Lecture # 1

Introduction

In different sectors of engineering we talk about precision and control. Every aspect of our activities is affected by some type of control system. Control systems forms an integral part of sectors such as; industry, space technology, weapon systems, transportation systems, power systems, robotics etc,

Control System

In general, a system is an organized set of components, coordinated to accomplish a certain task. A control system specifically is characterized by an input that has some relationship with correcting the output when commanded in order to maintain certain precision with regard to the desired output despite any disturbance. A control system constitute of subsystems, each defining a process for attaining the required objective. Any control system must be characterized by input (desired response), output (actual response) and plant (describing a control process). The input is generally called the reference input, while the output can be called the controlled variable and the system is characterized by a process modelled by mathematical function called a transfer function. Transfer function is the backbone for control system analysis and design. Figure (1) shows a simplest possible diagram of a control system.
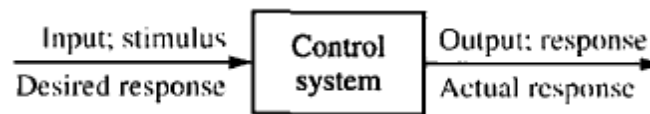

Figure 1: A Simple Control System

In general, the objective of the control system is to control the outputs in some prescribed manner by the inputs through the elements of the control system. Two important entities of control system performance are its speed (fast acting) and accuracy (precision), evaluated as transient and the steady-state response respectively. Control systems are classified as open-loop and closed-loop with respect to dependency of input on output.

Open-Loop System

In an open-loop system the input is independent of output. That means the input does not know about the changes undergoing in the output at a particular instant is. This type of system can go out of bounds in the event of disturbance. For example, the plant can be a crude type of air conditioning system, where the output variable is temperature and input variable is the supply voltage. In the case it is open-loop then the temperature will keep on dropping so long the supply voltage is applied and so has no limits. The simple schematic is same as shown in Figure (1).

The main disadvantage of open-loop systems is the inability to correct for desired temperature due to lack of communication between the output and input. The changes in the output from the desired value is therefore not corrected and so the input cannot adjust itself to the changes in output and the output is thus not brought to the desired value.

Closed-Loop Systems

A closed-loop system is the one in which the input depends on the output. Thus whatever is desired by the output variable, the input has to correct itself to follow the output. The general schematic diagram of a closed-loop system is shown in Figure (2).
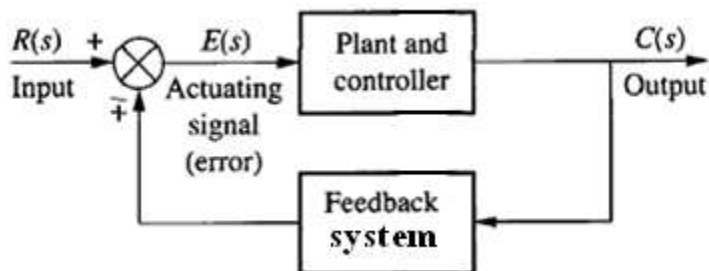

Figure 2: A Simple Closed-Loop System

A more understandable and appropriate schematic of a closed-loop control system is shown in Figure (3) in the controller and the plant are separated. Controller is usually incorporated in the forward path before the plant. The function of controller is to regulate the signal for the plant so that the plant processing is controlled.
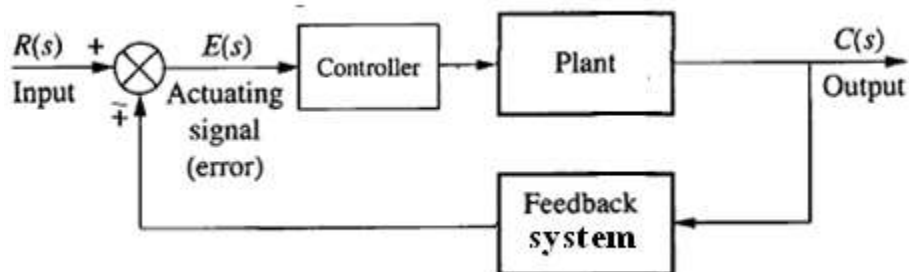

