

Lecture # 15

Frequency Response Plots

My dear students, in the world of electronics and communication, we are dealing with range of frequencies from few Hz to MHz or even GHz. The magnitude of signal variations over such wide range of frequencies is tedious exercise, but such characteristics are often necessary in the design and evaluating the system performance. The frequency response characteristic of a system can be found from frequency response plots, referred to as Bode plot, and plotted on a semi-log sheet with frequency as independent variable. The complete frequency response plots consists of magnitude-frequency Bode plot with magnitude in dB (Decibel) and frequency in rad/sec, and phase-frequency Bode plot with phase-angle in degrees and frequency in rad/sec. Frequency may also be considered in Hz.

The Decibel

In communication engineering it is convenient to represent the magnitude of, for example, signal strength in comparison with a reference signal. This makes tasks easier, just like students are graded relatively in comparison, so that they can be categorized as good, average and below average. In obtaining the frequency-response characteristics of communication systems or amplifiers over a wide range of frequencies, it makes sense to use a logarithmic scale for frequencies as well as magnitude. Speaking about signal power, the dB is:

$$\text{dB} = 10 \log \frac{P_2}{P_1} \quad 1$$

It is important to mention that the logarithm is not the natural of exponential. The logarithm is that with base-10. The base-10 refers to a decade due to the factor of 10. However, I will be using log for \log_{10} . Thus each time the power increases by a factor of 10, the power ratio in dB increases linearly by 10dB. A decade is therefore change in frequency by a factor of 10. For example, 1 kHz is a decade above 100Hz and a decade below 10kHz. In most cases we are interested in voltages, so that the dB can be expressed as:

$$\text{dB} = 20 \log \frac{V_2}{V_1} \quad 2$$

As a number increases by a factor of 10, the decibel value increases by 20dB. The number 100 is 10 times as large as the number 10, and its decibel value is 20dB more. In control systems the transfer functions are always of the form of a rational polynomial and mostly representing voltage or current ratio, so we will use $20 \log |T(j\omega)|$ to compute the magnitude in dB. In MATLAB, dB of a magnitude (M) are calculated by using: **$20 * \log_{10}(M)$** . The semi-log paper has divisions on the horizontal axis in terms of cycle. A cycle consists of 10 divisions with scale increases by 10^0 , 10^1 , 10^2 , 10^3 factors and so on depending on how much cycles are on the semi-log paper. Thus a 3-cycle semi-log graph will have 10 divisions for each 10^0 , 10^1 and 10^2 . Figure (1) is a 4-cycle semi-log graph paper rotated to fit the page, so that the vertical axis is actually the horizontal axis. Each cycle in-turn is divided into 10 un-equal divisions that follow logarithmic characteristics.

