Lecture # 2

Transfer Function of Mechanical Systems

The transfer function of the mechanical systems likewise can be obtained from the governing differential equations describing the system. Mechanical systems are classified as:

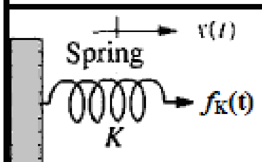
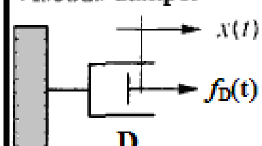
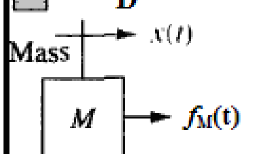
1. Translational
2. Rotational

Like electrical systems, mechanical systems have driving sources and passive elements. We will discuss both in addition to electro-mechanical systems.

Mechanical Translational System

Mechanical translational system is characterized by driving source, which is Newtonian force and three passive, linear components; mass, spring (stiffness) and damper (dissipation). Mass and spring are energy-storage devices, whereas the damper dissipates energy. We will see later that the two energy-storage devices are analogous to the two electrical energy-storage devices, whereas the damper is analogous to electrical resistance. Mechanical systems obey Newton's law; that the sum of the forces equals zero or that the sum of the applied forces must be equal to the sum of the reactive or transmitted forces. This is analogous to the Kirchoff's laws as applicable to electrical circuits, in which case the sum of the applied voltages must be equal to the voltage drops in a loop (KVL) and currents summing up at node is zero (KCL). Table (1) lists the mechanical translational system elements and their equations.

Table 1

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f_K(t) = K\int_0^t v(\tau)d\tau$	$f_K(t) = Kx(t)$	K
Viscous damper 	$f_D(t) = \mathbf{D}v(t)$	$f_D(t) = \mathbf{D}\frac{dx(t)}{dt}$	$\mathbf{D}s$
Mass 	$f_M(t) = M\frac{dv(t)}{dt}$	$f_M(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Mass: The mass *M* is the inertial element and the displacements at both its ends are the same. A force applied to a mass produces acceleration '*a*' of the mass. The reaction force f_M is equal to the product of mass and acceleration, characterizing the equation of motion and is opposite in direction to the applied force that is:

$$f(t) = Ma(t)$$

Since acceleration is the time rate of change of velocity and velocity is time rate of change of displacement, the above equation can be expressed in terms of the velocity *v*(t) and displacement *x*(t), and can be written as:

$$f(t) = M\frac{dv(t)}{dt} = M\frac{d^2x(t)}{dt^2} \qquad 1$$

Damper: The damping *D* also represents viscous fluid friction characterizes the damper, element that absorbs and dissipates energy. Damping may be added to a system by use of a dashpot. This element has a constructional feature of a housing filled with an incompressible fluid. The basic operation of a dashpot is that the applied force acting on the piston causes the piston to do work against a fluid, thus resulting in its displacement. Friction in mechanical systems is characterized by an equivalent damping effect, since it dissipates energy. The damping force required to move the piston inside the housing is proportional to the velocity. The reaction damping force f_D is approximated by the product of damping *D* and the relative velocity of the two ends of the dashpot. The assumption that the viscous friction is linear simplifies the expression, which is:

$$f(t) = Dv(t) = D\frac{dx(t)}{dt} \qquad 2$$

Spring: The component spring obeys Hooke's law; thus if stretched, the spring tries to contract; if compressed, it tries to expand to its normal length. The stiffness, *K* provides the necessary restoring force. The displacement at the two ends of the spring is different, unless one of its ends is fixed at some reference point, where its displacement is then zero. The reaction force on each end of the spring is the same and is equal to the product of the stiffness *K* and the amount of deformation (change in the length, governed by the end displacement) of the spring. The force equation, in accordance with Hooke's law, is then:

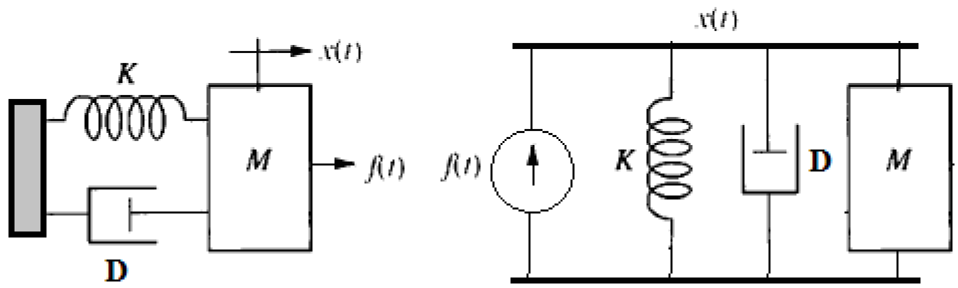
$$f(t) = Kx(t) \qquad 3$$

Steps for obtaining the Transfer Function

1. The equivalent mechanical network is drawn, which comprise of a straight horizontal line as reference surface and nodes (displacements) are placed suitably above this reference line.
2. Differential equations are formed for each displacement node using Newton's Law in conjunction with KCL. The force equation is written for each node or displacement by equating the sum of the applied forces with the sum of transmitted forces at each node. The equations are similar to the nodal equations in an electric circuit, with force analogous to current, displacement analogous to voltage, and the mechanical elements with their appropriate operators analogous to admittance or impedances.

3. The differential equations are then expressed in s-domain by applying laws of Laplace Transform.
4. Output in the form of displacement is defined and the transfer function is finally obtained by manipulation and elimination of all other displacements except the defined output displacement. The ratio of output to input (driving force) in s-domain is the transfer function.

Example 1: Consider a linear mechanical translational system shown in Figure (1). Obtain its governing differential equation and the transfer function.



(a) **(b)**

Figure 1: (a) Mechanical System (b) Equivalent Mechanical Network

Solution: From the given mechanical system of Figure (1a), a mechanical network is formed as shown in Figure (1b) in which all the elements are connected as mechanical impedances. The force is applied to the mass *M* causing a displacement *x*(*t*) acts on the spring and the dashpot so that there is only a single displacement. A node representing the displacement *x*(*t*) is laid as a line over a reference line of zero displacement. One end of the mass is always connected to the reference, while the other is connected to the node where the mass suffers a displacement. Mass always rests on the reference because it constitutes the mass of the system as a whole. All the other elements of the mechanical system are connected in the network in accordance with the displacements at their ends. For example, in this case, one end of both the spring and the dashpot has the displacement of the mass and their other ends are connected to the reference. The force is applied to the mass *M* causing a displacement *x*(*t*) acts on the spring and the dashpot. Thus according to Newton's law in conjunction with KCL:

$$f(t) = f_M(t) + f_D(t) + f_K(t)$$

The differential equation for node *x*(*t*) is:

$$f(t) = M\frac{d^2x(t)}{dt^2} + D\frac{dx(t)}{dt} + Kx(t)$$

Taking the Laplace transform of both sides:

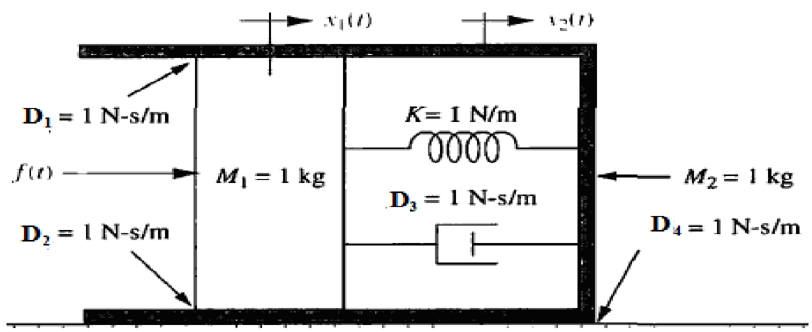
$$F(s) = s^2MX(s) + sDX(s) + KX(s)$$

The transfer function is:

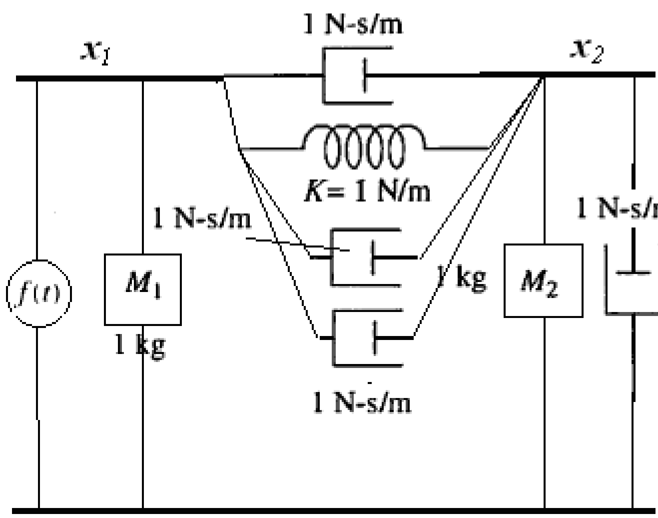
$$\frac{X(s)}{F(s)} = \frac{1}{s^2 M + Ds + K}$$

Or
$$\frac{X(s)}{F(s)} = \frac{1/M}{s^2 + (D/M)s + (K/M)}$$

Example 2: Obtain the differential equation and transfer function: $\frac{X_2(s)}{F(s)}$ of the mechanical system shown in Figure (2a).



(a)



(b)

Figure 2: Mechanical System of Example (2)

Solution: The system can be viewed as a mass M_1 pushed in a compartment or housing of mass M_2 against a fluid, offering resistance. Or it can be a box in which a rigid object is being pushed against a "cushion", offering resistance. The resistance being offered by the fluid or "cushion" provides the necessary restoring force. While pushing mass M_2, and due to the resistance being offered by fluid or "cushion" to the mass M_1, the compartment or the housing or the box so suffers a displacement on a friction surface, which can be a table or floor. The mass M_1 is also moved along friction surfaces which are the walls of the compartment or box. In forming the mechanical network of the system, a reference line of zero displacement is drawn and since there are two displacements, so that two nodes are placed above the reference line. One end of masses M_1 and M_2 are connected to the reference with their other ends connected to the respective nodes where they suffer displacement. The other components are connected between nodes, which corresponds to the ends where they suffer displacements in the actual system. The mechanical network obtained as such is shown in Figure (2b). The differential equations are formed for each displacement node [referring to Figure (2b)]. For node $x_1(t)$, we have:

$$f(t) = M_1 \frac{d^2x_1(t)}{dt^2} + (D_1 + D_2 + D_3)\frac{d[x_1(t) - x_2(t)]}{dt} + K[x_1(t) - x_2(t)]$$

Or $$f(t) = \frac{d^2x_1(t)}{dt^2} + 3\frac{dx_1(t)}{dt} + x_1(t) - 3\frac{dx_2(t)}{dt} - x_2(t) \qquad 4$$

For node $x_2(t)$, we have:

$$0 = M_2 \frac{d^2x_2(t)}{dt^2} + (D_1 + D_2 + D_3)\frac{d[x_2(t) - x_1(t)]}{dt} + K[x_2(t) - x_1(t)] + D_4\frac{dx_2(t)}{dt}$$

Or $$0 = \frac{d^2x_2(t)}{dt^2} + 4\frac{dx_2(t)}{dt} + x_2(t) - 3\frac{dx_1(t)}{dt} - x_1(t) \qquad 5$$

Expressing Eq (4) and Eq (5) in s-domain, we have:

$$F(s) = s^2X_1(s) + 3sX_1(s) + X_1(s) - 3sX_2(s) - X_2(s) \qquad 6$$

And $$0 = s^2X_2(s) + 4sX_2(s) + X_2(s) - 3sX_1(s) - X_1(s) \qquad 7$$

From Eq (7), the expression for $X_1(s)$ in terms of $X_2(s)$ is:

$$X_1(s) = \frac{(s^2 + 4s + 1)}{3s + 1} X_2(s)$$

Substituting in Eq (6) to replace of $X_1(s)$, simplifying and expressing it as a transfer function: $\frac{X_2(s)}{F(s)}$ that is:

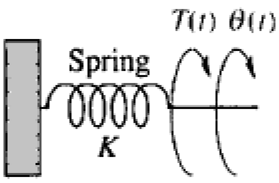
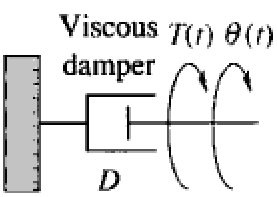
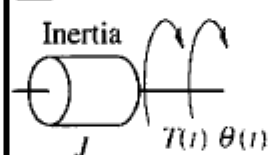
$$\frac{X_2(s)}{F(s)} = \frac{3s + 1}{s(s^3 + 7s^2 + 5s + 1)}$$

