Lecture # 8

Root Locus

Dear students, this lecture is very important as it covers more than 50% control system design and analysis. It is therefore essential that you understand this lecture; not only browsing through lecture notes but going through similar topics in control system book(s).

Remember that the main objective of control system is to keep the desired output within allowable limits with maintaining stability. Not only must the system be stable in steady-state, but also the transients must die out in a sufficiently short time for the system to settle to the desired steady-state value. The stability can be ascertained from the characteristic polynomial; 1 + $G(s)H(s)$ of the closed-loop transfer function of the form: $\dfrac{G(s)}{1 + H(s)G(s)}$. Equating the characteristic polynomial to zero will yield the characteristic equation that is: $1 + H(s)G(s) = 0$. Solving the characteristic equation will give the roots or closed-loop poles of the system. Any closed-loop pole placed in the RHP (right-half plane) will lead to the fact that the given system is unstable. Forming Routh table from the characteristic polynomial and applying Routh-Hurwitz criterion, it is possible to determine stability without solving for the roots of the characteristic equation. However, Routh-Hurwitz criterion only provides information about the absolute stability. The control system engineer is interested in relative stability of the system. The answer lies in the Root Locus.

Root locus is a graphical method, pioneered by Walter R. Evans, for stability analysis and control system design that includes plotting of the closed-loop poles by varying the system gain. Root locus plot allows a control system engineer to observe ranges of stability, and the conditions that cause a system to break into oscillation and resorting to unstable operation.

Root Locus of Second-order System

A second-order system is a simple system that is mostly used as a benchmark in control system studies. Consider a second-order system represented by the transfer function: $\dfrac{K}{s^2 + as + K}$. The characteristic equation from the closed-loop transfer function is: $s^2 + as + K = 0$, and the roots of the characteristic equation (closed-loop poles) are:

$$s = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4K} \qquad\qquad 1$$

When K is varied from $-\infty$ to $+\infty$, Eq (1) gives infinite roots. However, values of $K > 0$ are considered. Thus for $a > 0$ and small values of K the roots or poles will be real and for larger values of K the roots will be complex conjugate. Supposing $a = 10$ is selected, then for various values of K the poles of the transfer function will have range of values that are listed in Table (1). Thus for each pole, there is a collection of point for different values of gain forming a locus on which the characteristic roots or closed-loop poles of the system lies as shown in Figure (1). Thus the designer will be able to choose the closed-loop poles which give a desired transient performance for a given value of gain.

The root-locus technique is limited to only one variable parameter in K. In many control-systems problems, the effects of varying several parameters should be investigated. When more than one parameter varies continuously from $-\infty$ to $+\infty$, the root loci are referred to as the root contours. Root contours still possess the same properties as the single-parameter root loci, so that the methods of construction thus far are all applicable.

Table 1

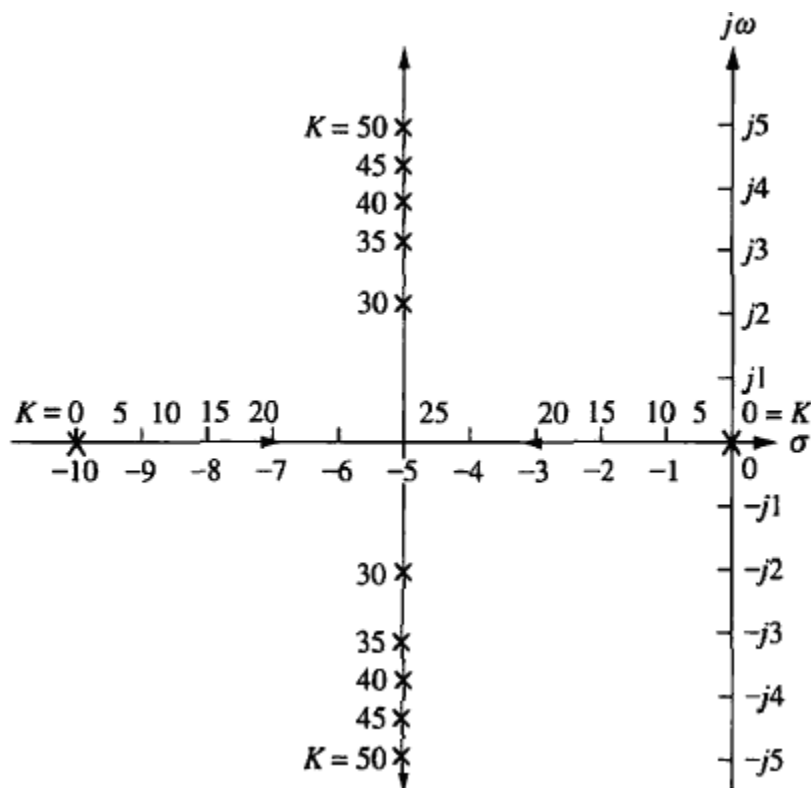| K	First pole	Second pole
0	− 10	0
5	− 9.47	− 0.53
10	− 8.87	− 1.13
15	− 8.16	− 1.84
20	− 7.24	− 2.76
25	− 5	− 5
30	− 5 + j2.24	− 5 − j2.24
35	− 5 + j3.16	− 5 − j3.16
40	− 5 + j3.87	− 5 − j3.87
45	− 5 + j4.47	− 5 − j4.47
50	− 5 + j5	− 5 − j5

Figure 1

The root-locus method can also be extended to the analysis and design of higher-order systems.

