Lecture # 14

Frequency Response

My dear students, so far as the system behavior is concerned, we have understood the transient and steady-state response and have known their parameters and quantities that are essential in the *t*-domain analysis using some standard inputs applied, namely; step, ramp and parabolic. The step function input is generally preferred by engineers in time-domain analysis in order to evaluate the performance based on time specifications of control systems. The steady-state performance is evaluated in terms of steady-state error to a specific standard input mentioned above depending on whether the system is Type-0, 1 or 2.

On the contrary in the analysis and design of communication systems the frequency response is more important, since most of the standard signals are sinusoids with magnitude, frequency and phase. Frequency-response techniques provide a comprehensive study of a system based on frequency response Bode plots.

The *t*-domain analysis is facilitated by Laplace transform that converts a *t*-domain function into frequency domain (*s*-domain). Solution of *t*-domain differential equations becomes lot more easy using Laplace transform technique. The Fourier transform (FT) is another powerful method for representing signals and systems in the frequency domain. The FT of a function $f(t)$ is designated by $F(j\omega)$ and is given by Fourier integral as:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad 1$$

According to Fourier an aperiodic signal can be thought as a periodic signal with a period, sufficiently large, extending from $-\infty$ to $+\infty$. Fourier transform is usually considered as special type of bilateral Laplace transform, with the real part of the complex frequency set equal to zero. FT has wider applications as compared to Laplace transform in communication and signal processing world. Likewise, the Inverse Fourier Transform (IFT) is used to convert from $j\omega$ to *t*-domain and is given by integral:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega \qquad 2$$

The closed-loop transfer function $T(s)$ of a system is then a function of the complex variable $j\omega$ and is represented as $T(j\omega)$ that has a magnitude and phase angle. The basic disadvantage of the frequency-response method for analysis and design is the indirect link between the frequency-domain and time-domain.

The Basis Concept

Frequency response means the response of a system to a sinusoidal input. A sinusoidal quantity has three ingredients; magnitude, phase-angle and frequency. It must be remembered that in the steady-state, sinusoidal inputs to a linear system produces sinusoidal response of the same frequency in the output, so that there is no change in the frequency whereas both the magnitude and phase-angle are subject to changes. The changes and differences in magnitude and phase at the input and output are, however, functions of frequency. This is illustrated in Figure (1).

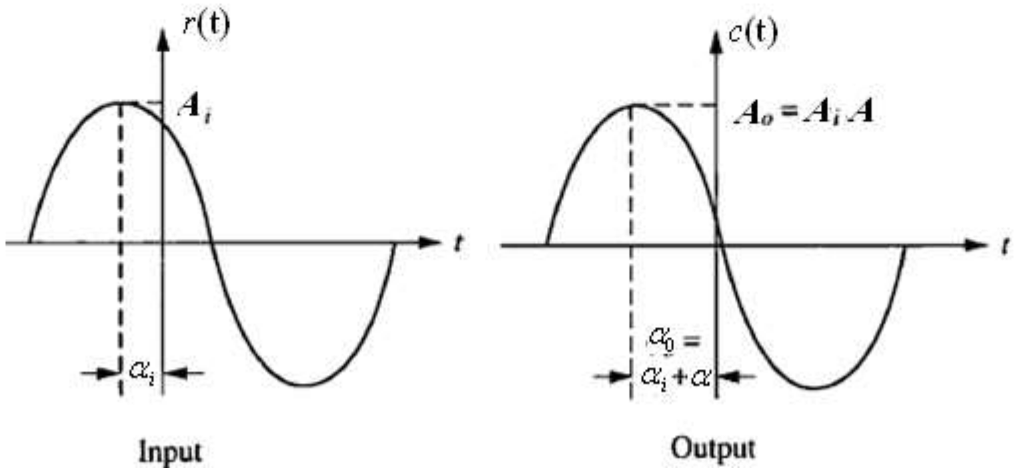

Figure 1

Sinusoids are dealt under the subject of complex quantities. The value of the complex quantity is the magnitude whereas its angle α represents the phase. A sinusoid of the form: $A\cos(\omega t + \alpha)$ can be represented as $A\angle\alpha$ (polar) or $Ae^{j\alpha}$ (exponential). Since a system causes both the amplitude and phase angle of the input to be changed, the system itself can be represented by a complex system function, defined such that the product of the input phasor and the system function yields the phasor representation of the output. Consider a system shown in Figure (2), the input is sinusoidal, the steady-state output response of the system is also sinusoidal and having the same frequency as the input.

