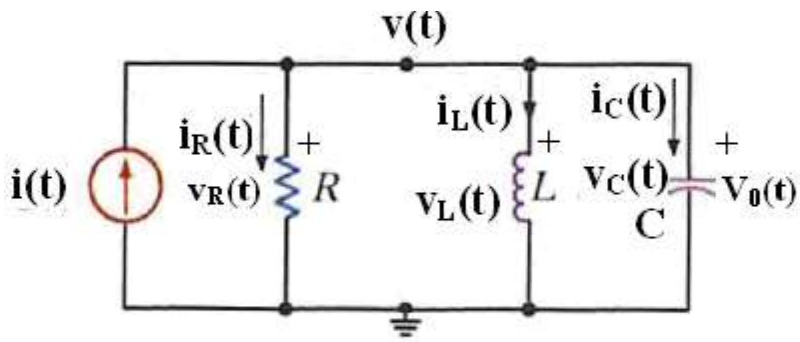

Figure 1: RLC Parallel Combination

Let us assume that the output is taken across the capacitor as a voltage $v_0(t)$. The state variables in this case are the quantities; inductor current $i_L(t)$ and the capacitor voltage $v_C(t)$, since these quantities exist in derivative forms when defining the inductor voltage and capacitor current. In the circuit of Figure (1), applying KCL at the node where the voltage is $v(t)$, we have:

$$i(t) = i_R(t) + i_L(t) + i_C(t)$$

Or
$$i(t) = \frac{v(t)}{R} + i_L(t) + C\frac{dv_C(t)}{dt}$$

Since the node voltage is appearing as the same across each element connected in parallel, therefore this voltage is same as the voltage appearing across the capacitor, which is one of the state variables. Thus $v(t) = v_C(t)$. Therefore:

$$i(t) = \frac{v_C(t)}{R} + i_L(t) + C\frac{dv_C(t)}{dt}$$

Or
$$\frac{dv_C(t)}{dt} = -\frac{1}{C}i_L(t) - \frac{1}{RC}v_C(t) + \frac{1}{C}i(t) \qquad 10$$

Since the other state variable is inductor current, whose derivative, when multiplied with L is the inductor voltage is:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

Since the voltage is same across the parallel combination, therefore: $v_L(t) = v_C(t)$, so that:

$$v_C(t) = L\frac{di_L(t)}{dt}$$

Or $$\frac{di_L(t)}{dt} = \frac{1}{L}v_C(t) \qquad 11$$

Eq (4.7) and Eq (4.8) forms one set of the state equations, which can be expressed in matrix form as:

$$\begin{bmatrix} \frac{di_L(t)}{dt} \\ \frac{dv_C(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + i(t)\begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} \qquad 12$$

Let us consider the output to be the voltage across the capacitor. Thus: $v_0(t) = v_C(t)$. So that:

$$v_0(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + 0 \qquad 13$$

Thus: $\boldsymbol{A} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $\boldsymbol{D} = [0]$

Example 1: Obtain the state-space representation of the electrical system shown in Figure (2a).

Solution: We will proceed according to the following steps:

Step 1: The first step is to define the state variables, which are quantities, associated with the energy storage devices and must be those quantities which exist in first derivative form (except the forcing function; input). Thus the state variables are: $i_L(t)$, $v_{C1}(t)$, $v_{C2}(t)$ and the input is $v(t)$. The governing equations for energy-storage devices in derivative form are:

$$v_L(t) = \frac{di_L(t)}{dt} \qquad 14$$

$$i_{C1}(t) = \frac{dv_{C1}(t)}{dt} \qquad 15$$

$$i_{C2}(t) = \frac{dv_{C2}(t)}{dt} \qquad 16$$

Step 2: In this step all the branch currents and voltages across the energy-storage devices are indicated. The labeled circuit diagram of the given system is shown in Figure (2b) where the reference node is grounded.

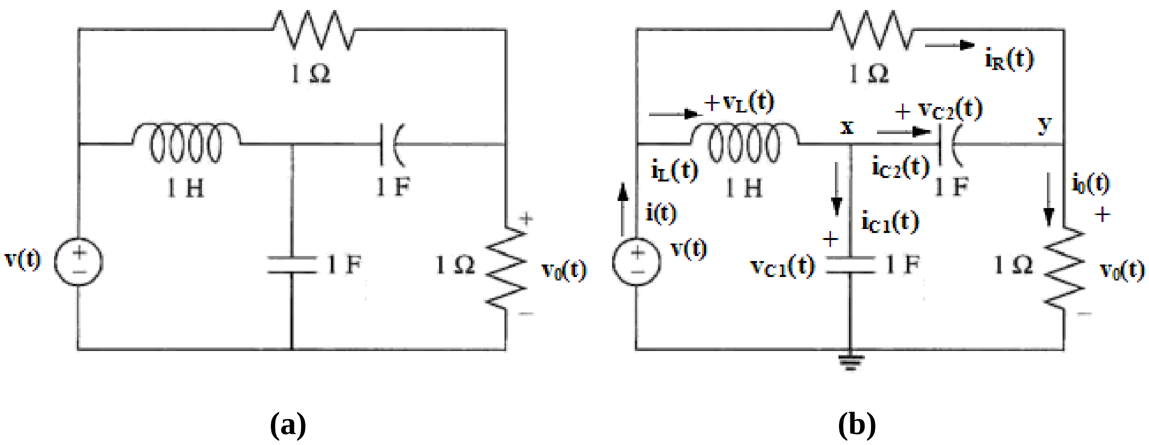

Figure 2: Electrical System of Example (1)