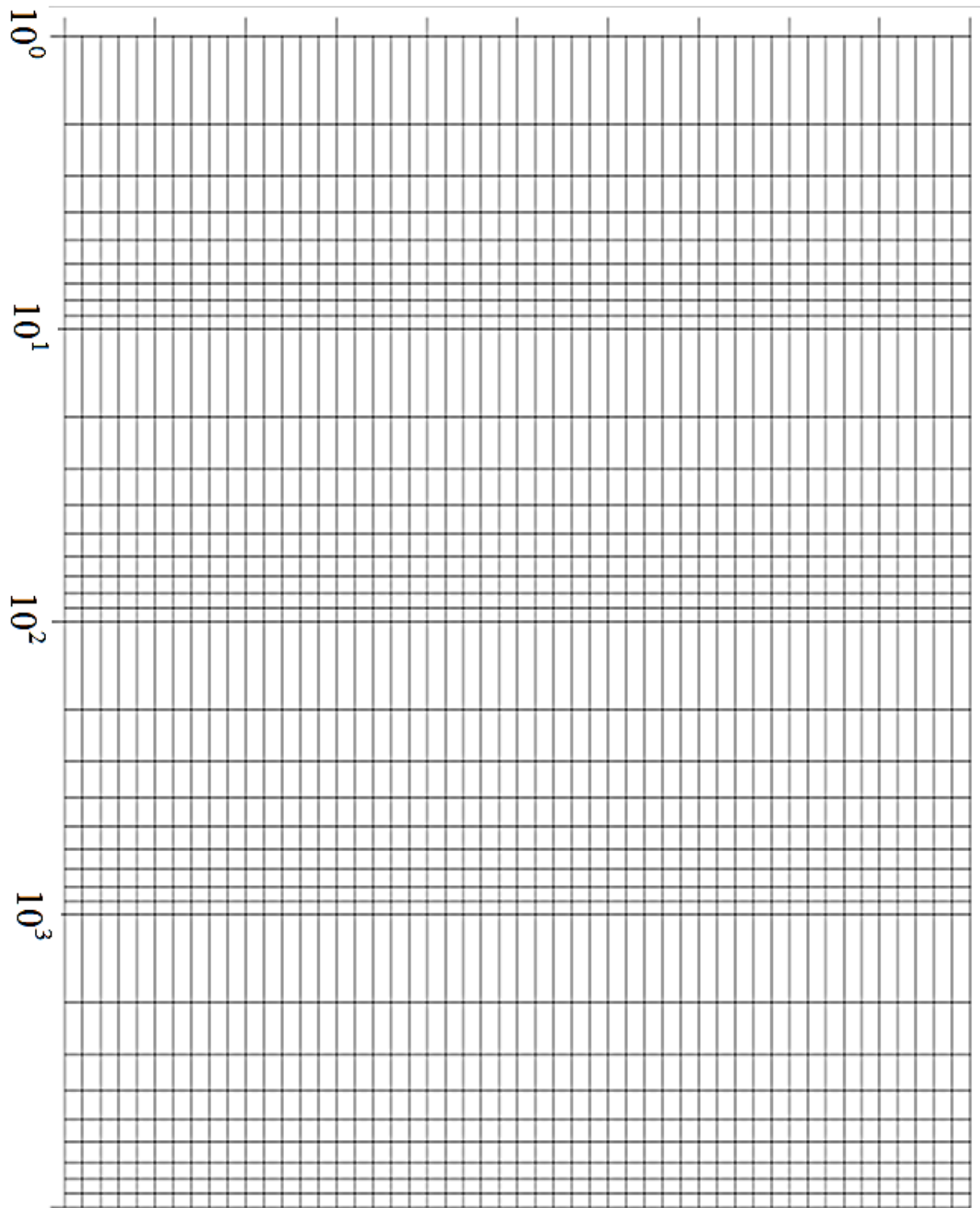


Figure (1)

Example 1: Obtain the expression for magnitude and phase belonging to a second-order unity feedback system, whose forward function is given as: $G(s) = \frac{10}{s(s+4)}$. Plot the frequency response of the system with both magnitude and phase.

Solution: The given forward transfer function is: $G(s) = \frac{10}{s(s+4)}$. The closed-loop transfer function according to: $T(s) = \frac{KG(s)}{1+KG(s)H(s)}$ is then:

$$T(s) = \frac{10}{s^2 + 4s + 10}, H(s) = 1 \text{ (unity feedback system)}$$

$$T(s)|_{s=j\omega} = T(j\omega) = \frac{10}{(j\omega)^2 + 4(j\omega) + 10} = \frac{10}{(10 - \omega^2) + j4\omega}$$

The magnitude $|T(j\omega)| = M$ is:

$$M = \frac{10}{\sqrt{(10 - \omega^2)^2 + (4\omega)^2}} \quad 3$$

The phase of the function can be obtained from the argument of $G(j\omega)$ expression as:

$$\text{Phase} = \alpha = -\tan^{-1}\left(\frac{4\omega}{(10 - \omega^2)}\right) \quad 4$$

We can find the magnitude and phase-angle of the given system at any frequency, simply by substituting the value of desired frequency in Eq (3) and (4). In order to plot the magnitude-frequency response plot, we make use of Eq (3). In Eq (3) the magnitude is calculated for each frequency considered and converted to dB. Let us plot a few points for the plot.

$$\text{At } \omega = 1: \quad M = \frac{10}{\sqrt{(9)^2 + (4)^2}} = 1.015 \quad dB = \log M = \log(1.015) = 0.0065$$

$$\text{At } \omega = 10: \quad M = \frac{10}{\sqrt{(10 - 10^2)^2 + (40)^2}} = 0.25 \quad dB = \log M = -0.6$$

$$\text{At } \omega = 100: \quad M = \frac{10}{\sqrt{(10 - 100^2)^2 + (400)^2}} = 0.001 \quad dB = \log M = -3$$

$$\text{At } \omega = 500: \quad M = \frac{10}{\sqrt{(10 - 500^2)^2 + (2000)^2}} = 4 \times 10^{-5} \quad dB = \log M = -4.4$$

$$\text{At } \omega = 1k: \quad M = \frac{10}{\sqrt{(10 - 1000^2)^2 + (4000)^2}} = 1 \times 10^{-5} \quad dB = \log M = -5$$

$$\text{At } \omega = 5k: \quad M = \frac{10}{\sqrt{(10 - 5000^2)^2 + (20000)^2}} = 4 \times 10^{-7} \quad dB = \log M = -6.4$$

$$\text{At } \omega = 10k: \quad M = \frac{10}{\sqrt{(10 - 10000^2)^2 + (40000)^2}} = 1 \times 10^{-7} \quad dB = \log M = -7$$