Open-Loop Transfer Function

Stability analysis from closed-loop transfer function is more difficult to handle in terms of finding the closed-loop poles, unless the system's characteristic polynomial is factorized. Further, the closed-loop poles change with changes in system gain. On the other hand open-loop transfer function is friendlier for determination of open-loop poles and their location do not change with system gain. The open-loop transfer function is the ratio of feedback signal $B(s)$ to the error signal $E(s)$, and is easily obtained from the closed-loop transfer function. Consider a closed-loop feedback control system as shown in Figure (2).
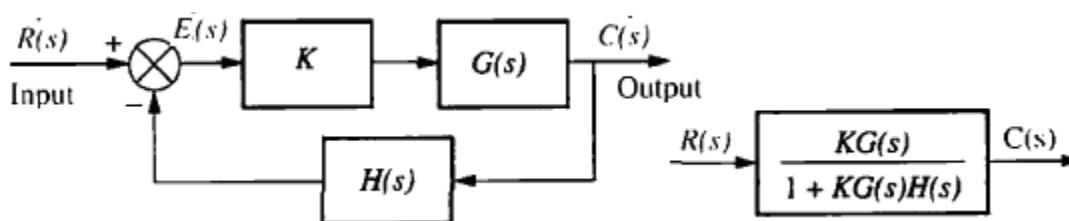

Figure 2

From Figure (2), the closed-loop transfer function is: $\dfrac{KG(s)}{1+KG(s)H(s)}$ and from which the open-loop transfer function is the term: $KG(s)H(s)$ of the denominator (characteristic polynomial). It must be remembered that the zeros of the open-loop transfer function are the zeros of $G(s)$ and $H(s)$ and the poles of open-loop transfer function are the poles of $G(s)$ and $H(s)$, whereas; closed-loop transfer function $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$. For example, if:

$$G(s) = \frac{K(s+1)}{s(s+2)}$$

And $\qquad H(s) = \dfrac{(s+3)}{(s+4)}$

Then the open-loop transfer function is:

$$G(s)H(s) = \frac{K(s+1)(s+3)}{s(s+2)(s+4)} \qquad\qquad 2$$

The poles of $G(s)H(s)$ are 0, –2 , and –4. The zeros of $G(s)H(s)$ are –1 and –3. Now consider a closed-loop transfer function:

$$T(s) = \frac{K(s+1)(s+4)}{s^3 + (K+6)s^2 + (4K+8)s + 3K} \qquad\qquad 3$$

Thus, the zeros of $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$. The poles of $T(s)$ are not immediately known without factoring the denominator since they are a function of K. Since the transient response and stability of the system are dependent upon the poles of $T(s)$, we cannot know about the performance of the system unless the denominator for specific values of K is factorized. The root locus can make it easy.

Every transfer function should have equal number of poles and zeros if we include the infinite poles and zeros as well as the finite poles and zeros. The number of poles and zeros are balanced, therefore a missing zero or pole in a transfer function is an indication that it is lying at infinity. Thus the root locus must start at a point where the system gain is zero (open-loop pole) and will end up at a point where the system gain is infinite (open-loop zero).

Angle-Magnitude Criteria

Consider the control system as shown in Figure (2) for which the closed-loop transfer function is:

$$T(s) = \frac{KG(s)}{1+KG(s)H(s)}$$

The closed-loop poles of the system can be determined from the characteristic equation that is:

$$1+KG(s)H(s)=0$$

Or $KG(s)H(s)=-1=1\angle(2k+1)180^{o}$ 4

Where $k = 0, \pm1, \pm2, \pm3,$ and so on. Remember s is complex in nature and so $KG(s)H(s)$ is a complex quantity with a magnitude and an angle. Therefore for $K > 0$, the magnitude criterion is expressed as:

$$| KG(s)H(s)|=1$$ 5

The angle criterion is:

$$KG(s)H(s)=(2k+1)180^{o}$$ 6

Simply: $KG(s)H(s)=-1$

Or $G(s)H(s)=-\dfrac{1}{K}$

Or $K=\dfrac{1}{|G(s)H(s)|}$

For $K < 0$, the magnitude criterion is same as that for $K > 0$, whereas the angle criterion is:

$$KG(s)H(s)=(k)360^{o}$$

Points to remember (the 10 commandments):

1. Plotting the root locus requires the open-loop transfer function of the form: $KG(s)H(s)$.
2. The root locus starts at open-loop pole and terminate (end) at open-loop zero. The open-loop poles and open-loop zeros are **not included** in the root locus.
3. Any point that must lie on the root locus will be a system's closed-loop pole at a particular value of K, and at that point both the angle and magnitude criteria are satisfied.
4. At open-loop poles, the value of $K = 0$. Thus $K = 0$ are points on the root loci on which poles of $G(s)H(s)$ will lie.
5. At open-loop zeros, the value of $K = \pm \infty$. Thus the $K = \pm \infty$ are points on the root loci which are the zeros of $G(s)H(s)$.
6. The poles and zeros referred to here also include those at infinity, if any.
7. As the magnitude of K approaches zero, $G(s)H(s)$ approaches infinity, so s must approach the poles of $G(s)H(s)$.
8. As the magnitude of K approaches infinity, the zeros of $G(s)H(s)$ must also approach infinity.
9. When a system has zero(s) lying at infinity, the system will have infinite number of closed-loop poles.
10. When the number physical of poles and zeros of open-loop transfer function are equal, the system will have finite closed-loop poles.

