# Lecture # 9

## Solved Examples of Plotting Root Locus

Dear Student this lecture is based on the technique of sketching the root locus as described in Lecture 8. Remember that for plotting the root locus, the open-loop transfer function is required. Here three examples are considered. In Example 1, the open-loop transfer function is given so everything is crystal clear; open-loop poles and open-loop zeros are found directly, which are necessary to start drawing the root locus. In Example 2, the closed-loop transfer function is given, from which the open-loop transfer function is obtained, which can be factorized for obtaining open-loop poles and zeros. So things are not directly. Example 3, deals with the situation when we encounter complex open-loop poles and zeros. In this case a few conventional parameters; such as breakaway point and break-in points and sometimes $j\omega$-crossing points are missing. However, two new parameters are introduced; angle of departure from complex open-loop pole (replacing the breakaway point) and angle of arrival at a complex open-loop zero (replacing the break-in point).

**Example 1:** Plot the root locus for a system whose open-loop transfer function is: $\dfrac{K(s+1)}{s(s+2)(s+3)}$

**Solution:** The system's open-loop poles are 0, –2 and –3 and open-loop zero is: –1, and there are two zeros lying at infinity.

**Number of loci:** $N$ = number of open-loop poles = 3

**Real-axis loci:** From the location of open-loop poles and zeros in the *s*-plane, the real-axis loci will exist between 0 and –1 and between –2 and –3.

**Center of asymptotes:** $$\sigma_C = \frac{\sum p_i - \sum z_i}{p-z} = \frac{0-2-3-0}{3-1} = -2.5$$

**Angles of asymptotes:** $$\theta_C \frac{(2k+1)180^0}{p-z}$$

For $k$ = 0; $\theta_C = 90^0$, for $k$ = 1; $\theta_C = 270^0$. The angles are repeated in the same sequence for $k$ = 2, 3, …

**Breakaway point:** $$\frac{1}{\sigma_b - 1} = \frac{1}{\sigma_b} + \frac{1}{\sigma_b - 2} + \frac{1}{\sigma_b - 3}$$

From which: $$\sigma_b^{\,3} + 4\sigma_b^{\,2} + 5\sigma_b + 3 = 0$$

This gives: $\sigma_b = -2.465$

**Break-in point:** There is no break-in point since there is no real axis loci between two open-loop zeros. However, there are two open-loop zeros lying at infinity, and the root locus will tend to move to infinity from breakaway point to search for their zeros lying at infinity.

***jω*-crossing:** There is no *jω*-crossing since the root locus will depart from the real-axis at $90^o$ and $270^o$. The complete root locus is shown in Figure (1).

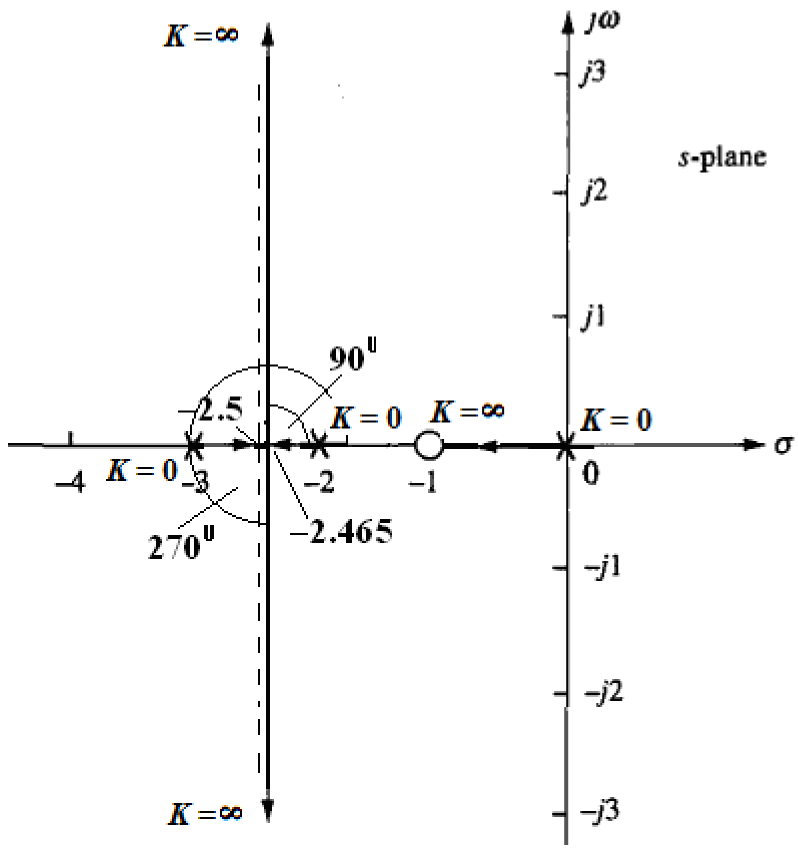



**Figure 1**

**Example 2:** Draw the root locus of a control system whose closed-loop transfer function is: $\dfrac{K}{s^3+6s^2+9s+K}$.