The magnitude frequency response plot is shown in Figure (2). Dear students you can use some “in between” values of frequency if you like, but I will suggest that the above values of frequency are enough.

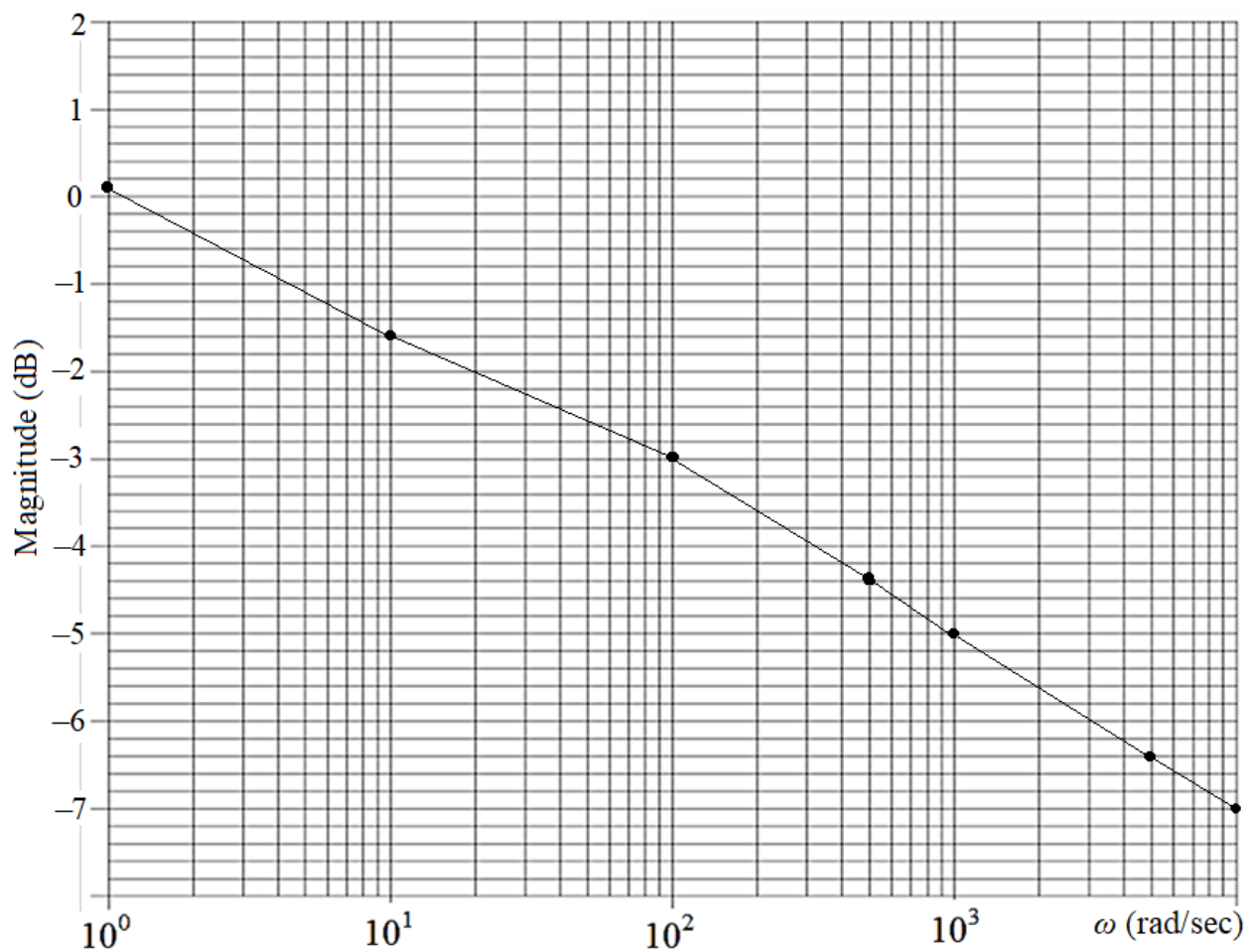


Figure (2)

Increasing the value of gain K will lift the entire magnitude-frequency response plot upward by an amount $\log(K)$. Thus in the above example, the gain $K = 100$ instead of 10, in the above plot each point corresponding to the vertical axis is multiplied by 2 because $\log(100) = 2$, so that the entire plot is shifted 2 units upward.