Step 3: We require three state equations (equal to the number of state variables) and one output equation (equal to the number of output). In state equations, the derivative of the state variable is on the LH-side and the state variables on the RH-side. In the all possible loop and nodal equations, all the quantities must be written in terms of the state variables. First the loop containing the voltage source is considered for the formation of loop equation by applying KVL. That is:

$$v(t) = v_L(t) + v_{C1}(t)$$

Or $$v_L(t) = -v_{C1}(t) + v(t) \qquad 17$$

Let us see whether we can get something in derivative form which can help us to form a state equation from the above first loop equation. The LH-side of Eq (17) is the inductor voltage and so that Eq (17) can be written as:

$$\frac{di_L(t)}{dt} = -v_{C1}(t) + v(t) \qquad 18$$

In Eq (18) we can see that there is only a state derivative and state variables. Thus Eq (18) is one of the state equations. Also by applying KVL to the output loop containing the voltage $v_0(t)$ and current $i_0(t)$, we have:

$$v_{C1}(t) = v_0(t) + v_{C2}(t)$$

Or $$v_0(t) = v_{C1}(t) - v_{C2}(t) \qquad 19$$

Eq (19) is the output state equation, since every element in the LH side contains only defined state variables. Considering node y and applying KCL, we have:

$$i_{C2}(t) + i_R(t) = i_0(t) \qquad 20$$

Where $i_R(t) = \dfrac{v(t) - v_0(t)}{1}$ and $i_0(t) = \dfrac{v_0(t)}{1}$. By substituting the expression of $i_R(t)$ and $i_0(t)$, Eq (20) can be expressed as:

$$i_{C2}(t) + v(t) - v_0(t) = v_0(t) \qquad 21$$

Substituting the value of $v_0(t)$ from Eq (19) in the above Eq (21), we have:

$$i_{C2}(t) + v(t) - [v_{C1}(t) - v_{C2}(t)] = v_{C1}(t) - v_{C2}(t)$$

Or
$$i_{C2}(t) = 2v_{C1}(t) - 2v_{C2}(t) - v(t) \qquad 22$$

Or
$$\frac{dv_{C2}(t)}{dt} = 2v_{C1}(t) - 2v_{C2}(t) - v(t) \qquad 23$$

In Figure (2b), applying KCL at node x, we have:

$$i_L(t) = i_{C1}(t) + i_{C2}(t)$$

Or
$$i_{C1}(t) = i_L(t) - i_{C2}(t) \qquad 24$$

Substituting the value of $i_{C2}(t)$ from Eq (21) in Eq (24), simplifying and rearranging, we have:

$$i_{C1}(t) = i_L(t) - [2v_0(t) - v(t)]$$

Substituting $v_0(t)$ from Eq (19) in the above equation, we have:

$$i_{C1}(t) = i_L(t) - 2[v_{C1}(t) - v_{C2}(t)] + v(t)$$

Or
$$\frac{dv_{C1}(t)}{dt} = i_L(t) - 2v_{C1}(t) + 2v_{C2}(t) + v(t) \qquad 25$$

Eqs (18), (23) and (25) are the state equations, which are re-written as follows:

$$\frac{di_L(t)}{dt} = -v_{C1}(t) + v(t)$$

$$\frac{dv_{C1}(t)}{dt} = i_L(t) - 2v_{C1}(t) + 2v_{C2}(t) + v(t)$$

$$\frac{dv_{C2}(t)}{dt} = 2v_{C1}(t) - 2v_{C2}(t) - v(t)$$

And the output equation is:

$$v_0(t) = v_{C1}(t) - v_{C2}(t)$$

Step 4: The state equations are finally expressed in state matrix form as:

$$\begin{bmatrix} \frac{di_L(t)}{dt} \\ \frac{dv_{C1}(t)}{dt} \\ \frac{dv_{C2}(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_{C1}(t) \\ v_{C2}(t) \end{bmatrix} + v(t) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

And $$v_0(t) = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_{C1}(t) \\ v_{C2}(t) \end{bmatrix} + v(t)[0]$$

Thus: $\boldsymbol{A} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$ and $\boldsymbol{D} = [0]$

Converting Differential Equation into State Space

System representation in classical time domain by differential equations can be converted to modern time domain (state-space) by using the identity: $s = \frac{d}{dt}$. The dependent variables and defined in terms of state variables and state derivatives.

Example 2: Represent the given differential equation: $\frac{d^2c(t)}{dt^2} + 4\frac{dc(t)}{dt} + 3c(t) = 2r(t)$ in state-space, with c(t) as dependent variable and r(t) as independent variable. Assume zero initial conditions.