Characteristics of Root Locus

The followings are the important characteristics of a root locus that are helpful in sketching the root locus.

1. The root locus starts at a point where the system gain: $K = 0$ (open-loop pole) and will end up at a point where the system gain: $K = \infty$. Therefore the root locus will exist beyond the real-axis segments in case of zero lying at infinity.
2. The root locus is symmetrical about the real axis. This is because that the closed-loop transfer function of all real-time physical systems have characteristic equation that have complex poles in the form of complex conjugate pairs. Thus the root locus will have a mirror image with respect to the real axis of the complex s-plane.

Rules for Plotting Root Locus

Poles and zeros:

1. For plotting root locus, the open-loop transfer function is necessary. If closed-loop transfer function is given, then the open-loop transfer function has to be obtained from the given closed-loop transfer function based on unity feedback system.
2. Obtain the open-loop poles and zeros from the open-loop transfer function

<u>**Real axis loci:**</u>

1. From the given open-loop transfer function $KG(s)H(s)$, plot all the open-loop poles and zeros in the s-plane.

2. The real axis loci will exist to the left of real axis odd number of pole or zero.
3. The real axis loci must exist between two real axis poles or two real axis zeros or between a real axis pole and a real axis zero.
4. In case a zero is lying at infinity, the loci will exist between a real axis pole and infinity (indicating a zero lying at infinity).

Breakaway and break-in point:

1. A breakaway point exists on the real axis loci formed between two poles.
2. A break-in point exists on the real axis loci formed between two zeros.
3. To find these points:
   a. Equate the open-loop transfer function $KG(s)H(s)$ to -1.
   b. Rearrange and take the derivative: $\dfrac{d}{ds}(K)$.
   c. Equate the derivative in part b equal to zero for $s = s_b$, forming an equation.
   d. Solve the equation for s_b to yield breakaway and break-in point. Suitable choice is necessary between multiple roots.
4. Alternatively the poles and zeros of the open-loop transfer function are arranged in the form of partial fractions as follows and then the breakaway point is obtained by solving the equation for s. For example, consider the open-loop transfer function: $KG(s)H(s) = \dfrac{K(s+2)}{s(s+4)(s+6)}$. The partial fraction type of equation with poles on one side and zeros on the other side in the form of fractions is formed below as:

$$\frac{1}{s+2} = \frac{1}{s} + \frac{1}{s+4} + \frac{1}{s+6}$$

Centre and angles of asymptotes:

1. The centre of asymptotes σ_C can be obtained using: $\sigma_C = \dfrac{\sum p - \sum z}{n}$, where: n is the difference of the number of poles and zeros.
2. Angle of asymptotes can be obtained using: $\theta_C = \dfrac{\pm 180(2k+1)}{n}$. Repetitions are not considered.

Imaginary axis crossing points:

1. Obtain the closed-loop transfer function from the given open-loop transfer function.
2. Get the characteristic equation from the closed-loop transfer function: $1 + KG(s)H(s) = 0$
3. Obtain the Fourier transform of the characteristic equation that is: $1 + KG(jw)H(jw) = 0$.
4. Equate separately the imaginary and real components of the Fourier transformed characteristic equation to zero and solve them for w and K. The quantity w gives the $\pm jw$ points symmetrically above and below the origin with the value of K for which the

system is marginally stable. Alternatively use the Routh table and apply the all-zero row rule to get the auxiliary equation. Solving the auxiliary equation will give w for jw crossing and the condition of equating to zero on the expression containing K at the calculated value of w will give the gain at jw crossing.

Angle of departure from a complex pole:

1. In the given open-loop transfer function, remove the complex pole of the form: $s = -a + jb$ at which the angle of departure is to be calculated.
2. Find the angle of the resultant open-loop transfer function for: $s = -a + jb$. Let this angle be α.
3. The angle of departure θ_D is then: $\theta_D = 180 + \alpha$.

Angle of arrival at a complex zero:

1. In the given open-loop transfer function, remove the complex zero of the form: $s = -c + jd$ at which the angle of arrival is to be calculated.
2. Find the angle of the resultant open-loop transfer function for: $s = -c + jd$. Let this angle be β.
3. The angle of arrival θ_A is then: $\theta_A = 180 - \beta$.

