Lecture # 10

Simple Gain Controller

My dear students the previous lecture was based on plotting the root locus, which I mentioned that it is very important in the analysis and design of control systems. In this lecture we will study the use of root locus plot to design a simple gain controller for maintained the percentage overshoot at a fixed value while the settling time is altered in order to make the system (plant) fast acting in rejecting disturbances therefore improving its transient performance. Remember that the settling time is less when the operating point (desired closed-loop pole on the root locus shifts to the left along with the entire root locus. The operating point must lie on the root locus.

In general control systems are represented by a simplified unity feedback configuration with the open-loop transfer function: $KG(s)H(s)$ [$H(s) = 1$ for unity feedback systems]. However, we can separate the K and $G(s)$ and show them as separate blocks as shown in Figure (1). In Figure (1) K represents a simple gain controller in cascade with the plant represented by $G(s)$.

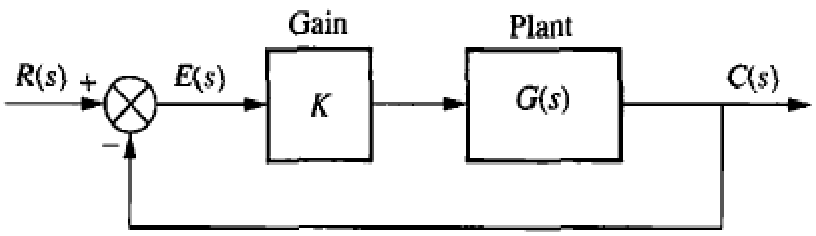

Figure 1: General unity feedback configuration

The controller function is very important as it form an actuating signal for the plant to maintain stability and operate at a desired closed-loop pole. However, in the case of simple gain controller the parameter of interest is percentage overshoot rather than settling time. In many systems it is also desired to limit the overshoot rather than settling time so a simple gain controller will serve the purpose. Please remember that there are two important parameters of a system; natural frequency ω_n and damping coefficient ζ (zeta). The settling time depends on both while the overshoot depends only on ζ. For this purpose the damping line on the root locus plot plays an important role.

Steps in Designing of Simple Gain Controller using Root Locus

Root locus is a very powerful technique, used not only for stability analysis but also widely used for designing various controllers for control systems. The most important is the simple gain controller. The followings are important for designing simple gain controller.

Location of the dominant complex pole:

1. Given the open-loop transfer function, obtain the closed-loop transfer function.
2. Solve the characteristic equation: $1 + KG(s)H(s) = 0$ for a given design value of K.
3. In case the design value of K is not given, a suitable design value of K can be obtained from the range of K values for stability from Routh table. Preferably, that value of K at

which the gain margin should be around 5. The **gain margin** is the ratio of the value of K at $j\omega$ crossing point to the design value of K.

4. The complex root: $-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$ of the characteristic equation in step 2 is then the dominant closed-loop complex pole.

Drawing the damping line:

1. The damping line is the line that extends from the origin that passes through the dominant closed-loop pole.
2. Alternatively if the value of ζ is known, the damping line is drawn at the origin with an angle: $\beta = 180 - \cos^{-1}\zeta$.
3. The dominant closed-pole is then the point of intersection of the damping line on the root locus.

Setting the design value of *K*:

1. Using the magnitude criteria, equate the magnitude of the open-loop transfer function to unity that is: $|KG(s)H(s)| = 1$.
2. Find the design value of K from the magnitude criteria above by substituting the dominant complex pole: $-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$ for the value of s.

The Damping Line

To get started a root locus based on the given open-loop transfer function of a unity feedback system as in Figure (1) is drawn. My dear students by this time you would have developed good skills of plotting the root locus. Once a root locus is drawn, the next step is to draw a damping line by knowing the value of damping coefficient ζ corresponding to a particular overshoot of interest. Let us say that we are not interested for the time being in settling time but the overshoot is of concern. Remember most of the systems are under-damped with ζ lying between 0 and 1. Overshoot is an inverse function of ζ between 0 and 1. Overshoot is maximum with ζ close to 0 and is minimum when ζ approaches 1. The damping line is a straight line drawn at an angle ϕ at the origin with respect to the real axis as illustrated in Figure (2). The angle ϕ for the damping line is determined as:

$$\phi = 180^0 - \cos^{-1}\zeta \qquad 1$$

From Eq (1) it can be seen that when ζ is less (near 0), the cosine term is large and the difference on the RHS of Eq (1) is small therefore the value of ϕ is small and nears 90°. On the other hand with ζ closer to unity, cosine term is smaller and the deference on RHS gets larger with angle ϕ approaching 180°. So we can end up with variety of angles to draw the damping line. However, we have to stick to how much overshoot we are interested in.