Figure 3: Closed-Loop System

A sensor, measures the output response and any undesirable deviation is reported back as a suitable signal through a feedback element to the summing junction or comparator where it is compared with the reference input. Any resulting error acts as an actuating signal (driving signal) to set the process in the plant in order to make sure that the output is maintained within the required bounds (stability margins). Let us consider the same crude type of air conditioning system, a special sensing element called thermostat (temperature measuring, typically a device

whose electrical resistance changes with temperature) is incorporated with a feedback system that include a potentiometer to adjust or cut-off the input voltage

The closed-loop systems differ from open-loop system that the former incorporates a feedback mechanism. Open-loop system can be converted to closed-loop systems by incorporating feedback. Feedback is either positive or negative. The motivation for using feedback is to reduce the error between the reference input and the system output. A closed-loop system is therefore also called a feedback control system. In general, we can state that whenever a closed sequence of cause-and-effect relationships exists among the variables of a system, feedback is said to exist.

Terminologies used in Control Systems

Control Variable: Control variable refers to the output of control system and is usually denoted by c.

Reference Input: It is the input to a control system derived directly from the source or through a transducer. It is generally denote by r. Reference input is also referred to as set-point (desired value).

Actuating Signal: Actuating signal is denoted by e and is obtained from a comparison measuring device and is equal to the reference input minus the primary feedback.

Disturbance: Disturbance is denoted by d and is the unwanted signal that tends to affect the controlled variable. The disturbance may be introduced into the system at many places. Disturbances may be internal to the system or may be from an external source.

Feedback Element: Feedback element H produces the primary feedback b from the controlled variable. This is generally a proportionality device but may also modify the characteristics of the controlled variable.

Process: Also referred to as plant; the process refers to a system that has the controlled variable as its property and has some means to influence the controlled variable.

Sensor: The sensor is an apparatus to measure the controlled variable and make the measurement result available to the controller. Most sensors are some form of transducers (that convert one form of energy to other).

Controller: The controller is a device that evaluates the control deviation and computes an appropriate control action. The controller measures the deviation of the controlled variable from the desired value and produces a corrective action, which is coupled to the input of the process and is used to drive the output variable toward the desired value.

The Transfer Function

The mathematical model in s-domain (Laplace transformed) representing a system or a plant is called the transfer function. A differential equation describing the relationship between the input and output of a system is developed (just like we did in circuit analysis) which can also be then called governing equation. Applying Laplace transform concepts to the differential equation, the transfer function (cause and effect relationship between input and output), can be

devised. Unlike the differential equations, the transfer function will allow separation of the input, system, and output into three separate and distinct parts. Consider a general n^{th}-order, linear differential equation as:

$$A_n \frac{d^n c(t)}{dt^n} + A_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + ... + A \frac{dc(t)}{dt} + A_0 c(t) = B_m \frac{d^m r(t)}{dt^m}$$
$$+ B_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + ... + B \frac{dr(t)}{dt} + B_0 r(t)$$

     1

Taking the Laplace transform of both sides of the above differential equation and simplifying, we have:

$$(A_n s^n + A_{n-1} s^{n-1} + ... + As + A_0)C(s) = (B_m s^m + B_{m-1} s^{m-1} + ... + Bs + B_0)R(s)$$

Expressing the above expression as the ratio $C(s)$ to $R(s)$, gives the required transfer function:

$$\frac{C(s)}{R(s)} = \frac{(B_m s^m + B_{m-1} s^{m-1} + ... + Bs + B_0)}{(A_n s^n + A_{n-1} s^{s-1} + ... + As + A_0)}$$

     2

A system with a transfer function can be represented as a block diagram, as shown in Figure (4), with the input on the left indicated by a terminating arrow, the output on the right as an originating arrow, and the system transfer function inside the block.
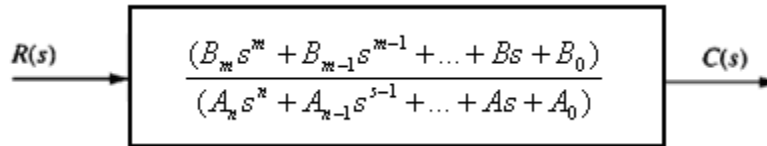