**Solution:** The given system can be converted to a unity feedback configuration by considering the given transfer function and manipulating it as:

$$\frac{K/(s^3+6s^2+9s)}{1+K/(s^3+6s^2+9s)} = \frac{G(s)}{1+H(s)G(s)}$$

Thus: $$G(s)H(s)=\frac{K}{s(s^2+6s+9)}=\frac{K}{s(s+3)^2}$$

The system poles are: 0, –3 and –3. There are three zeros lying at infinity.

**Number of loci:** $N$ = number of open-loop poles = 3.

**Real-axis loci:** From the location of open-loop poles, the real-axis loci will exist between 0 and –3 and from –3 to –∞.

**Center of asymptotes:** $$\sigma_C=\frac{\sum p_i-\sum z_i}{p-z}=\frac{0-3-3-0}{3-0}=-2$$

**Angles of asymptotes:** $$\theta_C=\frac{(2k+1)180^0}{p-z}=\frac{(2k+1)180^0}{3-0}$$

For $k$ = 0; $\theta_C=60^0$, for $k$ = 1; $\theta_C=180^0$ and for $k$ = 2; $\theta_C=300^0$. The angles are repeated in the same sequence for $k$ = 3, 4, 5, …

**Breakaway point:** From the characteristic equation: $s^3+6s^2+9s+K$ we have:

$$\frac{dK}{ds}=-3s^2-12s-9=0$$

This yield: $$3\sigma_b^{\ 2}+12\sigma_b+9=0$$

Which give real roots; –1 and –3. The real root: –1 is within the real-axis loci between 0 and –3, therefore the real root; $\sigma_b=-1$ is the breakaway point.

**Break-in point:** There is no break-in point since the root locus will tend to move to infinity from breakaway point to search for zeros lying at infinity.

***jω*-crossing:** The $j\omega$-crossing point can be obtained by considering the characteristic polynomial of the closed-loop transfer function: $s^3+6s^2+9s+K$ from which the Routh table is formed below:

| | | |
|---|---|---|
| $s^3$ | 1 | 9 |
| $s^2$ | 6 | $K$ |

| | | |
|---|---|---|
| $s^1$ | $\frac{54-K}{6}$ | 0 |
| $s^0$ | $K$ | 0 |

Forcing the term of the first column of $s^1$ row to be zero, we can form an auxiliary equation of the preceding even row. Thus for $K$ = 54, we will have an all-zero row. The auxiliary equation of the preceding row of even power of $s$ is then: $6s^2 + 54 = 0$, which yields:

$$s = \pm j3$$

Which are the required points of $j\omega$-crossing. The complete root locus is shown in Figure (2).