Mechanical Rotational System

The driving sources of the rotational mechanical systems and translational mechanical system have the same effect that is to cause motion, except that torque replaces force. The components undergo rotation instead of translation and so angular displacement replaces translational displacement and translational velocity is replaced by angular velocity. The mechanical components for rotational systems have the same characteristics as those for translational systems, except that the mass is rotating inertial mass accounting for moment of inertia and Hooke's spring is torsional spring, which refers to the ability to resist the twisting of, for example a shaft while rotating. The viscous damping accounts for friction and energy dissipation. The parameters *K*, *D*, and *J* are called spring constant, coefficient of viscous friction, and moment of inertia, respectively.

Likewise, writing the differential equations is simplified by first drawing the mechanical network for the system. Then the torque equation is written for each node by equating the sum of the torques at each node to zero. Table (2) shows the components along with the relationships between torque and angular velocity, as well as angular displacement.

Table 2

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t)=K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K (N-m/rad)
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds (N-m.s/rad)
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2 (kg-m^2)

Inertia: The torque applied to rotate an object with having moment of inertia *J* will produce angular acceleration α that is:

$$T(t) = J\alpha$$

In terms of angular velocity $\omega(t)$ and angular displacement $\theta(t)$, the relation between torque and moment of inertia can be expressed as:

$$T(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2} \quad 8$$

Damper: Damping occur whenever a body moves through a fluid, which may be a liquid or a gas such as air. To produce motion of the body, a torque must be applied to overcome the reaction damping torque that depends on the viscous friction coefficient *D*. The angular displacement of the two ends of the dashpot is different because some of the energy is dissipated. Damper in mechanical rotational systems is also provided where there is friction, such as that at points where a shaft is passed through bearings or power loss between gears or where two rotating wheels, representing moment of inertia are sliding over each other during transmission of motion, for example in automobile clutch system. In case one end is fixed to the reference or the point of zero angular displacement, then the torque acting on the damper will produce an angular displacement $\theta(t)$ at the end where the torque is applied, so that:

$$T(t) = D\omega(t) = D\frac{d\theta(t)}{dt} \qquad 9$$

Spring: This component represents the torsion created in a rotating body such as shaft. In case when one end of the spring is connected to the reference with zero angular displacement, then the only displacement $\theta(t)$ will take place at the end where torque is applied, so that:

$$T_K(t) = K\theta(t) \qquad 10$$

Example 3: Obtain the differential equations and transfer function of the rotational mechanical system shown in Figure (4).

Solution: First of all a mechanical network of the given system is formed. A reference node of zero angular displacement is laid down as a base line. As seen in Figure (3) there are two angular displacements possible, so that two nodes are laid above the reference line (reference node). The components are then connected between nodes and the reference according to the angular displacements suffered on the application of torque as according to the system diagram given in Figure (3). The mechanical network is shown in Figure (4).

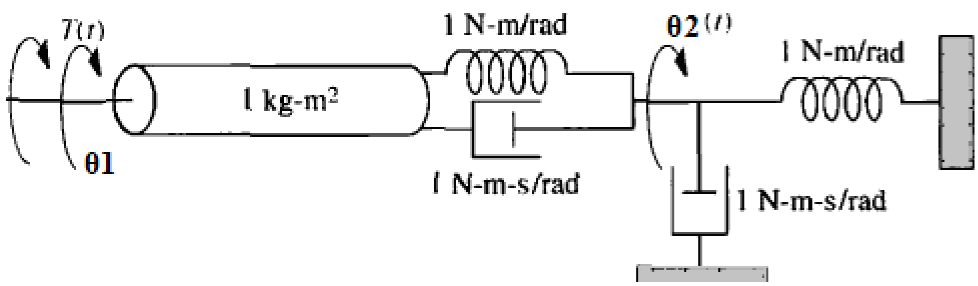


Figure 3: Mechanical System of Example (3)