Figure 4: A System Representation with Transfer Function

This ratio is $C(s)/R(s)$ is the closed-loop transfer function and is evaluated with zero initial conditions. The ratio of the feedback signal, $H(s)C(s)$ to the actuating error signal $[R(s)–B(s)]$ is called the open-loop transfer function. Here, $H(s)$ is the transfer function of the feedback element, $C(s)$ is the output signal and $R(s)$ is the reference signal. In general the transfer function of a system is thus defined as the ratio of the Laplace transform of the output function (response function) to the Laplace transform of the input function (driving or forcing function) under the assumption of all initial conditions set to zero. A more concise general form of transfer function is expressed as:

$$\frac{C(s)}{R(s)} = \frac{B_m[s^m + (B_{m-1}/B_m)s^{m-1} + ... + (B/B_m)s + (B_0/B_m)]}{A_n[s^n + (A_{n-1}/A_n)s^{s-1} + ... + (A/A_n)s + (A_0/A_n)]} = K \frac{P(s)}{Q(s)}$$

     3

In Eq (3), $K = \frac{B_m}{A_n}$, $P(s)$ is the polynomial expression of the numerator and $Q(s)$ is the polynomial expression of the denominator of the transfer function. $Q(s)$ is also called

characteristic polynomial, since the degree of $Q(s)$ defines the order of the system. The constant multiple K with the transfer function will be the gain of the system. K is non-zero and can characterize the system as follows:

Attenuator: $0 < K < 1$

Amplifier: $K > 1$

Buffer: $K = 1$

Inverter: $K < 0$

Transfer function finds wide applications in control system studies and is a basic tool for the analysis and design of control systems. Once the transfer function of a system is known it can provide valuable description of the dynamic characteristics of the system, which helps towards understanding the nature of the system.

In Eq (3) we have mth degree polynomial expression in the numerator $P(s)$, which when equated to zero will give us the mth degree polynomial equation. The roots of this equation are called zeros of the system. That is when any $P(s)$ root (zero) is substituted for s in the transfer function it will make the entire transfer function equal to zero. On the other hand we have nth degree polynomial expression in the denominator $Q(s)$, which when equated to zero will give us the nth degree polynomial equation. The roots of this equation are called poles of the system. When any root (pole) of $Q(s)$ is substituted for s in the transfer function, it will make the entire transfer function equal to infinity. Poles and zeros are supposed to be balanced. In case of any difference between physically present poles and zeros, then those entities are supposed to be lying at infinity. Poles and zeros may be real, imaginary or complex and therefore can be plotted on complex s-plane (real axis σ and imaginary axis $j\omega$), using an '**x**' for the pole and a '**o**' for the zero. The concept of poles and zeros forms the basis for the analysis and design of control systems. When all the closed-loop poles of the system lie in the left-half of s-plane the system is stable and if all the poles and zeros lie in the left-half of s-plane then the system is referred to as a minimum phase system.

Consider a general arrangement of a feedback control system as shown in Figure (5) having a forward block $G(s)$ representing a plant or a process, a feedback block $H(s)$ and a summing junction.
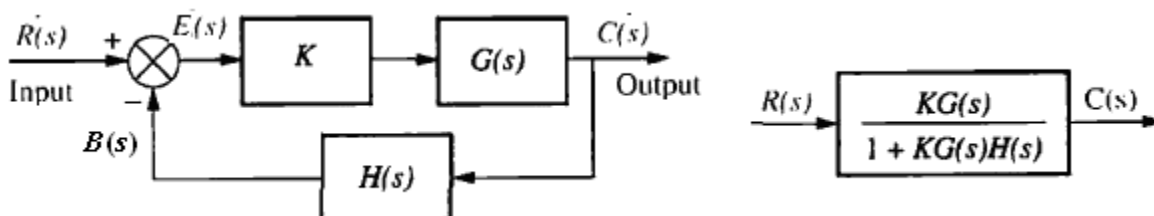