Let us now take an example of drawing the Root Locus for a unity feedback system of the form as shown in Figure E with forward transfer function of: $G(s) = \dfrac{K}{(s+1)(s+3)(s+5)}$.
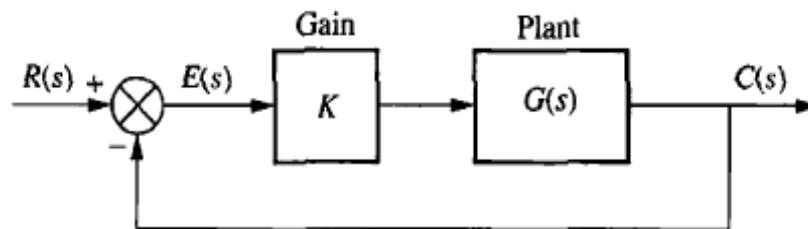

Figure E

Since: $H(s) = 1$ for a unity feedback system, the open-loop transfer function with gain takes the form:

$$KG(s)H(s) = \frac{K}{(s+1)(s+3)(s+5)}$$

The system open-loop poles are –1, –3 and –5. There are no physical zeros, so that the system has three zeros lying at infinity. Thus: $p - z = 3 - 0 = 3$. The number of loci will be equal to the number of open-loop poles, so that the complete root locus will consists of three loci.

In order to plot the root locus for the given system, we will proceed step-by-step according to the rules mentioned above.

Step 1: Obtain the open-loop poles and zeros from the open-loop transfer function and plot them on the s-plane; poles designated by **X** and zeros by **O**. In this example, there are three open-loop poles; –1, –3 and –5, and there are three open-loop zeros lying at infinity. They are plotted as shown in Figure (E1).
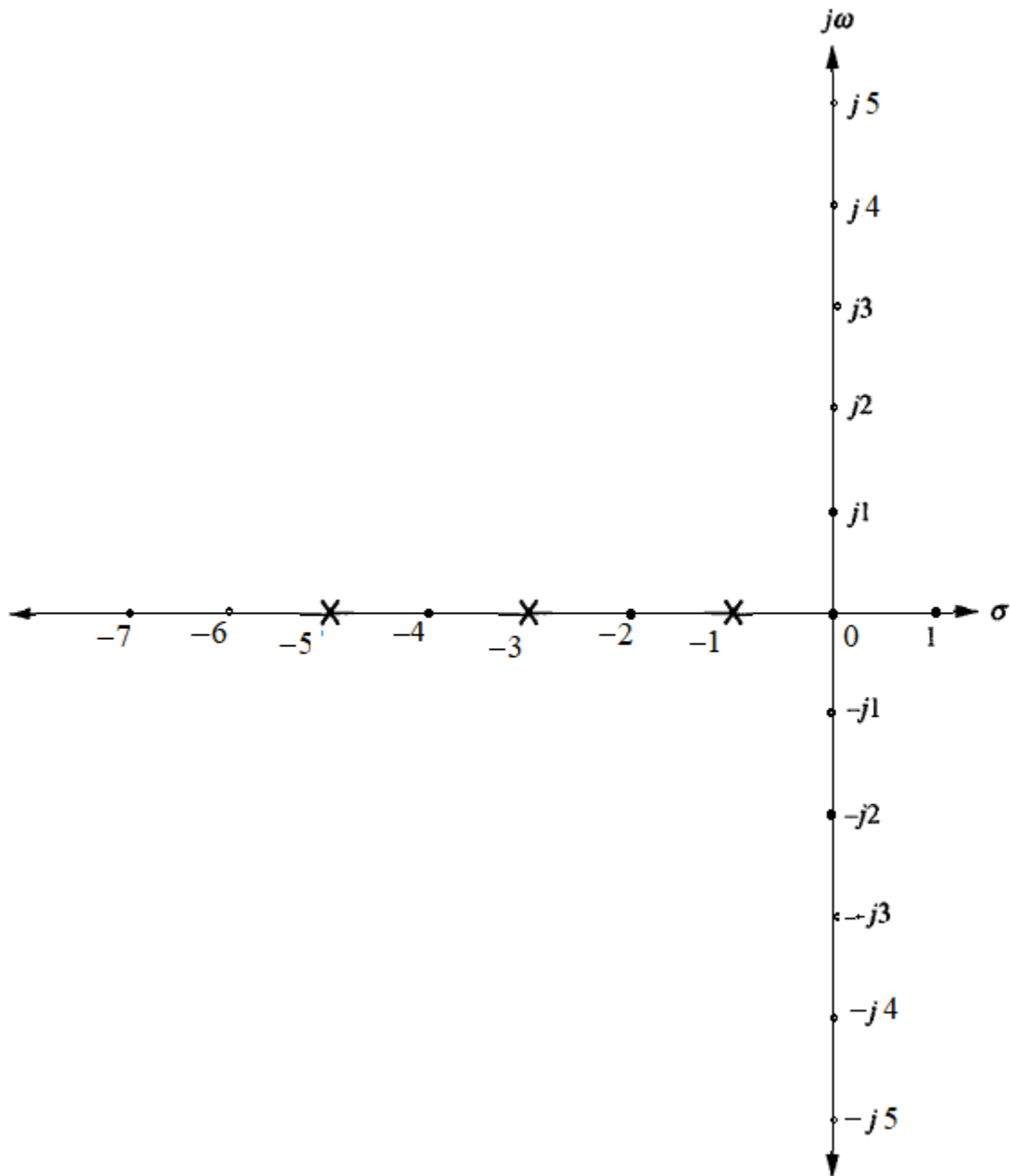