In order to check to plot the phase-frequency plot, we make use of Eq (4) and obtain the points for the same frequencies for the phase angle.

$$\text{At } \omega = 1: \quad \alpha = -\tan^{-1}\left(\frac{4\omega}{(10-\omega^2)}\right) = -\tan^{-1}\left(\frac{4}{9}\right) = -24^\circ$$

$$\text{At } \omega = 10: \quad \alpha = -\tan^{-1}\left(\frac{40}{-90}\right) = 24^\circ$$

$$\text{At } \omega = 100: \quad \alpha = -\tan^{-1}\left(\frac{400}{-9990}\right) = 2.3^\circ$$

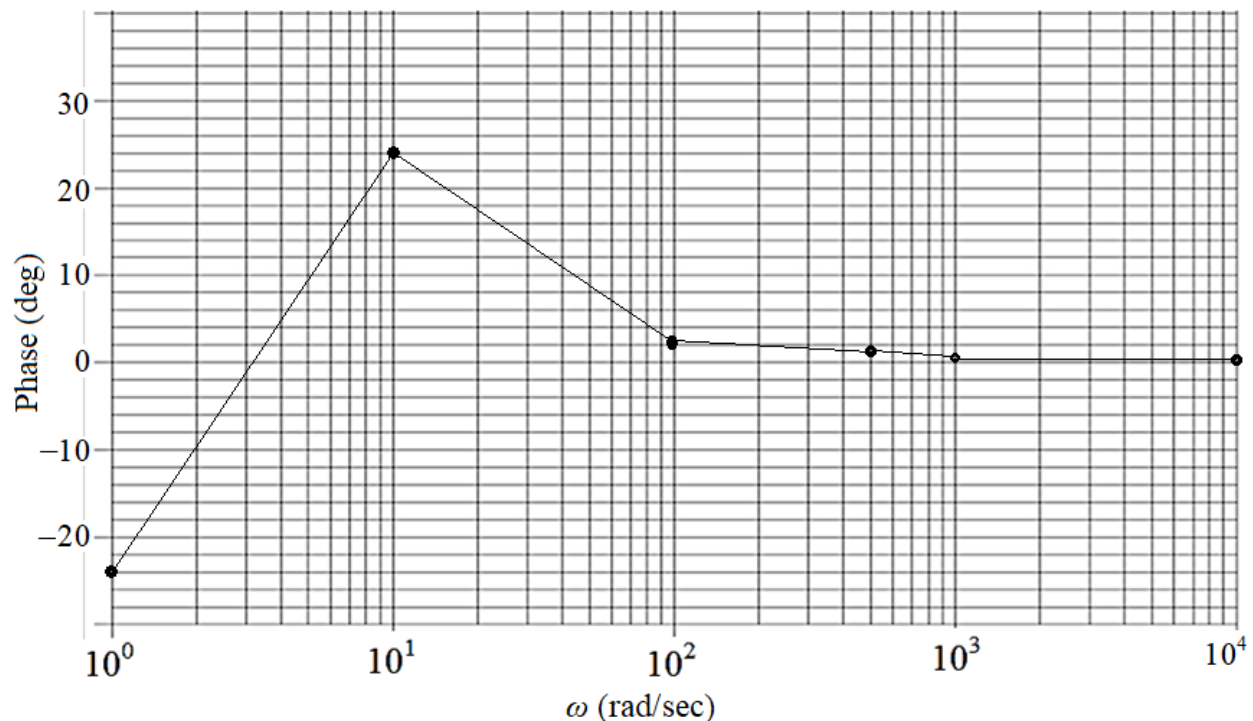
$$\text{At } \omega = 500: \quad \alpha = -\tan^{-1}\left(\frac{2000}{-249990}\right) = 0.45^\circ$$

$$\text{At } \omega = 1k: \quad \alpha = -\tan^{-1}\left(\frac{4000}{-999990}\right) = 0.23^\circ$$

$$\text{At } \omega = 5k: \quad \alpha = -\tan^{-1}\left(\frac{20000}{-24999990}\right) = 0.045^\circ$$

$$\text{At } \omega = 10k: \quad \alpha = -\tan^{-1}\left(\frac{40000}{-99999990}\right) = 0.023^\circ$$

The phase-frequency plot of the frequency response of the system is shown in Figure (3) below.