Figure 5: Feedback Control System

From Figure (5), the equation at the summing junction is:

$$E(s) = R(s) \mp B(s) \qquad\qquad 4$$

Using the definition of transfer function:

$$\frac{C(s)}{E(s)} = KG(s) \text{ (Forward transfer function)} \qquad\qquad 5$$

And $\qquad \dfrac{B(s)}{C(s)} = H(s)$ (Feedback transfer function) $\qquad\qquad 6$

Using Eq (5) and Eq (6), the quantities $E(s)$ and $B(s)$ in Eq (4) can be expressed in terms of $C(s)$ as:

$$\frac{C(s)}{KG(s)} = R(s) \mp C(s)H(s)$$

Simplifying and expressing the above expression as a ratio $C(s)/R(s)$ that is:

$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 \pm KG(s)H(s)} \qquad\qquad 7$$

The expression on the RH-side of Eq (7) is the closed-loop transfer function and is denoted by $T(s)$. Thus: $T(s) = \dfrac{KG(s)}{1 \pm KG(s)H(s)}$. The term $KG(s)H(s)$ in Eq (7), which is the ratio of feedback signal $B(s)$ to the actuating error signal $E(s)$ and is known as the open-loop transfer function. Where the expression; $1 \pm G(s)H(s)$ in the denominator of Eq (7) is termed as the characteristic polynomial, which when equated to zero will give us the characteristic equation. The roots of characteristic equation are referred to as poles.

In Eq (7), a positive sign in the denominator is used for a negative feedback system and negative sign for a positive feedback system. It must be remembered that the quantities $C(s)$ and $R(s)$ may be like or different; for example, in electrical system $R(s)$ may be a source voltage, whereas $C(s)$ may be current or a voltage drop across a particular element. Similarly in mechanical systems, $R(s)$ may be a force or torque applied to a system, whereas $C(s)$ may be force developed, velocity or displacement caused as a result of applied force or torque. Transfer function must not be confused with gain, which also the ratio of output to input, but in this case it is the ratio between similar quantities. System gain is usually included in the overall transfer function.

Example 1: Given a closed-loop transfer function: $T(s) = \dfrac{4s^2 + 12s + 8}{2s^3 + 10s^2 + 18s + 6}$. Find the order of the system, poles, zeros and gain, and characterize the system. Comment on the stability of the system.

Solution: First of all we express the given transfer function in standard form by taking out any constant multiple with the highest power of s in both the numerator and denominator.

$$T(s) = \frac{4(s^2 + 3s + 2)}{2(s^3 + 5s^2 + 9s + 3)} = 2\frac{(s^2 + 3s + 2)}{(s^3 + 5s^2 + 9s + 3)}$$

Factorizing the numerator and denominator or directly solving for roots when both are equated to zero. Thus:

$$T(s) = 2\frac{(s+1)(s+2)}{(s+0.4)(s+2.3 - j1.35)(s+2.3 + j1.35)}$$

The system is 3rd order, since the degree of $Q(s)$ is 3. The system poles are: –0.4, –2.3 + j1.35 and –2.3 – j1.35. The system physical zeros are –1 and –2 and one zero is lying at infinity (since there are three physical poles but two physical zeros, for balance, one zero is missing and that is lying somewhere at infinity). The system gain $K = 2$ and is greater than 1, it is therefore an amplifier. The poles and zeros are plotted in s-plane below:
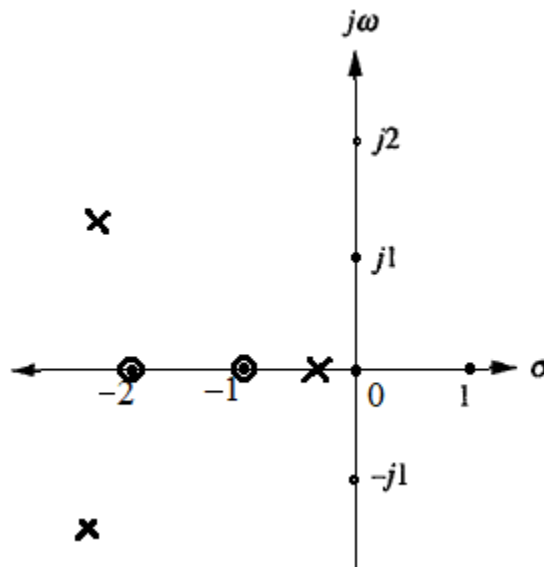