The differential equations are formed for each displacement node [referring to Figure (4)]. For node representing the angular displacement $\theta_1(t)$, we have:

$$T(t) = \frac{d^2\theta_1(t)}{dt^2} + \frac{d\theta_1(t)}{dt} + \theta_1(t) - \frac{d\theta_2(t)}{dt} - \theta_2(t) \qquad 11$$

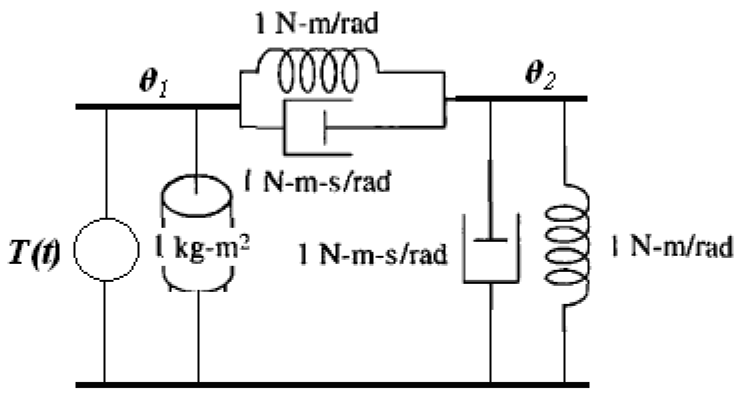


Figure 4: Mechanical Network

For node representing the angular displacement $\theta_2(t)$, we have:

$$0 = 2\frac{d\theta_2(t)}{dt} + 2\theta_2(t) - \frac{d\theta_1(t)}{dt} - \theta_1(t) \qquad 12$$

Expressing Eq (11) and Eq (12) in *s*-domain, we have:

$$T(s) = (s^2 + s + 1)\theta_1(s) - (s+1)\theta_2(s) \qquad 13$$

And $$0 = (2s+2)\theta_2(s) - (s+1)\theta_1(s) \qquad 14$$

From Eq (14), the expression for $\theta_1(s)$ in terms of $\theta_2(s)$ is obtained as:

$$\theta_1(s) = 2\theta_2(s) \qquad 15$$

Substituting in Eq (13) and expressing it as a transfer function as:

$$\frac{\theta_2(s)}{T(s)} = \frac{1}{2s^2 + s + 1}$$

Transfer Function of Electromechanical System

Electromechanical systems consist of electrical and mechanical subsystems that find extensive applications in modern control systems, for example robots. Majority of electromechanical systems incorporate a servomotor, which is an electromechanical component that yields a displacement output for a voltage input, that is, a mechanical output generated by an electrical input. The servomotor simply refers to a motor that is used for the function of control. Figure (5) is a simple outlay of a servomotor comprising of a magnetic field that constitutes the stator (stationary component). A rotating member, called the armature comprise of winding fixed in slots. The working is same as any electric motor, which you would have studied in electrical machines course.

Thus the machine can be modelled for its rotating member (armature) only since it carries mechanical load. The armature is composed of the current-carrying conductors that have total resistance R_a and inductance L_a, and are considered as series combination. The rotating armature conductors under the influence of magnetic field produces a back emf $v_b(t)$, which is proportional to the angular speed $\omega(t)$. Thus:

$$v_b(t) = K_b \omega(t) = \frac{d\theta(t)}{dt} \qquad 16$$

K_b is a constant of proportionality called the back emf constant and $\theta(t)$ is the angular displacement at the motor shaft. Taking the Laplace transform of Eq (16), we obtain:

$$V_b(s) = K_b s\theta(s) \qquad 17$$

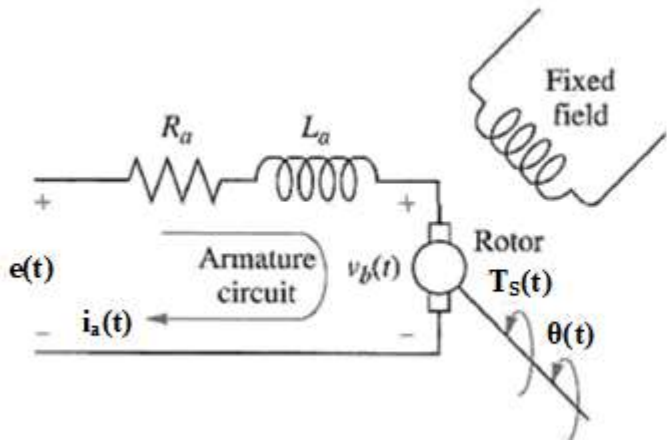


Figure 5: Electric Servo-Motor

The relationship between the armature current, $i_a(t)$, the applied armature voltage, $e_a(t)$, and the back emf, $v_b(t)$, is formed by writing a loop equation based on KVL around the Laplace transformed armature circuit having resistance R_a and inductance L_a that is:.