Let us now consider another example with the same transfer function but with the addition of a zero ($s + 2$) with the plant. This can be viewed as a PD controller, if we compare its transfer function: $sK_D + K$ with the factor of added zero, in which case (controller gain) $K_D = 1$ and (proportional added gain) $K = 2$. Remember that a PD controller or lead compensator is used to improve the transient response of a system by shifting the root locus to the left so that the response becomes quick in rejecting the disturbances. Let us check the frequency response side, as to what actually a PD controller does to the system's frequency response.

Example 2: Obtain the expression for magnitude and phase belonging to a second-order unity feedback system, whose forward function is given as: $G(s) = \frac{10(s+2)}{s(s+4)}$. Plot the frequency response of the system with both magnitude and phase.

Solution: The forward transfer function of the given unity feedback system is: $G(s) = \frac{10(s+2)}{s(s+4)}$.

The corresponding closed-loop transfer function: $T(s) = \frac{10(s+2)}{s^2 + 14s + 20}$. The Fourier transform of the closed-loop transfer function is:

$$T(s)|_{s=j\omega} = T(j\omega) = \frac{10(2 + j\omega)}{(20 - \omega^2) + j14\omega} \quad 5$$

The magnitude $|T(j\omega)| = M$ is:

$$M = \frac{10\sqrt{4 + \omega^2}}{\sqrt{(20 - \omega^2)^2 + 196\omega^2}} \quad 6$$

In order to obtain an expression for phase angle for the system in this example, we need to separate Eq (5) in real and imaginary parts. I hope you are all good in dealing with complex quantities in mathematics, so I leave all the steps to you. However, the end result accordingly is:

$$Phase = \alpha = -\tan^{-1}\left(\frac{8\omega + \omega^3}{40 - 16\omega^2}\right) \quad 7$$

For plotting the magnitude-frequency plot we have the following points at some of the well-defined frequencies using Eq (6):

$$\text{At } \omega = 1: \quad M = \frac{10\sqrt{5}}{\sqrt{(19)^2 + 196}} = 0.94 \quad dB = \log M = -0.02$$

$$\text{At } \omega = 10: \quad M = \frac{10\sqrt{104}}{\sqrt{(20 - 10^2)^2 + 19600}} = 0.61 \quad dB = \log M = -0.2$$

$$\text{At } \omega = 100: \quad M = \frac{10\sqrt{10004}}{\sqrt{(20-100^2)^2 + 1960000}} = 0.1 \quad dB = \log M = -1.0$$