Figure 7: Pole-zero plot

Since all the poles of the system transfer function are placed in the left-half of s-plane, the system is stable. Also since all the poles and zeros are lying in the left-half of s-plane, the system is thus a minimum phase system.

Transfer Function of Electrical Systems

Majority of control systems incorporates electrical subsystems comprising of active and passive elements. Combination of electrical components in different configurations and mode of connection will form electric circuits and networks. The concept of transfer function can be applied for the mathematical modeling of electrical systems comprises of three passive linear elements; resistors, capacitors, and inductors. Resistor is energy-dissipating device whereas; capacitor and inductor are the energy-storage devices. Table (1) summarizes the components and the relationships between voltage and current and between voltage and charge under zero initial conditions.

Table 1

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \dfrac{1}{C}\displaystyle\int_0^1 i(\tau)d\tau$	$i(t) = C\dfrac{dv(t)}{dt}$	$v(t) = \dfrac{1}{C}q(t)$	$\dfrac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \dfrac{1}{R}v(t)$	$v(t) = R\dfrac{dq(t)}{dt}$	R	$\dfrac{1}{R} = G$
Inductor	$v(t) = L\dfrac{di(t)}{dt}$	$i(t) = \dfrac{1}{L}\displaystyle\int_0^1 v(\tau)d\tau$	$v(t) = L\dfrac{d^2q(t)}{dt^2}$	Ls	$\dfrac{1}{Ls}$

Once the input and output is decided, the transfer function can be obtained. The basic principles are the use and application of Kirchhoff's laws, in which voltages are summed around loops (KVL) and currents are summed at nodes (KCL), depending on which technique involves the least effort in mathematical manipulation. From these relationships we can write the differential equations for the system. By taking the Laplace transforms of the differential equations and finally arriving at a transfer function is illustrated in the following examples.

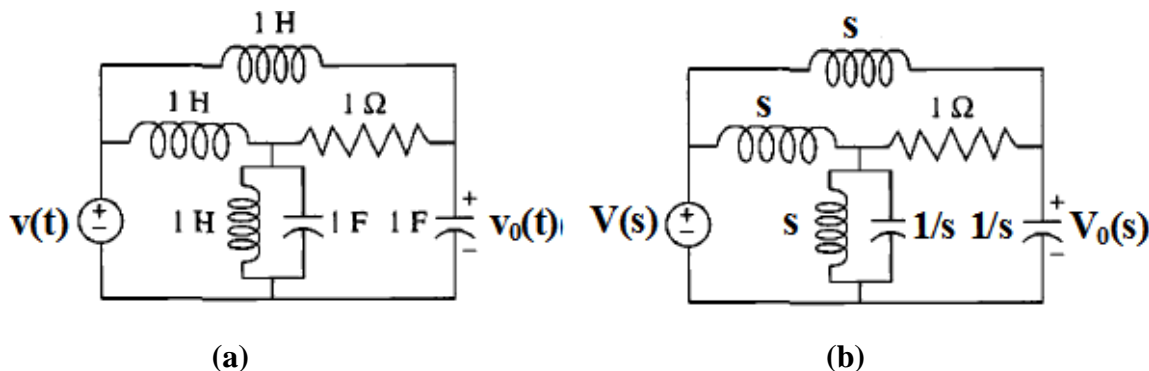**Example 2:** Obtain the transfer function of the electrical system shown in Figure (7) below:

(a)   **(b)**

Figure 7: Electrical System of Example (2)