Figure 2

Thus: $$A_0(\omega)\angle\alpha_0(\omega) = A(\omega)\angle\alpha(\omega).A_i(\omega)\angle\alpha_i(\omega)$$

Or $$A_0(\omega)\angle\alpha_0(\omega) = A(\omega).A_i(\omega)\angle[\alpha(\omega) + \alpha_i(\omega)]$$

Where the magnitude frequency response is:

$$A(\omega) = \frac{A_0(\omega)}{A_i(\omega)} \qquad 3$$

And the phase frequency response is:

$$\alpha(\omega) = \alpha_0(\omega) - \alpha_i(\omega) \qquad 4$$

In other words, the magnitude-frequency response is defined as the ratio of the output magnitude of sinusoid to the input magnitude. The phase-frequency response is defined as the difference in phase angle between the output and the input sinusoids. Both responses are a function of frequency and apply only to the steady-state sinusoidal response of the system. Most systems giving frequency response are termed as filters.

Example 1: Obtain the expression for magnitude and phase belonging to a second-order unity feedback system, whose forward function is given as: $G(s) = \dfrac{10}{s(s+4)}$.

Solution: The given forward transfer function is: $G(s) = \dfrac{10}{s(s+4)}$. The closed-loop transfer function according to: $T(s) = \dfrac{KG(s)}{1 + KG(s)H(s)}$ is then:

$$T(s) = \frac{10}{s^2 + 4s + 10}$$, with $H(s) = 1$ (unity feedback system)

$$T(s)|_{s=j\omega} = T(j\omega) = \frac{10}{(j\omega)^2 + 4(j\omega) + 10} = \frac{10}{(10-\omega^2) + j4\omega}$$

The magnitude $|T(j\omega)| = M$ is:

$$M = \frac{10}{\sqrt{(10-\omega^2)^2 + (4\omega)^2}} \qquad 5$$

The phase of the function can be obtained from the argument of $G(j\omega)$ expression as:

$$Phase = \alpha = -\tan^{-1}\left(\frac{4\omega}{(10-\omega^2)}\right) \qquad 6$$

We can find the magnitude and phase-angle of the given system at any frequency, simply by substituting the value of desired frequency in Eq (5) and (6). Thus at ω = 1 rad/s, we have:

Magnitude: $M = \dfrac{10}{\sqrt{(10-\omega^2)^2 + (4\omega)^2}} = \dfrac{10}{\sqrt{(9)^2 + (4)^2}} = 1.015$

Phase: $\alpha = -\tan^{-1}\left(\dfrac{4(1)}{[10-(1)^2]}\right) = -24°$

In order to check out which type of filter the system is, we can substitute extreme values of frequency; 0 and ∞. Therefore:

At $\omega = 0$: $M = \dfrac{10}{\sqrt{(10-\omega^2)^2 + (4\omega)^2}} = \dfrac{10}{\sqrt{(10)^2 + 0}} = 1$

At $\omega = \infty$: $M = \dfrac{10}{\sqrt{(\infty)^2 + (\infty)^2}} = 0$

The phase is:

At $\omega = 0$: $\alpha = -\tan^{-1}(0) = 0$

At $\omega = \infty$: $\alpha = -\tan^{-1}(\infty) = -90$

Apparently the filter behaves like a low-pass, since the magnitude is maximum at low frequency and attenuated to zero at high frequency. The system behavior therefore is that of a second-order low-pass filter.

Frequency Response Specifications

Direct analogy between the frequency response and the transient-response quantities is not very well defined. The *t*-domain transient response specifications or time specifications describes the constraints; the rise time, peak time, overshoot and settling time that are function of two vital parameters; ζ and ω_n. These specifications belonging to the *t*-domain are not directly used in the frequency domain. For frequency response analysis and design of the system alternate specifications are used. The following frequency-domain specifications:

1. **Gain margin K_g:** The gain margin is the measure of relative stability of a system that is the factor by which the gain must be altered in order to cause instability. K_g in most cases should be between 5 and 10 and is a function of phase cross-over frequency ω_ϕ.
2. **Phase margin *PM*:** The phase margin is the additional phase-lag at the gain crossover frequency required in order to cause instability. *PM* in most cases should be between 40° and 50° and is function of gain cross-over frequency ω_g.
3. **Resonance frequency ω_r:** Resonance frequency is that frequency at which the magnitude of the system transfer function is maximum.
4. **Resonant peak M_p:** The magnitude of the closed-loop transfer function at resonance frequency is called resonant peak.
5. **Bandwidth ω_b:** Bandwidth refers to the frequency range at which the magnitude is 0.707 as that of magnitude at resonance.