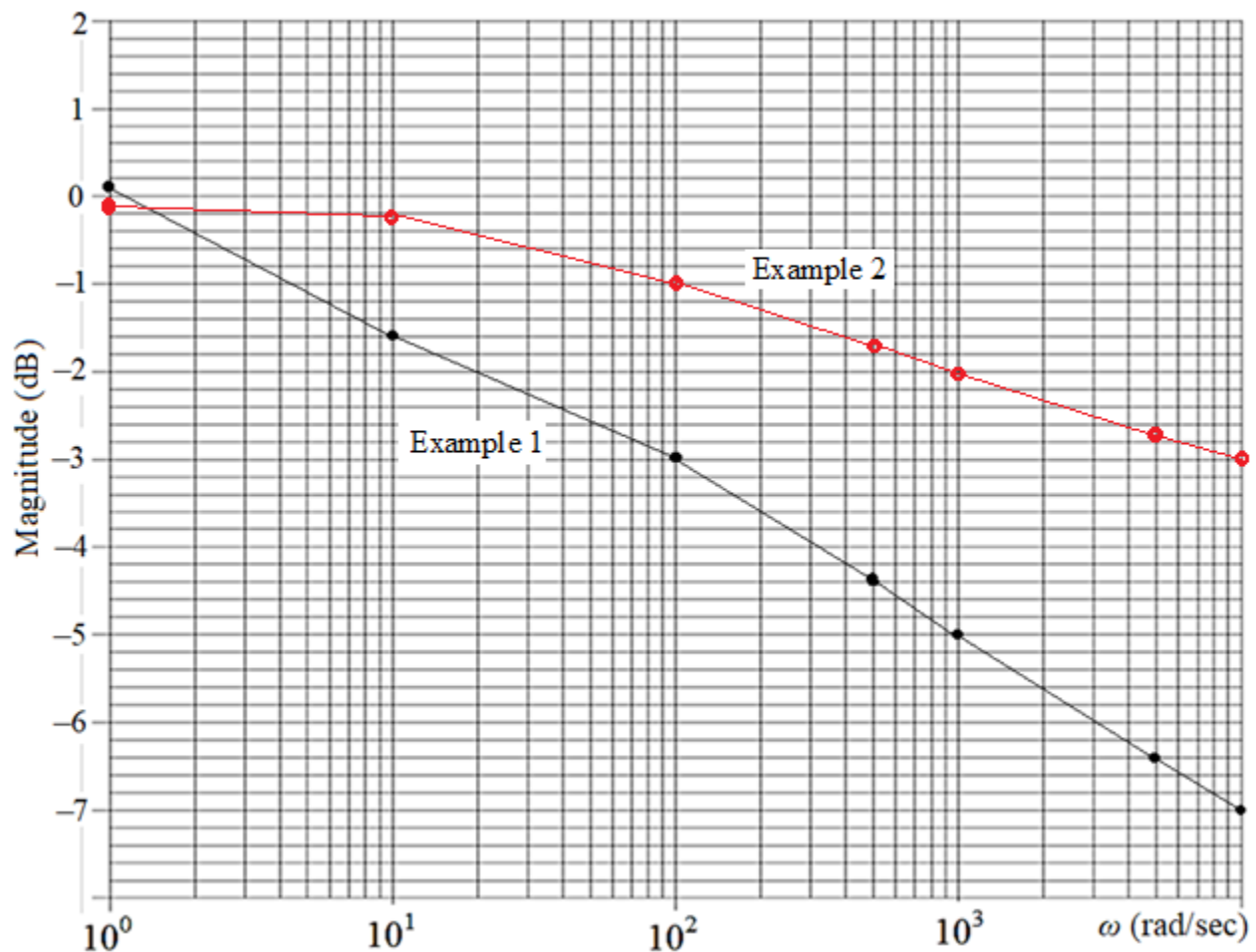
$$\text{At } \omega = 500: \quad M = \frac{10\sqrt{250004}}{\sqrt{(20-500^2)^2 + 49000000}} = 0.02 \quad dB = \log M = -1.7$$

$$\text{At } \omega = 1k: \quad M = \frac{10\sqrt{1000004}}{\sqrt{(20-1000^2)^2 + 196 \times 10^6}} = 0.01 \quad dB = \log M = -2$$

$$\text{At } \omega = 5k: \quad M = \frac{10\sqrt{25000004}}{\sqrt{(20-5000^2)^2 + 49 \times 10^8}} = 0.002 \quad dB = \log M = -2.7$$

$$\text{At } \omega = 10k: \quad M = \frac{10\sqrt{100000004}}{\sqrt{(20-10000^2)^2 + 196 \times 10^8}} = 0.001 \quad dB = \log M = -3$$

For the purpose of comparison, the magnitude-frequency response plot is plotted on the same graph for system in Example 1 and is shown in Figure (4) below.



It can be noticed that the addition of PD controller of a zero (not at the origin) seem to improve the magnitude frequency response of the system with tendency to flattening the curve, so that the system bandwidth is increased. However, in case where a zero is not incorporated, the cut-off is sharper, but with a reduced bandwidth.

In order to check to plot the phase-frequency plot, we make use of Eq (7) and obtain the points for the same frequencies to calculate the phase angles.

$$\text{At } \omega = 1: \quad \alpha = -\tan^{-1}\left(\frac{8(1) - (1)^3}{40 - 16(1)^2}\right) = -\tan^{-1}\left(\frac{7}{24}\right) = -16^\circ$$

$$\text{At } \omega = 10: \quad \alpha = -\tan^{-1}\left(\frac{-920}{-1560}\right) = -30^\circ$$

$$\text{At } \omega = 100: \quad \alpha = -\tan^{-1}\left(\frac{-999200}{-159960}\right) = -81^\circ$$