Solution: The electrical system shown in Figure (7a) is first transformed into equivalent s-domain system as shown in Figure (7b). The system can be further simplified by considering the parallel arrangement of s and $\dfrac{1}{s}$ as: $\dfrac{s}{s^2+1}$ shown in Figure (8). Referring to Figure (8), the loop equations are:

$$V(s) = \left(s + \frac{s}{s^2+1}\right)I_1(s) - \left(\frac{s}{s^2+1}\right)I_2(s) - sI_3(s)$$

$$0 = -\left(\frac{s}{s^2+1}\right)I_1(s) + \left(1 + \frac{1}{s} + \frac{s}{s^2+1}\right)I_2(s) - I_3(s)$$
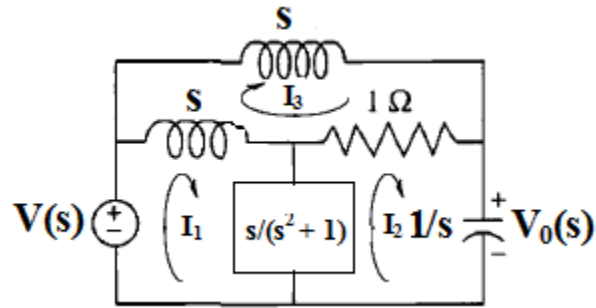
$$0 = -sI_1(s) - I_2(s) + (2s+1)I_3(s)$$

Figure 8: Simplified in *s*-Domain

Simplifying further: $V(s) = \left(\dfrac{s(s^2+2)}{s^2+1}\right)I_1(s) - \left(\dfrac{s}{s^2+1}\right)I_2(s) - sI_3(s)$

$$0 = -\left(\dfrac{s}{s^2+1}\right)I_1(s) + \left(\dfrac{s^3+3s^2+1}{s(s^2+1)}\right)I_2(s) - I_3(s)$$

$$0 = -sI_1(s) - I_2(s) + (2s+1)I_3(s)$$

Expressing the above equations in matrix form:

$$\begin{bmatrix} V(s) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{s(s^2+2)}{s^2+1} & -\dfrac{s}{s^2+1} & -s \\ -\dfrac{s}{s^2+1} & \dfrac{s^3+3s^2+1}{s(s^2+1)} & -1 \\ -s & -1 & 2s+1 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{bmatrix}$$

From Figure (8), the input voltage is $V(s)$ and the output voltage $V_0(s)$ is: $V_0(s) = I_2(s)/s$. Thus using Cramer's rule, $I_2(s)$ is:

$$I_2(s) = \dfrac{s(s^2+1)(s^2+2s+2)}{5s^6+3s^5+4s^4+7s^3+4s^2+s+2}V(s)$$

Substituting $I_2(s)$ in the expression for $V_0(s)$, simplifying and rearranging, we have the required transfer function as:

$$\dfrac{V_0(s)}{V(s)} = \dfrac{s^4+2s^3+3s^2+2s+2}{5s^6+3s^5+4s^4+7s^3+4s^2+s+2}$$

Operational Amplifiers

Operational amplifiers are widely used in control system design of electrical systems, since they can perform some important mathematical operations; inversion, summation, integration and differentiation. An operational amplifier, commonly referred to as op-amp ("nick name") is shown in Figure (9) is extensively used in controllers, compensators, sensors and comparators in combination with other active and passive elements. Op-amp have two input terminals; an inverting and non-inverting.
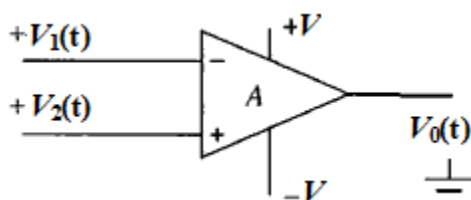