Figure E1

Step 2: Draw the real axis loci (see Figure E2). Real axis loci exist between: an open-loop pole and an open-loop zero, between two open-loop poles, and between two open-loop zeros. In this example, real axis loci will exist between –1 and –3, and between –5 and a zero at infinity. Thus the locus between –5 and infinity is finished as it had started from open-loop pole and has terminated on open-loop zero lying at infinity. On the other hand the loci between –1 and –3 need consideration.
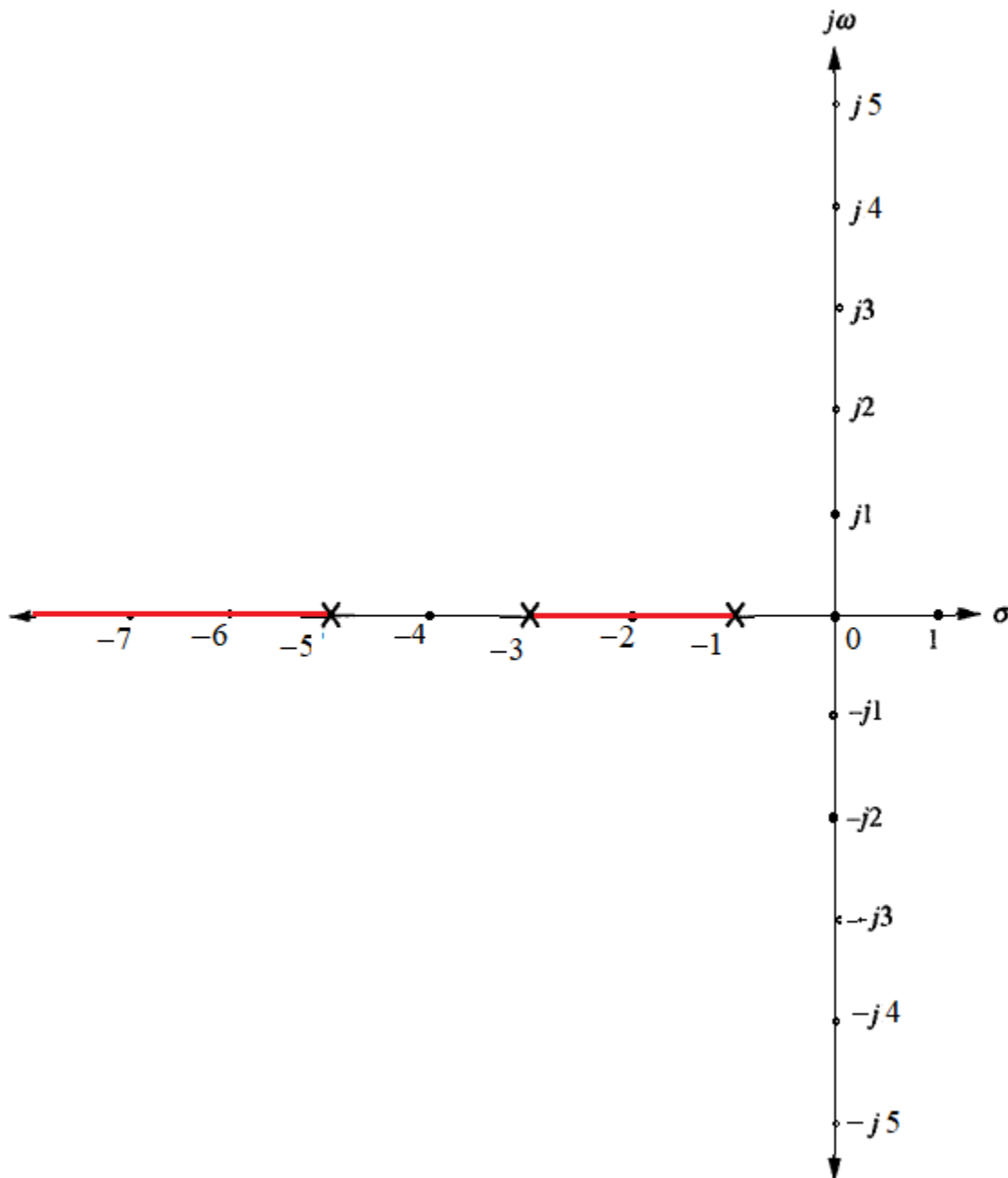

Figure E2

Step 3: There is no break-in point, however, there is one breakaway point (lying on the root locus between two consecutive open-loop poles). In order to determine the breakaway point, we proceed as follows:

Method 1: Using the gain-magnitude criteria: $KG(s)H(s) = -1$ on the open-loop transfer function, we have:

$$\frac{K}{(s+1)(s+3)(s+5)} = -1$$

Or $-K = s^3 + 9s^2 + 23s + 15$

Or $-\dfrac{dK}{ds} = 3s^2 + 18s + 23 = 0$

This yield: $s = -1.84$ and -4.15

The real root: -1.84 is within the real-axis loci between -1 and -3, therefore the real root; $s = -1.84$ is the breakaway point.