$$E(s) = I_a(s)(R_a + sL_a) + V_b(s)$$

Or

$$E(s) = I_a(s)(R_a + sL_a) + K_b s\theta(s) \qquad 18$$

When a current flow in the armature conductors, they are acted upon by a torque. The torque developed by the motor shaft or the shaft torque $T_S(t)$ is proportional to the armature current that is: $T_S(t) \propto i_a(t)$; thus:

$$T_S(t) = K_T i_a(t)$$

Where $T_S(t)$ is the torque developed at the motor shaft and K_T is a constant of proportionality, called the motor torque constant, which depends on the motor and magnetic field characteristics. In a consistent set of units, the value of K_T is equal to the value of K_b. Taking the Laplace transform of the above expression, we have:

$$T_S(s) = K_T I_a(s)$$

Or $$I_a(s) = \frac{1}{K_T} T_S(s) \qquad (19)$$

Substituting I_a(s) from Eq (19) in Eq (18), we have:

$$E(s) = \frac{1}{K_T}(R_a + sL_a)T_S(s) + K_b s\theta(s) \qquad (20)$$

Now we must find $T_S(s)$ in terms of $\theta(s)$ if we are to separate the input and output variables and obtain the transfer function of the form: $\frac{\theta(s)}{E(s)}$. Since the motor shaft has a mechanical load, which can be in the form of pulley mounted on the motor shaft, which is usually supported on bearing will have a combined effect of inertia and viscous damping that can be represented as shown in Figure (6). So that the mechanical system is modeled as:

$$T_S(s) = s^2 J\theta(s) + sD\theta(s) \qquad (21)$$

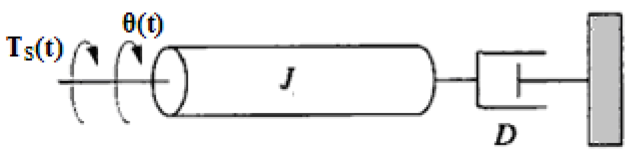


Figure 6: Representation of Mechanical Load

Substituting $T_S(s)$ from Eq (21) in Eq (20), we have:

$$E(s) = \frac{1}{K_T}(R_a + sL_a)(s^2 J + sD)\theta(s) + K_b s\theta(s) \qquad (22)$$

Or $$K_T E(s) = (R_a + sL_a)(s^2 J + sD)\theta(s) + K_T K_b s\theta(s)$$

Or $$\frac{\theta(s)}{E(s)} = \frac{K_T}{(R_a + sL_a)(s^2 J + sD) + sK_T K_b}$$

Or $$\frac{\theta(s)}{E(s)} = \frac{K_T / JL_a}{s^3 + s^2\left(\frac{JR_a + DL_a}{JL_a}\right) + s\left(\frac{DR_a + K_T K_b}{JL_a}\right)}$$

Or $$\frac{\theta(s)}{E(s)} = \frac{K_1}{s^3 + as^2 + bs}$$

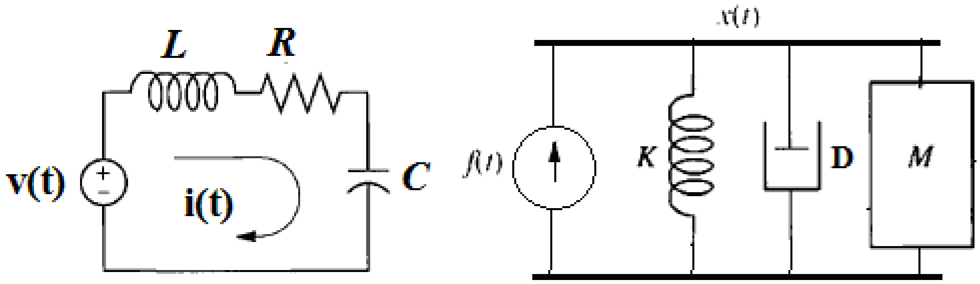
Where; $K_1 = \frac{K_T}{JL_a}$, $a = \left(\frac{JR_a + DL_a}{JL_a}\right)$ and $b = \left(\frac{DR_a + K_T K_b}{JL_a}\right)$.

