Lecture # 13

Improving Steady-State Performance

My dear students, in lecture 11 we studied compensation and by the time we covered lecture 12, we have understood the improvement of transient response through the use of cascade (series) lead compensator and simple gain controller. In either case the steady-state performance deteriorated in an attempt to improve transient performance. We have also designed a lead compensator for a Type-0 plant represented by a third order transfer function. In this lecture we will discuss the improvement of steady-state response using a cascade lag compensator of the same system. It must be remembered that the steady-state performance is improved by increasing the system gain. However, increasing the gain will shift the root locus towards the right approaching the instability. It must also be remembered that shifting the root locus to the right will deteriorate the transient response of the system by making its response slower. Just as a matter of interest, the word "lead" shifts the root locus to the left, whereas the word "lag" shifts the root locus to the right. Remember that compensators are controllers and are simply electric filters, when we speak of electrical systems.

The Lag Compensator

The purpose of the lag compensation is to improve the steady-state performance of the system by increasing the static error constant, which then reduces the steady-state error. We have discussed the effect of including a pure integrator in the forward path with the plant as shown in Figure (1). This improves the steady-state performance by shifting the root locus to the right so as to introduce more gain into the system. Although increasing gain, shift the entire root locus to the right, it is, however, not recommended because shifting the root locus to the right moves it more nearer to unstable region (RH-plane).

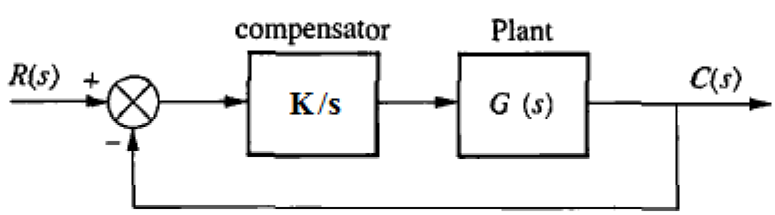

Figure 1: System with pure integrator

We have seen that by placing an integrator in series will modify the transfer function by adding a pole at the origin $s = 0$, so that the system Type is upgraded. In our example in one of the previous lectures, the original open-loop transfer function of the plant: $G(s) = \dfrac{K}{(s+2)(s+4)(s+6)}$ was modified to: $G(s) = \dfrac{K}{s(s+2)(s+4)(s+6)}$ by including a simple integrator in series. In this case we see that the system-type is upgraded from Type-0 to Type-1 with the gain remaining unchanged. For example, a Type-0 system responding to a step input with a finite error responds with zero error if the system-type is increased by one (Type-1).

Transfer Function of Lag Compensator

In practice lag compensators are used, having a general transfer function of the form: $G_C(s) = \dfrac{K_C(s+z_C)}{(s+p_C)}$. The pole and zero of the compensator are located very near to the origin so that the system-type remains the same and at the same time the flavor of integration is included satisfying the condition: $p_C < z_C$ and the gain Kc very close to unity. The pole and zero of the compensator are selected such that the angular contribution of the compensator zero and compensator pole nearly cancel out, and the point of interest is still on the root locus with the system-type unaffected. This can only be accomplished if both pole and zero are selected very near to the origin. The pole and zero of the compensator must then be chosen closer in such a manner so that the angle deviation of the lag network should be limited to about $\pm 5^{o}$. This improves the steady-state error without appreciably disturbing the transient response.

Consider that compensator has to be designed in order to improve the steady-state performance by reducing the steady-state error without appreciably altering the transient response. A lag compensator $G_C(s)$ can be designed using the root locus, given by the following expression:

$$G_C(s) = K_C \frac{s+z_C}{s+p_C} \qquad 1$$

With gain almost unity, the transfer function of lag compensator in Eq (1) can be expressed as:

$$G_C(s) = \frac{s+z_C}{s+p_C} \qquad 2$$

The transfer function of lag compensator is also expressed as:

$$G_C(s) = K_C \left[\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \right]$$

Where $z_C = \dfrac{1}{T}$ and $p_C = \dfrac{1}{\beta T}$ (β over here must not be confused with angle: $\beta = \cos^{-1}\zeta$). Thus:

$$G_C(s) = K_C \beta \left(\frac{Ts+1}{\beta Ts+1} \right)$$

With little mathematics and comparison, the above expression for the transfer function can be re-written as:

$$G_C(s) = K_C\left(\frac{s+z_C}{s+z_C/\beta}\right) \qquad 3$$

Steps for Designing Lag Compensator:

The steps necessary to design a lag compensator are as follows:

1. Draw the root locus of the given uncompensated system from the given open-loop transfer function.

2. Draw the damping line from the given value of damping ratio or alternatively if the percent overshoot is given, damping ratio can be determined. The damping line, whose angle with respect to the positive real axis at the origin defined by: $180^0 - \cos^{-1}\zeta$, which should cut the root locus at the desired closed-loop pole.
3. Obtain the closed-loop complex pole of the form: $-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$ from the closed-loop transfer function or from the point of intersection of damping line with the root locus.

4. Determine the corresponding value of the undamped natural frequency ω_n from the closed-loop complex pole. Thus the values of ζ and ω_n are both known.
5. Determine the static error constant; (say K_v) from the open-loop transfer function of the given uncompensated system by using in this case:

$$K_v = \lim_{S\to 0} sG(s)$$

6. Compute the steady-state error by using:

$$e(\infty) = \frac{1}{K_v}$$

7. In order to reduce the steady-state error to some desired value, the corresponding static error constant has to be increased. The new value of static error constant $K_v^{/}$, assuming almost unity gain, can be found as:

$$K_v^{/} = \beta K_v \qquad 4$$

8. Once β has been determined in step 7, then the transfer function of the compensator can be obtained as:

$$G_C(s) = K_C\frac{\left(s+\dfrac{1}{T}\right)}{\left(s+\dfrac{1}{\beta T}\right)}$$

9. Choose the appropriate values of zero and pole of a compensator close to origin say a zero at –0.01. Then the corresponding pole will be at $-0.01/\beta$, such that the angle contribution from the pole and the zero to the complex closed-loop pole does not exceed $\pm 5^o$. This will ensure that the pole and zero will approximately cancel out, leaving the transient performance almost unaffected.

10. The form of the transfer function of the compensator will then be:

$$G_C(s) = K_C \frac{s+0.01}{s+0.01/\beta}$$

11. The next step is to determine the gain of the compensator K_C. The overall gain of the system is the combined gain of the plant and compensator is K_1, and is: $K_1 = K_C K$. Form the combined open-loop transfer function of the system with compensator: $G_1(s) = G_C(s)G(s)$. Determine the gain K_1 from the combined open-loop transfer function by substituting the value of closed-loop pole obtained in step 2 for s.

12. Determine the gain of the compensator by using: $K_C = \dfrac{K_1}{K}$

Example: In lecture 10 based on designing a simple gain controller (see Example), a the unity feedback system in which the plant with transfer function is: $G(s) = \dfrac{K}{(s+2)(s+4)(s+6)}$ (Type-0 system), is operating at a closed-loop pole: $-2 + j2.8$ giving a damping ratio of 0.6 (10% OS) and that we are happy with the settling time of 2 sec at this operating point. However, at this operating point, the gain of the system is: $K = 47$ (as calculated previously), will have a static error constant based on Type-0 system as:

$$K_p = \lim_{S \to 0} G(s)$$

Or
$$K_p = \lim_{S \to 0} \left[\frac{47}{(s+2)(s+4)(s+6)} \right] = 0.98$$

The steady-state error is:

$$e(\infty) = \frac{1}{(1+K_p)} = \frac{1}{1.98} = 0.505$$

The steady-state error amounts to 50.5%, which is quite high. Let us say if it is desired to keep the steady-state error limited to approximately 10%, then we need to design a compensator for the purpose and then we will compare the results of the uncompensated and compensated systems. Let us follow the steps as mentioned above. However, step 1 to 6 are simple and it can be noted that we have already done those that is drawing of root locus, damping line, getting the operating point for percent OS and settling time, determination of static error constant and the steady-state error at the selected closed-loop pole on the root locus as our desired operating point for the transient response which we are satisfied with. So we start from step 7 as follows:

Step 7: In order to limit the steady-state error to 10% or $e^{/}(\infty) = 0.1$, we need to compute the new static error constant $K_p^{/}$ for this error, which for the present Type-0 system is obtained by manipulating the expression that relates error with error constant that is: $e^{/}(\infty) = \dfrac{1}{(1+K_p^{/})}$ or:

$$K_p^{/} = \frac{1-e^{/}(\infty)}{e^{/}(\infty)} = \frac{1-0.1}{0.1} = 9.$$

Thus: $\quad K_p^{/} = \beta K_p$

Or $\quad \beta = \dfrac{K_p^{/}}{K_p} = \dfrac{9}{0.98} = 9.18$

Step 8: The transfer function of compensator according to Eq (3) is: $G_C(s) = K_C\left(\dfrac{s+z_C}{s+z_C/9.18}\right)$.

Step 9: Arbitrary choosing the appropriate values of zero close to origin say a zero at –0.01. Then the corresponding pole will be at $-0.01/\beta \approx -0.01/9.18 = -0.001$, such that the angle contribution due to the placement of pole and the zero to the complex closed-loop pole is within the limit of $\pm 5^{o}$. Thus:

$$z_C = -0.01 \text{ and } p_C = -0.001.$$

Step 10: The transfer function of the compensator thus is: $G_C(s) = K_C \dfrac{s+0.01}{s+0.001}$. Experience has shown that if the value of $\beta \geq 10$, the angle deviation is less than 5°. Thus there is no need to find the angle deviation. However, just as a matter of interest let us find it by substituting the dominant closed-loop pole: $-2 + j2.8$ for s in the compensator transfer function above and obtain the angle of numerator and denominator that is:

$$\angle G_C(s) = \frac{(-2+j2.8)+0.01}{(-2+j2.8)+0.001} = \frac{\angle 125.40}{\angle 125.52} = -0.12^{o}$$

That proves it.

Step 11: The next step is to determine overall gain of the compensated system, which is considered as: K_1, and is: $K_1 = K_C K$. Applying the magnitude criteria on the overall combined open-loop transfer function of the system with compensator with dominant pole $-2 + j2.8$ substituted for s in: $G_1(s) = G_C(s)G(s)$. Thus:

$$G_1(s) = \frac{K_1(s+0.01)}{(s+0.001)(s+2)(s+4)(s+6)}$$

Or

$$\frac{K_1(-2+j2.8+0.01)}{(-2.+j2.8+0.001)(-2+j2.8+2)(-2+j2.8+4)(-2+j2.8+6)} = 1$$

Taking the magnitude of each bracketed term in the above expression and cross-multiplying, we have:

$$K_1 = \frac{3.44 \times 2.8 \times 3.44 \times 4.88}{3.43} = 47.16$$

Step 12: The gain of the compensator is then determined using: $K_C = \frac{K_1}{K} = \frac{47.16}{47} = 1.003$. Thus it can be seen that the compensator gain is almost unity and so there is very small or no gain contribution to the system from lag compensator. The root locus therefore will only slightly shift to the right from its initial position thereby not affecting the transient behavior of the system. Thus it is expected that the percentage overshoot and settling time will be almost unchanged. Thus the transfer function of the compensated system is:

$$G_1(s) = G_C(s)G(s) = \frac{47.16(s+0.01)}{(s+0.001)(s+2)(s+4)(s+6)}$$

The system is still Type-0 but the order of the system has changed. That does not matter as long as the system Type and steady-state error is correctly maintained. Let us find out whether our compensated system can furnish the desired steady-state error or not. Let us first find the static error constant.

$$K_p{}' = \lim_{S \to 0} G_1(s) = \frac{47.16(0.01)}{(0.001)(2)(4)(6)} = 9.82$$

The steady-state error is thus:

$$e'(\infty) = \frac{1}{(1+K_p{}')} = \frac{1}{1+9.82} = 0.0924 \text{ or } 9.24\%$$

Most of the calculations are approximated so there is always a likelihood of small error. Well I suppose that we have achieved our objective of improving steady-state error while the expected change in the transient response is very small since the angle deviation in step 10 is very small. However, many systems are robust and slight variations in transient response specifications are not of concern.

Important: In electrical world, a lead compensator is essentially a high pass filter because its gain increases at high frequencies. On the other hand, a lag compensator is essentially a low pass filter since its gain is high at low frequencies. The low and high frequencies are defined with respect to a center or cut-off frequency.