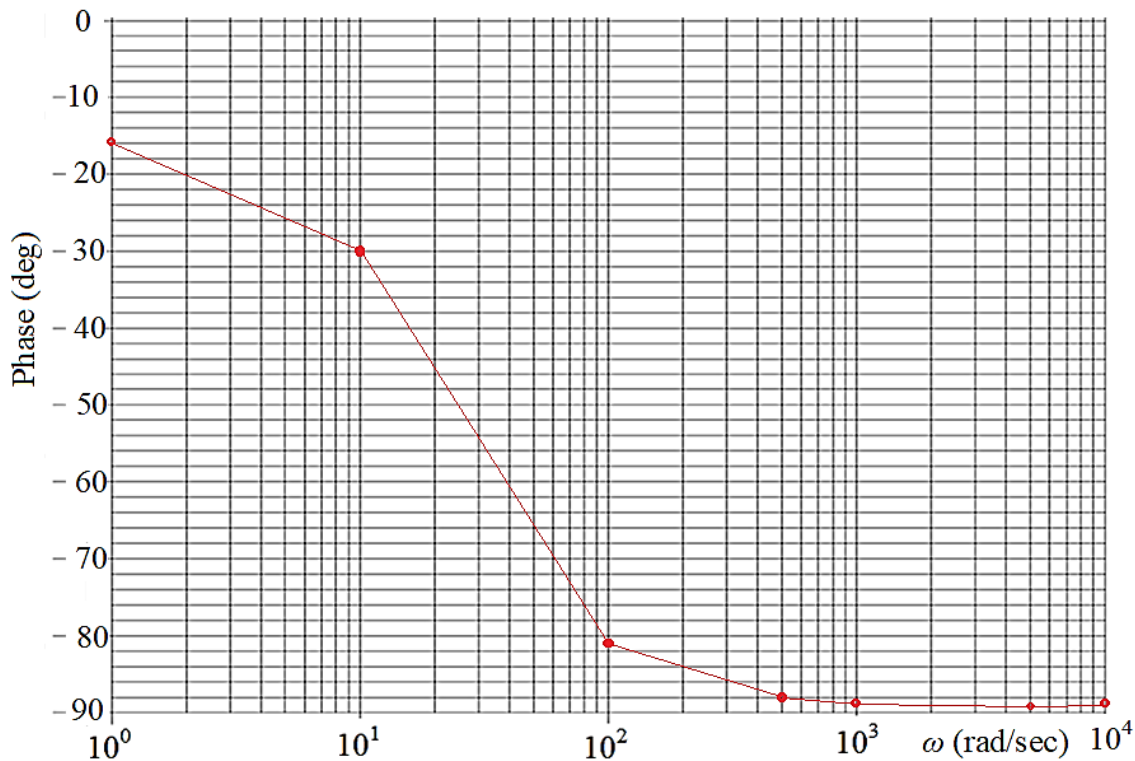
$$\text{At } \omega = 500: \quad \alpha = -\tan^{-1}\left(\frac{-124996000}{-3999960}\right) = -88^\circ$$

$$\text{At } \omega = 1k: \quad \alpha = -\tan^{-1}\left(\frac{-99992000}{-15999960}\right) = -89^\circ$$

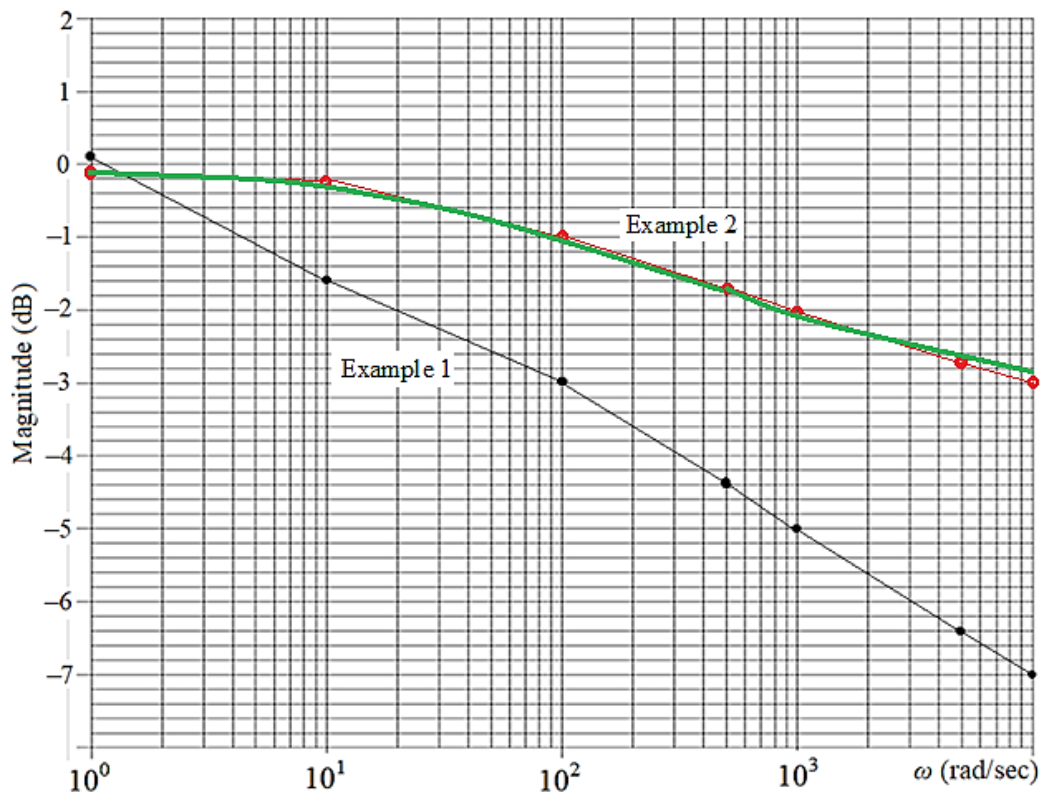
$$\text{At } \omega = 5k: \quad \alpha = -\tan^{-1}\left(\frac{-12499960000}{-399999600}\right) = -88^\circ$$