In frequency-response approach, the transient-response performance is specified indirectly. That is, gain margin, phase margin and resonant peak magnitude provides a rough estimate of the damping in a system. The gain crossover frequency, resonant frequency, bandwidth give a rough estimate of the speed of transient response. In order to understand the evaluation of the system response parameters for frequency response analysis, we consider an example.

Example 2: Evaluate the frequency response specifications of a unity feedback control system as shown in Figure (3).

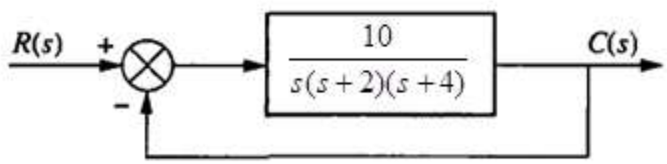

Figure 3

Solution: The forward function of the transfer function of the plant is: $G(s)=\dfrac{10}{s(s+2)(s+4)}$.

The corresponding closed-loop transfer function is then: $T(s)=\dfrac{10}{s^3+6s^2+8s+10}$. We will proceed in sequence and follow the steps.

A. Gain Margin: The steps are:

1. Take the Fourier transform of both $G(s)$ and $T(s)$ by replacing s with $j\omega$ and obtain their magnitudes.
2. In the T(s) expression, consider the characteristic polynomial and replace s with $j\omega$ and equate it to zero to form characteristic equation. Equating separately the imaginary part of the characteristic equation to zero, obtain the frequency $\omega = \omega_\phi$ (phase crossover frequency).
3. Obtain $|G(j\omega_\phi)|$ for substituting $\omega = \omega_\phi$ in $|G(j\omega)|$.
4. The reciprocal of $|G(j\omega_\phi)|$ is K_g.

Step 1: $G(j\omega)=\dfrac{10}{j\omega(j\omega+2)(j\omega+4)}$ (with $s = j\omega$ replacement)

Therefore: $|G(j\omega)|=\dfrac{10}{\sqrt{36\omega^4+(8\omega-\omega^3)^2}}$

And $\angle G(j\omega)=-\tan^{-1}\left(\dfrac{8\omega-\omega^3}{6\omega^2}\right)$

Step 2: The characteristic polynomial of the corresponding closed-loop transfer function is: $s^3+6s^2+8s+10$. The corresponding characteristic equation is:

$$s^3+6s^2+8s+10=0$$

Let us substitute: $s = j\omega$ for $\omega = \omega_\phi$ in the characteristic equation:

$$(j\omega_\phi)^3+6(j\omega_\phi)^2+8(j\omega_\phi)+10=0$$

Separately equating the imaginary part to zero that is:

$$-j\omega_\phi^{\ 3}+j8\omega_\phi=0$$

Or $\omega_\phi = 2.83$ rad/sec

Step 3: Substituting the value ω_ϕ = 2.83 rad/sec in the expression for the $|G(j\omega)|$, at $\omega = \omega_\phi$ we have:

$$|G(j\omega_\phi)| = \frac{10}{\sqrt{36(2)^4 + [8(2.83) - (2.83)^3]^2}} = 0.58$$

Step 4: The gain margin is therefore:

$$K_g = \frac{1}{|G(j\omega_\phi)|} = \frac{1}{0.58} = 1.72$$

B. **Phase Margin:** The followings are the steps:

1. Obtain the Fourier transform $G(j\omega)$ and get its magnitude: $|G(j\omega)|$ and angle $\angle G(j\omega)$.
2. Equate: $|G(j\omega)| = 1$, solve for ω, which gives ω_g (gain crossover frequency).
3. Evaluate $\angle G(j\omega)$ for $\omega = \omega_g$.
4. The $PM = 180^0 + \angle G(j\omega)$.