Figure 9: Basic Operational Amplifier

by virtue it can be used as inverting and non-inverting amplifiers respectively as shown in Figure (10a) and Figure (10b) respectively.
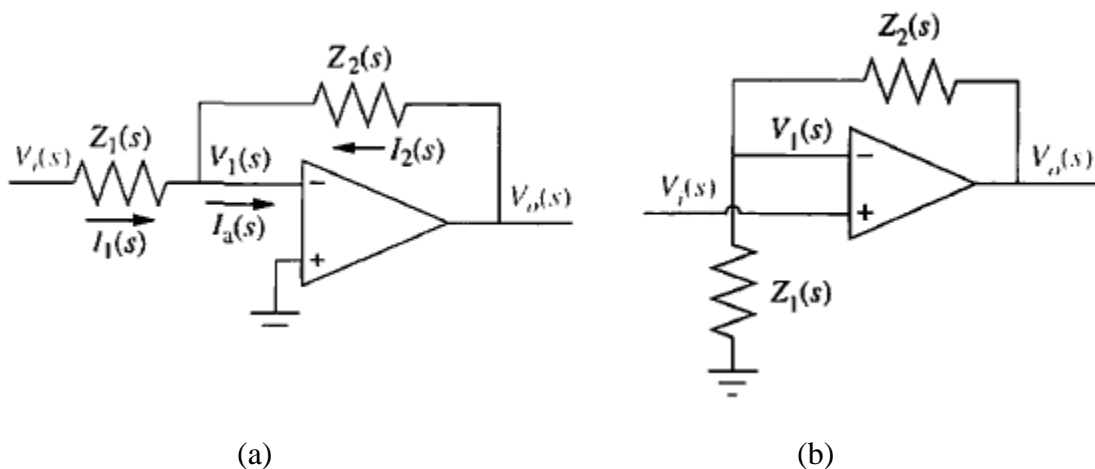

(a)   (b)

Figure 10: (a) Inverting mode (b) Non-inverting mode

The transfer function of op-amp when used in inverting mode is:

$$\frac{V_0(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} \qquad\qquad 8$$

The negative sign should be dropped when finding the transfer function. The negative sign only refers to the mode. The transfer function of non-inverting op-amp circuit in s-domain is:

$$\frac{V_0(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)} \qquad\qquad 9$$

Example 3: Obtain the transfer function of the op-amp circuits shown in Figure (11a) and Figure (11b).

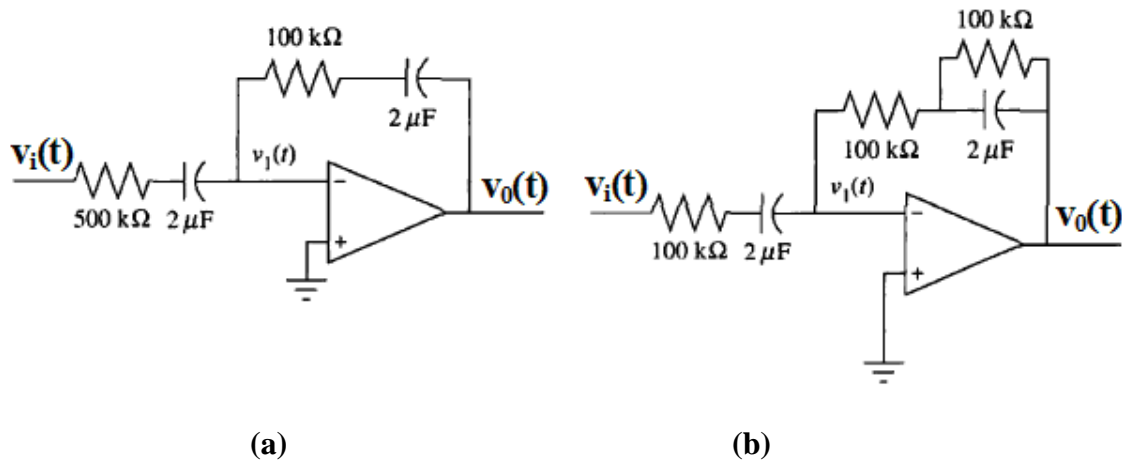(a)   (b)

Figure 11: Operational Amplifier Circuit for Example (3)