$$\text{At } \omega = 10k: \quad \alpha = -\tan^{-1}\left(\frac{-9999920000}{-159999960}\right) = -89^\circ$$

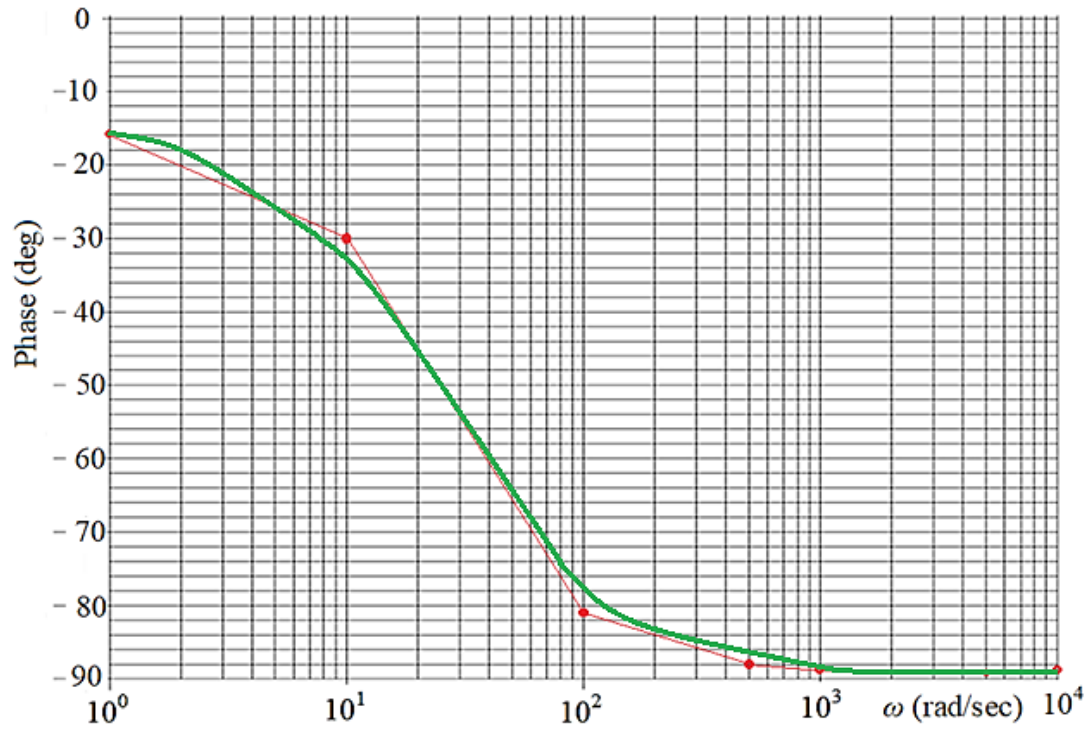
The phase-frequency plot is shown in Figure (5) on the next page.



Smooth curves can be drawn (green) as illustrated below for Example 2.



Magnitude-frequency plot



Phase-frequency plot

If we compare the two phase plots, it can be noted that a system transfer function without a zero has tendency to settle its phase to 0 degree, whereas with the inclusion of zero tends to settle the phase of the system to -90 degrees. Thus a zero or a PD controller introduces a phase-shift in the phase-frequency response of a system.