Method 2: Arranging the open-loop poles and zeros in partial fraction form, with poles on one side and zeros on the other side, we have:

$$0 = \frac{1}{(s+1)} + \frac{1}{(s+3)} + \frac{1}{(s+5)}$$

From which: $s^2 + 18s^2 + 23s = 0$

This gives: $s = -1.84$ and -4.15 (same as above)

The breakaway point is located on the real axis loci between -1 and -3, at s $= -1.84$ as shown in Figure E3.
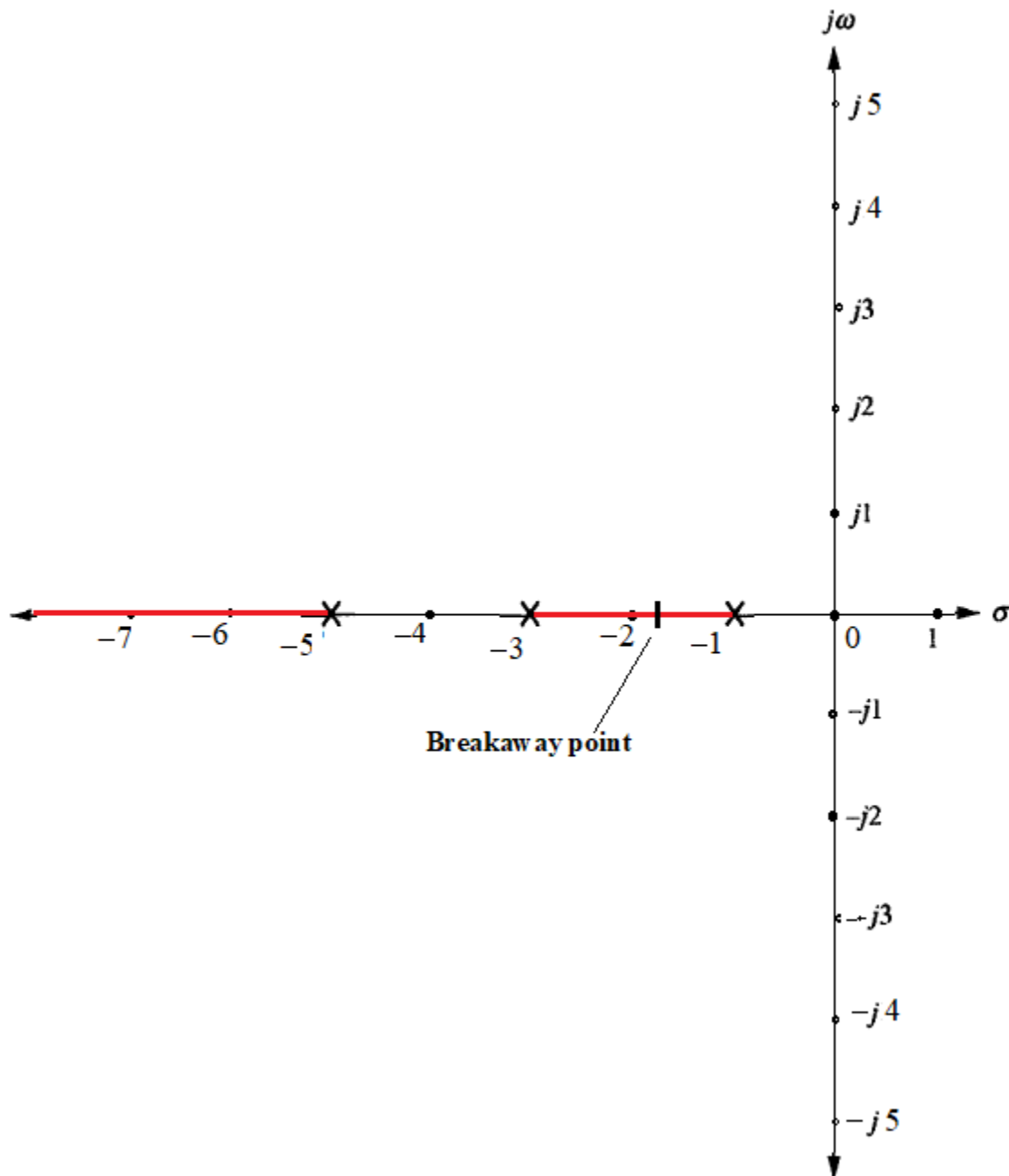
Figure E3

Step 4: In this step we compute the center and angles of asymptotes as follows

Center of asymptotes: $\sigma_C = \dfrac{\sum p_i - \sum z_i}{p - z} = \dfrac{(-1-3-5)-0}{3-0} = -3$

Angles of asymptotes: $\qquad \theta_C = \dfrac{(2k+1)180^0}{p-z} = \dfrac{(2k+1)180^0}{3-0}$