Solution: Figure (11a) is inverting op-amp circuit, converting to s-domain, we have:

$$Z_1(s) = 5 \times 10^5 + \frac{10^6}{2s}$$

$$Z_2(s) = 10^5 + \frac{10^6}{2s}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{Z_2(s)}{Z_1(s)} = \frac{10^5 + 10^6/2s}{5 \times 10^5 + 10^6/2s} = \frac{10^5[1 + 10/2s]}{10^5[5 + 10/2s]}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{(2s + 10)}{(10s + 10)} = \frac{0.2(s + 5)}{s + 1}$$

Figure (11b) is inverting op-amp circuit, converting to s-domain:

$$Z_1(s) = 10^5 + \frac{10^6}{2s}$$

$$Z_2(s) = 10^5 + \frac{10^6}{(2s + 10)}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{Z_2(s)}{Z_1(s)} = \frac{10^5 + 10^6/(2s + 10)}{10^5 + 10^6/2s} = \frac{10^5[1 + 10/(2s + 10)]}{10^5[1 + 10/2s]}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{s(s+10)}{(s+5)^2} = \frac{s(s+10)}{s^2+10s+25}$$

Example 4: Obtain the transfer function of op-amp shown in Figure (12a) and Figure (12b).
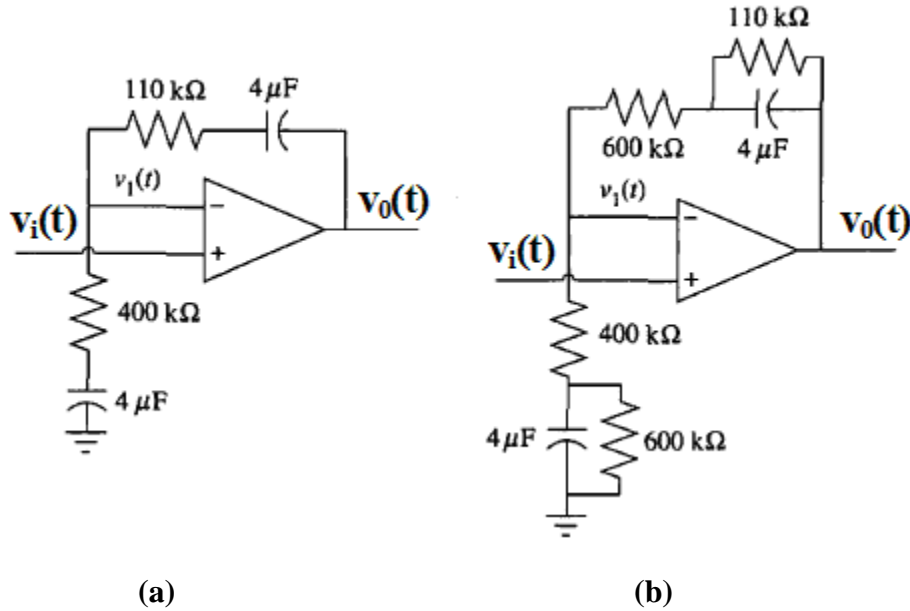

(a)   **(b)**

Figure 12: Operational Amplifier Circuit for Example (4)

Solution: Figure (12a) is a non-inverting op-amp circuit, converting to s-domain and finding the required impedance, so that:

$$Z_1(s) = 4 \times 10^5 + \frac{10^6}{4s}$$

$$Z_2(s) = 11 \times 10^4 + \frac{10^6}{4s}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)} = \frac{[4 \times 10^5 + 10^6/4s] + [11 \times 10^4 + 10^6/4s]}{4 \times 10^5 + 10^6/4s}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{1.275(s+0.98)}{s+0.625}$$

Likewise in Figure (12b) we have:

$$Z_1(s) = 4 \times 10^5 + \frac{60 \times 10^5}{24s+10}$$

$$Z_2(s) = 6 \times 10^5 + \frac{110 \times 10^5}{(44s + 100)}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)} = \frac{[4 \times 10^5 + 60 \times 10^5 /(24s + 10)] + [6 \times 10^5 + 110 \times 10^5 /(44s + 100)]}{4 \times 10^5 + 60 \times 10^5 /(24s + 10)}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{2.5(s^2 + 3.18s + 1.62)}{s^2 + 3.31s + 2.36}$$