Solution: Let us suppose that: $c(t) = x_1(t)$, then:

$$\frac{dc(t)}{dt} = \frac{dx_1(t)}{dt} = x_2(t)$$

And $$\frac{d^2c(t)}{dt^2} = \frac{d}{dt}\frac{dx_1(t)}{dt} = \frac{dx_2(t)}{dt} = x_3(t)$$

The given differential equation can therefore be written as:

$$\frac{d^2c(t)}{dt^2} = -4\frac{dc(t)}{dt} - 3c(t) + 2r(t)$$

Or $$\frac{dx_2(t)}{dt} = -4x_2(t) - 3x_1(t) + 2r(t)$$

Thus the state equations are:

$$\frac{dx_1(t)}{dt} = x_2(t)$$

And $$\frac{dx_2(t)}{dt} = -4x_2(t) - 3x_1(t) + 2r(t)$$

The state-space representation is:

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + r(t) \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

And $$c(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 0$$

State Space to Transfer Function

Transfer function is regarded as a powerful tool in many control problems. State space representation can be conveniently converted to transfer function. Systems can be conveniently modeled by a mathematical expression, known as transfer function. To convert the state-space representation into a transfer function, the state equations are re-written in *s*-domain followed by simplification and rearrangement to obtain the ratio of the output to input variable. Consider the general form of state equation and output equation.

$$\frac{dx(t)}{dt} = \boldsymbol{A}x(t) + \boldsymbol{B}r(t)$$

$$c(t) = \boldsymbol{C}x(t) + \boldsymbol{D}r(t)$$

Taking the Laplace transform of the above equations to express them in *s*-domain:

$$sX(s) = \boldsymbol{A}X(s) + \boldsymbol{B}R(s) \qquad 26$$

And $$C(s) = \boldsymbol{C}X(s) + \boldsymbol{D}R(s) \qquad 27$$

From Eq (26), we have:

$$X(s)[s\boldsymbol{I} - \boldsymbol{A}] = \boldsymbol{B}R(s)$$

Or $$X(s) = [s\boldsymbol{I} - \boldsymbol{A}]^{-1}\boldsymbol{B}R(s) \qquad 28$$

Where $\boldsymbol{I}$ is the identity matrix. Substituting $X(s)$ from Eq (28) in Eq (27), we have:

$$C(s) = \boldsymbol{C}[s\boldsymbol{I} - \boldsymbol{A}]^{-1}\boldsymbol{B}R(s) + \boldsymbol{D}R(s)$$

Or $$C(s) = (\boldsymbol{C}[s\boldsymbol{I} - \boldsymbol{A}]^{-1}\boldsymbol{B} + \boldsymbol{D})R(s)$$

The transfer function is then:

$$\frac{C(s)}{R(s)} = \boldsymbol{C}[s\boldsymbol{I} - \boldsymbol{A}]^{-1}\boldsymbol{B} + \boldsymbol{D} \qquad 29$$

Example 3: From the given state representation of the system, obtain the transfer function $C(s)/R(s)$. Also obtain the differential equation of the system.

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + r(t)\begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

And $$c(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + r(t)[0]$$

Solution: Given that: $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $\boldsymbol{D} = [0]$

Therefore: $$T(s) = \frac{C(s)}{R(s)} = \boldsymbol{C}[s\boldsymbol{I} - \boldsymbol{A}]^{-1}\boldsymbol{B} + \boldsymbol{D}$$

$$s\boldsymbol{I} - \boldsymbol{A} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Or $$s\boldsymbol{I} - \boldsymbol{A} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 3 & 2 & s+5 \end{bmatrix}$$

Therefore: $$[s\boldsymbol{I} - \boldsymbol{A}]^{-1} = \frac{1}{s^3 + 5s^2 + 2s + 3}\begin{bmatrix} s^2 + 5s + 2 & s+5 & 1 \\ -3 & s(s+5) & s \\ -3s & -(2s+3) & s^2 \end{bmatrix}$$

$$\boldsymbol{C}[s\boldsymbol{I}-\boldsymbol{A}]^{-1}\boldsymbol{B} = \frac{1}{s^3+5s^2+2s+3}\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix} s^2+5s+2 & s+5 & 1 \\ -3 & s(s+5) & s \\ -3s & -(2s+3) & s^2 \end{bmatrix}\begin{bmatrix}0\\0\\10\end{bmatrix}$$

Simplifying: $$\frac{C(s)}{R(s)} = \frac{10}{s^3+5s^2+2s+3} \qquad 30$$

The differential equation can be obtained once the transfer function is known. The transfer function is worked out by cross multiplying Eq (30), from which we have:

$$C(s)(s^3+5s^2+2s+3) = 10R(s)$$

Or $$s^3C(s)+5s^2C(s)+2sC(s)+3C(s) = 10R(s)$$

Transforming from *s*-domain to time-domain by using the identity: $s \equiv \frac{d}{dt}$, we have:

$$\frac{d^3c(t)}{dt^3}+5\frac{d^2c(t)}{dt^2}+2\frac{dc(t)}{dt}+3c(t) = 10r(t) \qquad 31$$