Step 1: $$G(j\omega) = \frac{10}{j\omega(j\omega+2)(j\omega+4)}$$

Therefore: $$|G(j\omega)| = \frac{10}{\sqrt{36\omega^4 + (8\omega - \omega^3)^2}}$$

And $$\angle G(j\omega) = -\tan^{-1}\left(\frac{8\omega - \omega^3}{6\omega^2}\right)$$

Step 2: Applying the condition for obtaining the gain crossover frequency: $\omega = \omega_g$ in the magnitude M_g of the open-loop function of the given system that is:

$$|G(j\omega)| = \frac{10}{\sqrt{36\omega^4 + (8\omega - \omega^3)^2}} = 1$$

Or $$36\omega^4 + (8\omega - \omega^3)^2 = 100$$

Simplifying: $$\omega^6 + 20\omega^4 + 64\omega^2 - 100 = 0$$

Supposing that $\omega^2 = x$, we have:

$$x^3 + 20x^2 + 64x - 100 = 0$$

Among the three roots, the reasonable value is:

$$x = 1.136 \text{ or } \omega = \omega_g = 1.06 \text{ rad/sec}$$

Step 3: Using this value of frequency; ω_g = 1.06 rad/sec, the angle contribution of the plant transfer function $G(s)$ is determined as follows:

$$\angle G(j\omega_g) = \frac{\angle 0}{\angle j1.06 \angle (j1.06+2) \angle (j1.06+4)}$$

Or $$\angle G(j\omega_g) = \frac{\angle 0}{\angle 90^0 \angle 28^0 \angle 14.84^0}$$

Thus: $$\angle G(j\omega_g) = \angle -132.84^0$$

Thus the phase margin is: $PM = 180^0 + \angle G(j\omega_g)$

$$PM = 180^0 - 13284^0 = 47.15^0$$

The phase margin and the damping ratio are directly related. It has been observed that for a second-order system, phase margin PM and the damping ratio ζ are related approximately by a relationship: $\zeta = \frac{PM}{100}$. Thus a phase margin of 60^o will corresponds to a damping ratio of 0.6.

C. **Resonance Frequency ω_r**: The following steps helps to find the resonance frequency

1. Obtain the Fourier transform $T(j\omega)$ of the closed-loop transfer function $T(s)$ and get its magnitude $|T(j\omega)|$.
2. Take the derivative: $\frac{d}{d\omega}|T(j\omega)|$ and equate it to zero. The frequency obtained is the resonance frequency: $\omega = \omega_r$.

Step 1: Given: $G(s) = \frac{10}{s(s+2)(s+4)}$, the corresponding closed-loop transfer function is: $T(s) = \frac{10}{s^3 + 6s^2 + 8s + 10}$. The Fourier transform of $T(s)$ with $s = j\omega$ replacement is: $T(j\omega) = \frac{10}{-j\omega^3 - 6\omega^2 + j8\omega + 10}$. The magnitude of the Fourier transform of the closed-loop transfer function is: $|T(j\omega)| = \frac{10}{\sqrt{(10-6\omega^2)^2 + (8\omega - \omega^3)^2}}$

Step 2: The derivative of $|T(j\omega)|$ with equating it to zero is as follows:

$$\frac{d}{d\omega}|T(j\omega)| = \frac{d}{d\omega}\frac{10}{\sqrt{(10-6\omega^2)^2 + (8\omega-\omega^3)^2}} = 0$$

Or $$\frac{d}{d\omega}|T(j\omega)| = \frac{d}{d\omega}10[(10-6\omega^2)^2 + (8\omega-\omega^3)^2]^{-1/2} = 0$$

$$\frac{d}{d\omega}|T(j\omega)| = -5[(10-6\omega^2)^2+(8\omega-\omega^3)^2]^{-3/2}\frac{d}{d\omega}[(10-6\omega^2)^2+(8\omega-\omega^3)^2]=0$$

$$\frac{d}{d\omega}[(10-6\omega^2)^2+(8\omega-\omega^3)^2]=0$$

$$2(10-6\omega^2)(-12\omega)+2(8\omega-\omega^3)(8-3\omega^2)=0$$

$$2(10-6\omega^2)(-12\omega)+2(8\omega-\omega^3)(8-3\omega^2)=0$$

$$-120\omega+72\omega^3+64\omega-24\omega^3-8\omega^3+3\omega^5=0$$

$$3\omega^4+40\omega^2-56=0$$

Considering $\omega^2 = x$, the above expression can be re-written as:

$$3x^2+40x-56=0$$

Which gives: $x = 1.277$ as a reasonable root. Thus:

$$\omega_r = 1.13 \text{ rad/sec}$$

D. Resonant Peak; This is evaluated through the following steps:

1. Given the open-loop transfer function $G(s)$, obtain the Fourier transform of the closed-loop transfer function $T(j\omega)$ and get its magnitude: $|T(j\omega)|$.
2. Obtain the magnitude $|T(j\omega_r)| = Mp$ by substituting $\omega = \omega_r$ in $|T(j\omega)|$.

Step 1: Given that: $G(s)=\dfrac{10}{s(s+2)(s+4)}$, the corresponding closed-loop transfer function is: $T(s)=\dfrac{10}{s^3+6s^2+8s+10}$. The Fourier transform of $T(s)$ is: $T(j\omega)=\dfrac{10}{-j\omega^3-6\omega^2+j8\omega+10}$. The magnitude of the Fourier transform of the closed-loop transfer function is:

$$|T(j\omega)| = \frac{10}{\sqrt{(10-6\omega^2)^2+(8\omega-\omega^3)^2}}$$

Step 2: Substitute $\omega = \omega_r = 1.13$ rad/sec in the expression for $|T(j\omega)|$ above

$$|T(j\omega)| = \frac{10}{\sqrt{[10-6(1.13)^2]^2+[8(1.13)-(1.13)^3]^2}}$$

$$M_p = 1.25$$

A large value is an indication of the closed-loop poles having a small damping ratio, which results in larger overshoot. Thus we can say that *Mp* is somewhat an inverse function of damping coefficient.

E. **Bandwidth:** In order to find the bandwidth, the following steps are essential.

1. Given the open-loop transfer function *G*(s), obtain the closed-loop transfer function *T*(s). Get the magnitude of the Fourier transform of *T*(s) as $|T(j\omega)|$.

2. Equate $|T(j\omega)| = \frac{1}{\sqrt{2}}$ and solve for ω, which is ω_b.

Step 1: Given that: $G(s)=\frac{10}{s(s+2)(s+4)}$, the corresponding closed-loop transfer function is: $T(s)=\frac{10}{s^3+6s^2+8s+10}$. The Fourier transform of $T(s)$ is: $T(j\omega)=\frac{10}{-j\omega^3-6\omega^2+j8\omega+10}$. The magnitude of the Fourier transform of the closed-loop transfer function is:

$$|T(j\omega)|=\frac{10}{\sqrt{(10-6\omega^2)^2+(8\omega-\omega^3)^2}}$$

Step 2: Equating $|T(j\omega)|$ to $\frac{1}{\sqrt{2}}$ or 0.707 according to the definition:

$$\frac{1}{\sqrt{2}}=\frac{10}{\sqrt{(10-6\omega^2)^2+(8\omega-\omega^3)^2}}$$

Squaring both sides of the above expression and cross-multiplying, we have:

$$(10-6\omega^2)^2+(8\omega-\omega^3)^2=200$$

Or $$100+36\omega^4-120\omega^2+64\omega^2+\omega^6-16\omega^4=200$$

Or $$\omega^6+20\omega^4-56\omega^2-100=0$$

Letting: $\omega^2 = x$, the above expression can be re-written as:

$$x^3+20x^2-56x-100=0$$

The above third-order equation have three roots, the reasonable and positive is: $x = 3.566$.

Thus: $\omega_b^2 = 3.566$

Or $\omega_b = 1.88$ rad/sec

The bandwidth decreases with increasing damping coefficient ζ. Therefore the rise-time and bandwidth are inverse function of each other.

Dear students please check and correct the numerical calculations as you go through the lecture handout. As an exercise to keep you busy, consider a zero: (s + 2) included in the system open-loop transfer function. The transfer function modifies to: $G(s)=\dfrac{10(s+2)}{s(s+2)(s+4)}$. This zero ($s$ + 2) behaves like a PD controller, if you compare it with the general transfer function of the PD controller: $sK_D + K$. On comparison, it can be noted that controller gain: $K_D = 1$ and the proportional added gain is $K = 2$. Remember that a PD controller or lead compensator is used to improve the transient response of a system. You may get some interesting results. Good luck.