$$\%OS = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100 \qquad 2$$

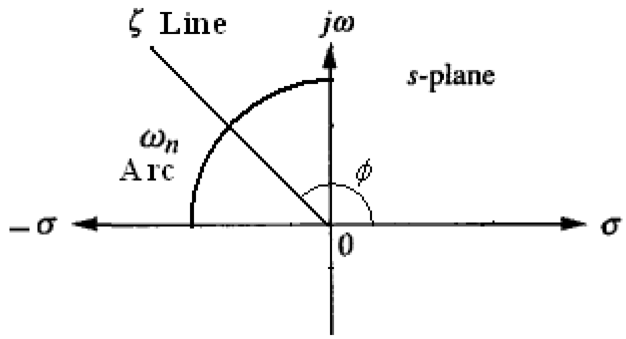

Figure 2: Damping Line

Considering that we are interested in 10% overshoot that corresponds to 0.1. Substituting this value in Eq (2), we have: $0.1 = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$ .Taking natural log of both sides:

$$-2.3 = \frac{-\pi\zeta}{\sqrt{1-\zeta^2}}$$

Squaring both sides of the above expression and simplifying we have:

$$\zeta = 0.6$$

Table (1) gives approximate values of percent overshoot corresponding to particular ζ through the relationship given in Eq (2) in addition to the damping line angle:

Table (1)

Overshoot	5%	10%	15%	20%
Zeta ζ	0.7	0.6	0.52	0.45
$\text{Cos}^{-1}(\zeta)$	45	53	59	63
ϕ	135	127	121	117

It can be seen in Table (1) that as that when overshoot reduces, the angle of damping line increases since the difference: $180^0 - \cos^{-1}\zeta$ increases due to reduction of the angle associated with the inverse cosine term in Eq (2). Figure (3) is an arbitrary root locus drawn with only the upper portion considered. If we draw damping lines with the angles as given in Table (1) and mark the intersection points of damping lines with the root locus as P_1, P_2, P_3 and P_4 corresponding to ζ; 0.45 (OS = 20%), 0.52 (OS = 15%), 0.6 (OS = 10%) and 0.7 (OS = 5%), we can see that the points of shifts relatively to the left as overshoot decreases or zeta increases towards unity. Thus P_4 is more towards left as compared with other points. At this point P_4 both the percentage overshoot and settling time are less as compared to other points, and we would desire that this may become operating point of our system. As the overshoot reduces, the settling time automatically reduces and the system gets rid of disturbances quickly. But remember the root locus plot in Figure (3) is that of a third order system with no zeros. In the case of second order system or a third order system with a zero, this will not be the case as the root locus is a straight vertical line in the complex plane at the breakaway point.

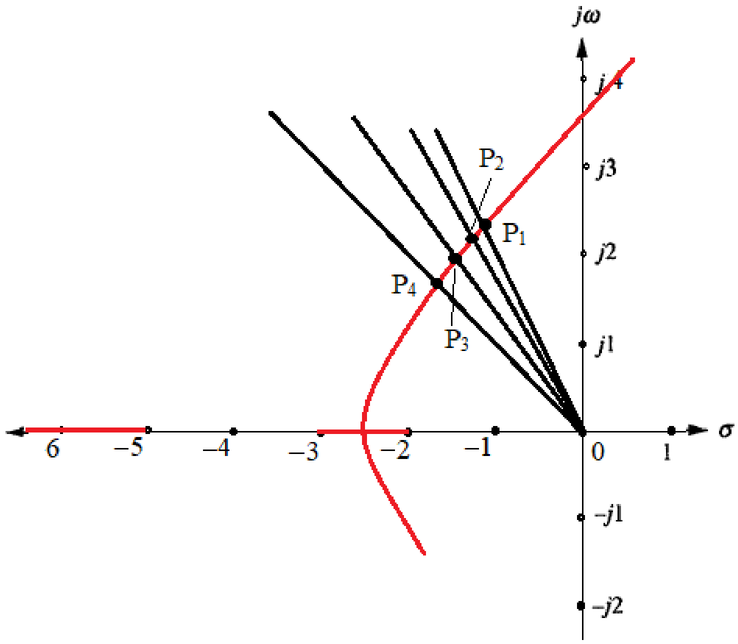

Figure 3: Root locus with damping lines

On the damping line as shown in Figure (2), the value of zeta remains constant all along its length and therefore this line is also called a constant damping line. The length of a point on the line from the origin is the un-damped natural frequency of the system. Therefore the natural frequency will vary at each point, selected on the damping line. If an arc is drawn with origin as center and operating point on the damping line, then the natural frequency will be constant on the this arc, which is also referred to as a constant ω_n arc. On this arc the value of zeta will vary. The damping line will cut the root locus at a complex closed-loop pole at which the controller's optimum gain K can be determined, which will yield the desired overshoot. Let us consider an example to understand setting of controller gain.