For $k = 0$; $\theta_C = 60^0$, for $k = 1$; $\theta_C = 180^0$ and for $k = 2$; $\theta_C = 300^0$. The angles are repeated in the same sequence for $k = 3, 4, 5, \ldots$ The angles of asymptotes are measured and drawn from the center of asymptotes with respect to the real axis counter-clockwise as illustrated in Figure E4.
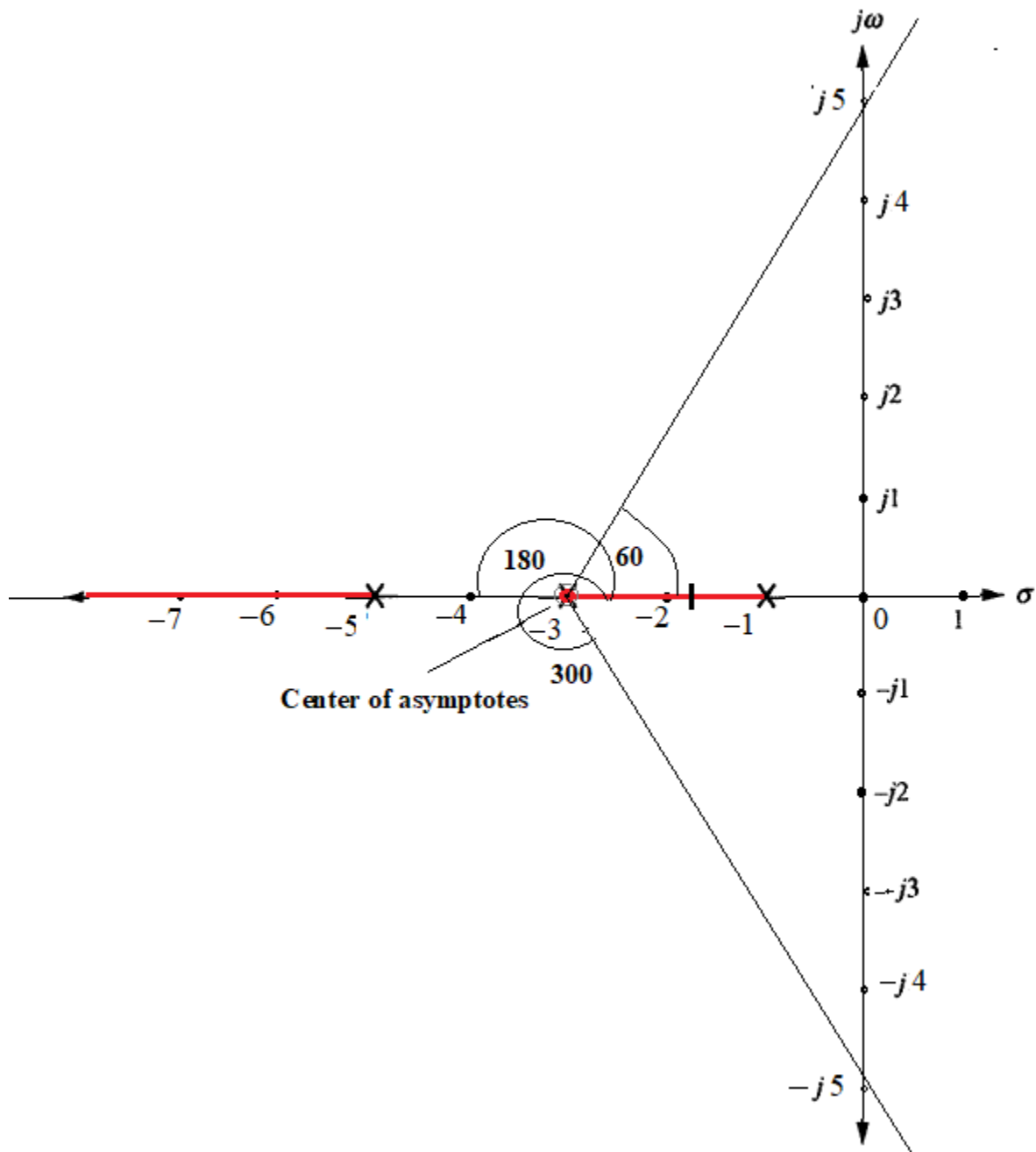

Figure E4

Step 5: The asymptotes shows that the root locus when drawn from the breakaway point will cross the imaginary axis ($j\omega$-axis) to search for the two other zeros lying at infinity. The $j\omega$ crossing point can be determined from the following two methods:

Method 1: The closed-loop transfer function of the given unity feedback system is: $\dfrac{K}{s^3 + 9s^2 + 23s + 15 + K}$. From which the characteristic equation is: $s^3 + 9s^2 + 23s + 15 + K$.
Replacing $s = j\omega$ in characteristic equation, we have:

$$- j\omega^3 - 9\omega^2 + j23\omega + 15 + K$$

Equating the imaginary parts in a group to zero, we have:

$$- j\omega^3 + j23\omega = 0$$

Which gives: $\omega = \pm 4.8$ or $j\omega = \pm j4.8$

Which give two symmetrical points; one above and other below on the imaginary axis at which the root locus will cross in search of the zeros lying at infinity. The root locus will follow the asymptotes, which are supposed to act as guide-paths.