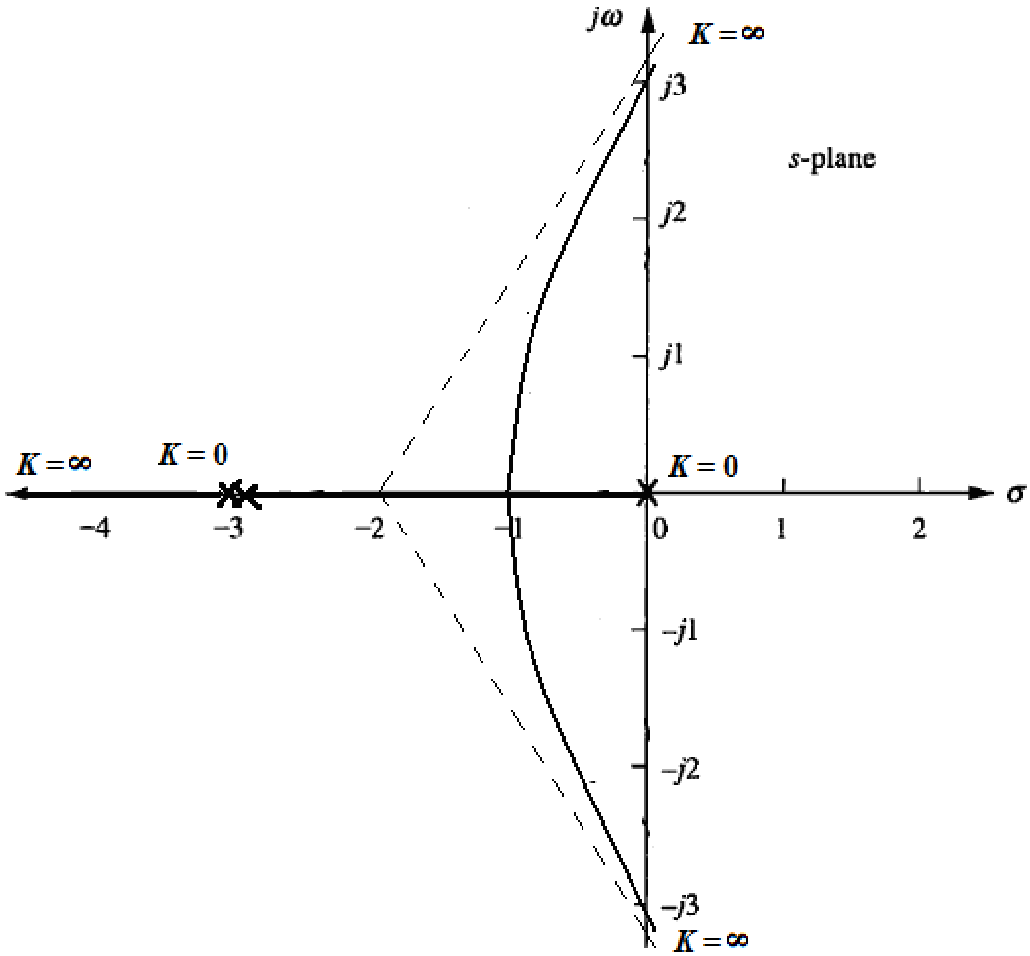



**Figure 2**

**Example 3:** Plot the root locus of the unity feedback system whose forward transfer function is given as: $G(s) = \dfrac{K(s^2 + 4s + 13)}{(s+4)(s^2 + 2s + 5)}$.

**Solution:** The open-loop transfer function of the given unity feedback system is:

$$G(s)H(s) = \frac{K(s^2 + 4s + 13)}{(s+4)(s^2 + 2s + 5)}$$

The open-loop transfer function can be conveniently factorized as:

$$G(s)H(s) = \frac{K(s + 2 + j3)(s + 2 - j3)}{(s+4)(s + 1 + j2)(s + 1 - j2)}$$

The system poles are: –4, and $-1 \pm j2$. The complex zeros are: $-2 - j3$ and $-2 + j3$. There is a zero lying at infinity.

**Number of loci:** Number of loci ($N$) = number of open-loop poles = 3

**Real-axis loci:** From the position of open-loop poles and zeros, the real-axis loci exist at –4 and moves towards negative infinity direction in search of its zero lying at infinity. Since the real-axis loci tends to move towards infinity in search of the zero lying at infinity and the loci starting from the complex poles will terminate on the complex zeros, therefore there is no breakaway and break-in point on the real-axis. Thus the centre and angles of asymptotes will not exist. The root locus in the $s$-plane will terminate on the complex zeros therefore there will be no $j\omega$-crossing point.

**Angle of departure at complex poles:** Considering a complex pole $s = -1 + j2$, the angle of departure $\theta_D$ can be determined by ignoring the angle contribution from the factor of the complex pole considered and then finding the angle contribution of the remaining poles and zeros at: $s = -1 + j2$. This angle contribution is then: $\angle K[G(s)H(s)]^{/}$. Where the function: $K[G(s)H(s)]^{/}$ is the open-loop transfer function in which the angle contribution of the complex pole of interest has been ignored. Thus:

$$K[G(s)H(s)]^{/} = \frac{K(1 + j5)(1 - j1)}{(3 + j2)(j4)}$$

Therefore: $$\angle K[G(s)H(s)]^{/} = \frac{\angle 78.7^0 \angle -45^0}{\angle 33.7^0 \angle 90^0} = -90^0$$

Therefore: $$\theta_D = 180^0 + \angle K[G(s)H(s)]^{/} = 180^0 - 90^0 = 90^0$$

**Angle of arrival at a complex zero:** Since in this example there are two complex zeros, the loci from the complex poles will terminate on the complex zeros. The angle of arrival at a complex zero ($s=-2+j3$), can be determined from the given open-loop transfer function by ignoring the angle contribution from the zero of interest ($s=-2+j3$). The open-loop transfer function in which the factor of the considered zero is ignored will be: $K[G(s)H(s)]^{//}$. The angle contribution of $K[G(s)H(s)]^{//}$ is the argument of this function, which is obtained as follows:

$$K[G(s)H(s)]^{//} = \frac{K(s+2+j3)}{(s+4)(s+1+j2)(s+1-j2)}$$

At $s=-2+j3$, the function will be:

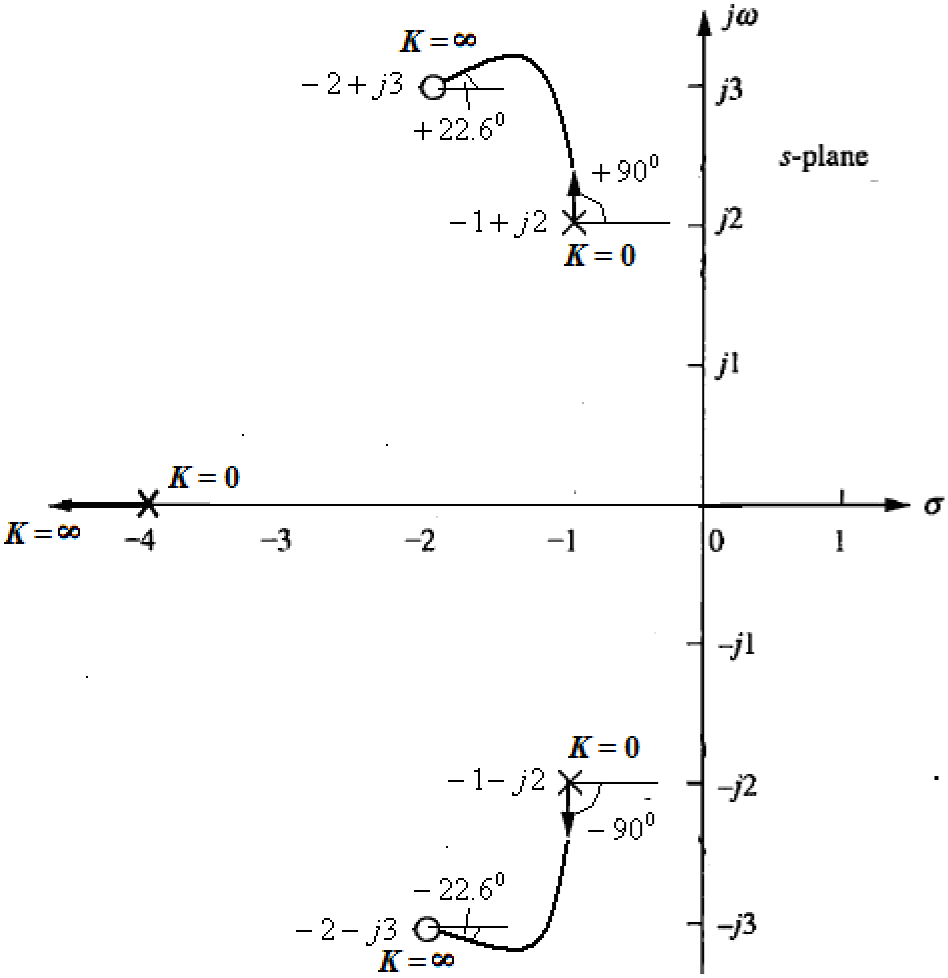
$$K[G(s)H(s)]^{//} = \frac{K(j6)}{(2+j3)(-1+j5)(-1+j1)}$$

$$\angle K[G(s)H(s)]^{//} = \frac{\angle 90^0}{\angle 56.3^0 \angle 101.3^0 \angle 135^0} = -202.6^0$$

Therefore the angle of arrival at the complex pole of interest is:

$$\theta_A = 180^o - \angle K[G(s)H(s)]^{//} = 180^0 + 202.6^0 = 382.6^0 \text{ or } 22.6^0$$

Since the function contains two complex poles and two complex zeros in the LHP, the loci from the complex poles will terminate on the corresponding complex zeros, therefore there will be no $j\omega$-crossing point. The other zero is lying at infinity and the loci from the real-axis pole –4 will move to infinity in search of the zero lying at infinity. The complete root locus is shown in Figure (3).



jω
K = ∞
− 2 + j3
+ 22.6°
j3
s-plane
+ 90°
− 1 + j2
j2
K = 0
j1
K = 0
K = ∞
−4
−3
−2
−1
0
1
σ
−j1
K = 0
− 1 − j2
−j2
− 90°
− 22.6°
− 2 − j3
−j3
K = ∞


**Figure 3**