Or $$\frac{\theta(s)}{E(s)} = \frac{K_1}{s(s^2 + as + b)}$$ 23

Analogous Systems

Analogy between electrical and mechanical system and their components parameters and quantities can be drawn by comparing the governing equation of each element in addition to the loop or nodal equations of an electrical system with the equation of motion of a mechanical system. Consider an *RLC* series circuit as shown in Figure (7a). The source voltage is $v(t)$ which drives a current $i(t)$ that flows in a single loop forming series combination of *R*, *L* and *C*. Applying KVL, the loop equation is:

$$v(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)dt$$ 24



(a) **(b)**

Figure 7: Series RLC Circuit and its Mechanical Analog

Since, electric current is the rate of flow of charges that is: $i(t) = \frac{dq(t)}{dt}$, Eq (24) can then be expressed as:

$$v(t) = L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t)$$ 25

Now let us consider that the three mechanical components *M*, *D* and *K* are connected in a mechanical system in such a way so as to form a mechanical network shown in Figure (7b). By applying Newton's law, the equation of motion is:

$$f(t) = M\frac{d^2x(t)}{dt^2} + D\frac{dx(t)}{dt} + Kx(t)$$ 26

Since: $v(t) = \frac{dx(t)}{dt}$, Eq (26) can also be expressed as:

$$f(t) = M\frac{dv(t)}{dt} + Dv(t) + K\int v(t)dt$$ 27

Comparing Eq (24) and Eq (27), an analogy between electrical system and mechanical system can be drawn. The driving mechanical quantity (force) is analogous to driving electrical quantity (voltage) and the mechanical quantity, velocity is analogous to the electrical quantity, current. On the other hand, comparing Eq (25) and Eq (26), it can be noted that the driving mechanical quantity (force) is analogous to driving electrical quantity (voltage) and the mechanical quantity, displacement is analogous to the electrical charge.

Thus it can be seen that in both cases the electrical element with L (inductance) is analogous to mechanical element M (mass). In mechanical systems mass is the measure of inertia, so that inductance can be regarded as the measure of magnetic inertia in electrical systems. Now consider a simple *RLC* parallel circuit as shown in Figure (8).

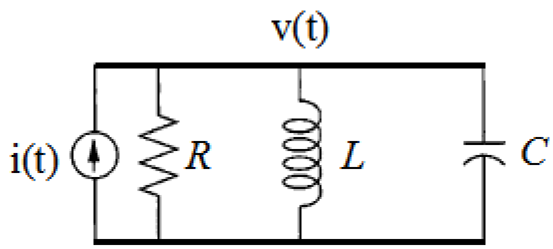


Figure 8: Simple RLC Parallel Circuit

Assuming that the node voltage is $v(t)$, the nodal equation is:

$$i(t) = C\frac{dv(t)}{dt} + \frac{1}{L}\int v(t)dt + \frac{v(t)}{R} \qquad 28$$

Another analogy can be drawn by comparing the force-velocity Eq (27) to the current-voltage Eq (28). Here the analogy is between force and current and between velocity and voltage. In this case the spring is analogous to the inductor, the viscous damper is analogous to the resistor, and the mass is analogous to the capacitor. Table (3) summarizes the analogies between mechanical system and series and parallel electric circuit.

Table 3

Property	**Mechanical**	**Electrical**	**Series Analogy**	**Parallel Analogy**
Inertia	Mass (M)	Inductance (L)	M (Henry)	1/K (Henry)
Dissipation	Viscous Damper (D)	Resistance (R)	D (Ohm)	1/D (Ohm) (Conductance)
Stiffness	Spring (1/K)	Capacitance (C)	1/K (Farad)	M (Farad)
Dependent quantity	Velocity		Loop current	Node Voltage
Independent quantity	Applied Force		Voltage source	Current source