Method 2: The $j\omega$-crossing point can also be obtained by considering the characteristic polynomial of the closed-loop transfer function: $s^3 + 9s^2 + 23s + 15 + K$ from which the Routh table is formed below:

s^3	1	23
s^2	9	$K + 15$
s^1	$\dfrac{192 - K}{9}$	0
s^0	$K + 15$	0

Forcing the term of the first column of s^1 row (odd row) to be zero, we can form an auxiliary equation of the preceding even row. Thus for K = 192, we will have an all-zero row. The auxiliary equation of the preceding row of even power of s is then: $9s^2 + K + 15 = 0$. Substituting $K = 192$ in this equation will result in: $9s^2 + 207 = 0$ which yields:

$$s = \pm j4.8$$

Which are the required points of $j\omega$-crossing. The root locus is drawn in the form of a smooth curve between breakaway point and the $j\omega$ points above and below the real axis. The smooth curve should follow the asymptotes without touching and extended beyond the $j\omega$ points becoming almost in-parallel and closer to the asymptotes. The asymptotes and the root locus finally touch each other at infinity where zero is located for termination. The complete labeled root locus plot of the given system is shown in Figure E5.
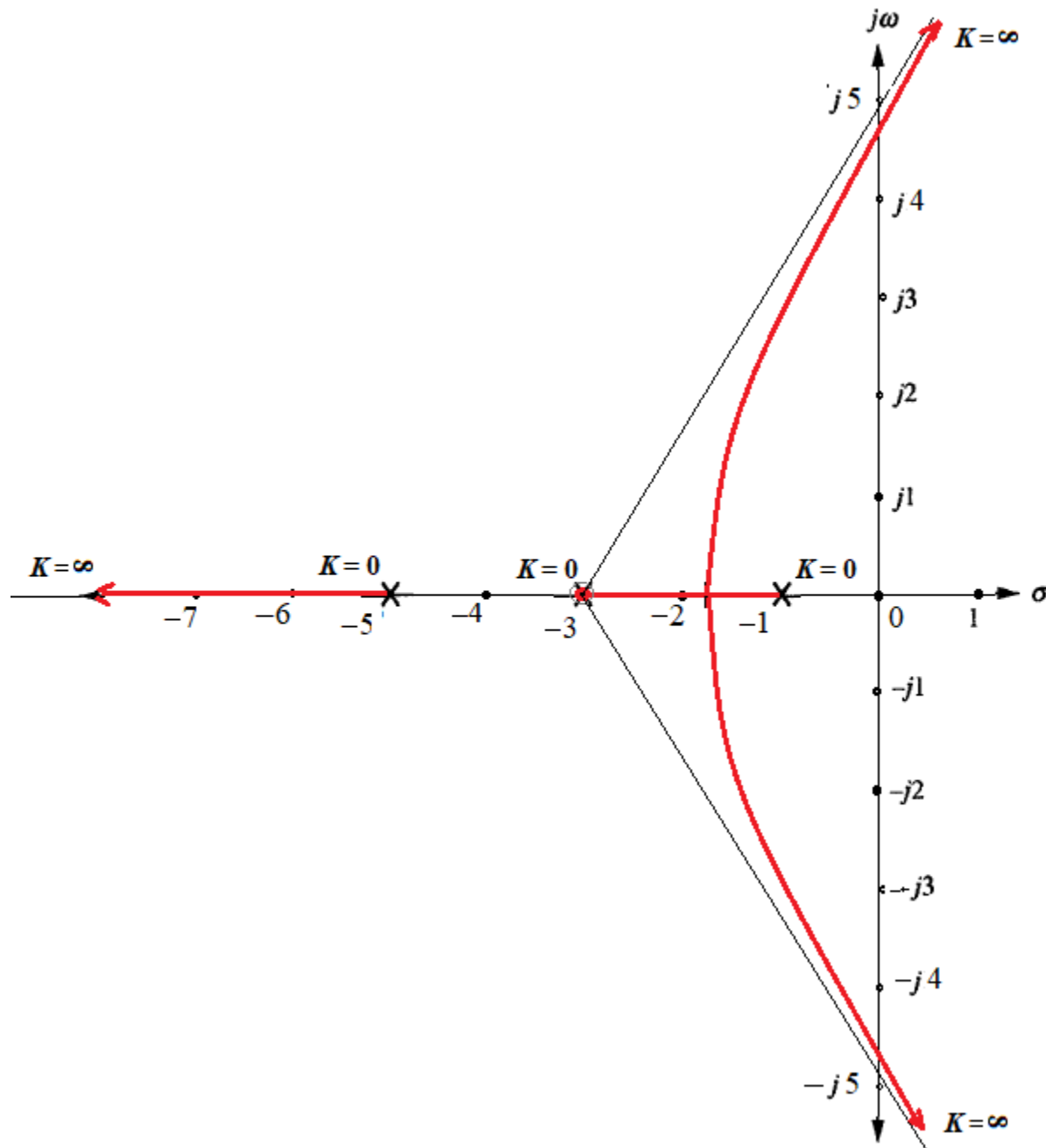

Figure E5