Example 1: A unity feedback system is shown in Figure (4). Estimate the value of K for maintaining 10% overshoot. Find also the setting time at this overshoot.

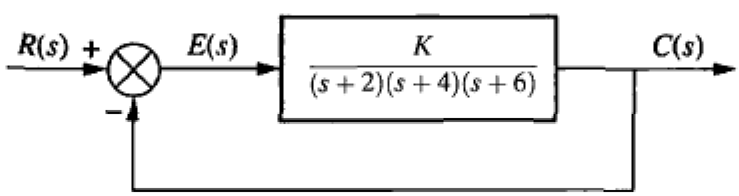

Figure 4

Solution: The system poles are: –2, –4 and –6. There are no zeros. Thus three zeros are lying at infinity. The number of loci are 3, equal to the number of open-loop poles.

Real-axis loci: From the location of open-loop poles in the *s*-plane, the real-axis loci is between –2 and –4 and then extends from –6 onward to –∞.

Breakaway point: $0 = \frac{1}{\sigma_b + 2} + \frac{1}{\sigma_b + 4} + \frac{1}{\sigma_b + 6}$

From which, on simplification:

$$3\sigma_b^{\ 2} + 24\sigma_b + 44 = 0$$

Which gives: $\sigma_b = -2.845$ and -5.154. The breakaway point is –2.845 since it lie on the real-axis loci.

Center of asymptotes: $\sigma_C = \frac{\sum p_i - \sum z_i}{p - z} = \frac{-2-4-6-0}{3-0} = -4$

Angles of asymptotes: $\theta_C = \frac{(2m+1)180^0}{p-z} = \frac{(2m+1)180^0}{3-0}$

For *m* = 0; $\theta_C = 60^0$, for *m* = 1; $\theta_C = 180^0$, for *m* = 2; $\theta_C = 300^0$. The angles are repeated in the same sequence for *m* = 3, 4, 5 …

The *jω*-crossing: The *jω*-crossing points can be obtained from the characteristic polynomial of the closed-loop transfer function. The closed-loop transfer function is:

$$T(s) = \frac{K}{s^3 + 12s^2 + 44s + 48 + K} \qquad 3$$

The characteristic polynomial is: $s^3 + 12s^2 + 44s + (48 + K)$ in which we substitute $s = j\omega$ and then equate it to zero that is:

$$-j\omega^3 - 12\omega^2 + 44j\omega + (48 + K) = 0 \qquad 4$$

Equating the imaginary terms collectively to zero, we have: $-j\omega^3 + 44j\omega = 0$. From which we have: ω = 6.63 rad/sec. Thus: $j\omega = \pm j6.63$. Equating real parts collectively to zero of Eq (4), we have:

$$-12\omega^2 + 48 + K = 0$$

Or $$-12(6.63)^2 + 48 + K = 0$$

Or $$K = 480$$

This is the gain at *jω*-crossing point where the system will be purely oscillatory and will be on the verge of moving into RH-plane (unstable region). The root locus is shown plotted in Figure (5). The damping line can be drawn at the origin with an angle of: $\phi = 180^0 - \cos^{-1}(0.6) = 127^0$.

This line is shown on the root locus of Figure (5), which intersect the root locus at point *P* allowing us to locate the dominant complex closed-loop pole of the form: $s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$. At point *P* the dominant complex pole from Figure (5) is: $s = -2 + j2.8$. Thus comparing with the general expression of complex closed-loop pole, we have: $\zeta\omega_n = 2 \Rightarrow \omega_n = \dfrac{2}{0.6} = 3.33$ rad/sec. The peak time for the first overshoot is:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{3.33 \times \sqrt{1-0.6^2}} = 1.17 \text{ sec.}$$

The settling time of transients is: $T_S = \dfrac{4}{\zeta\omega_n} = \dfrac{4}{0.6 \times 3.33} = 2$ sec.

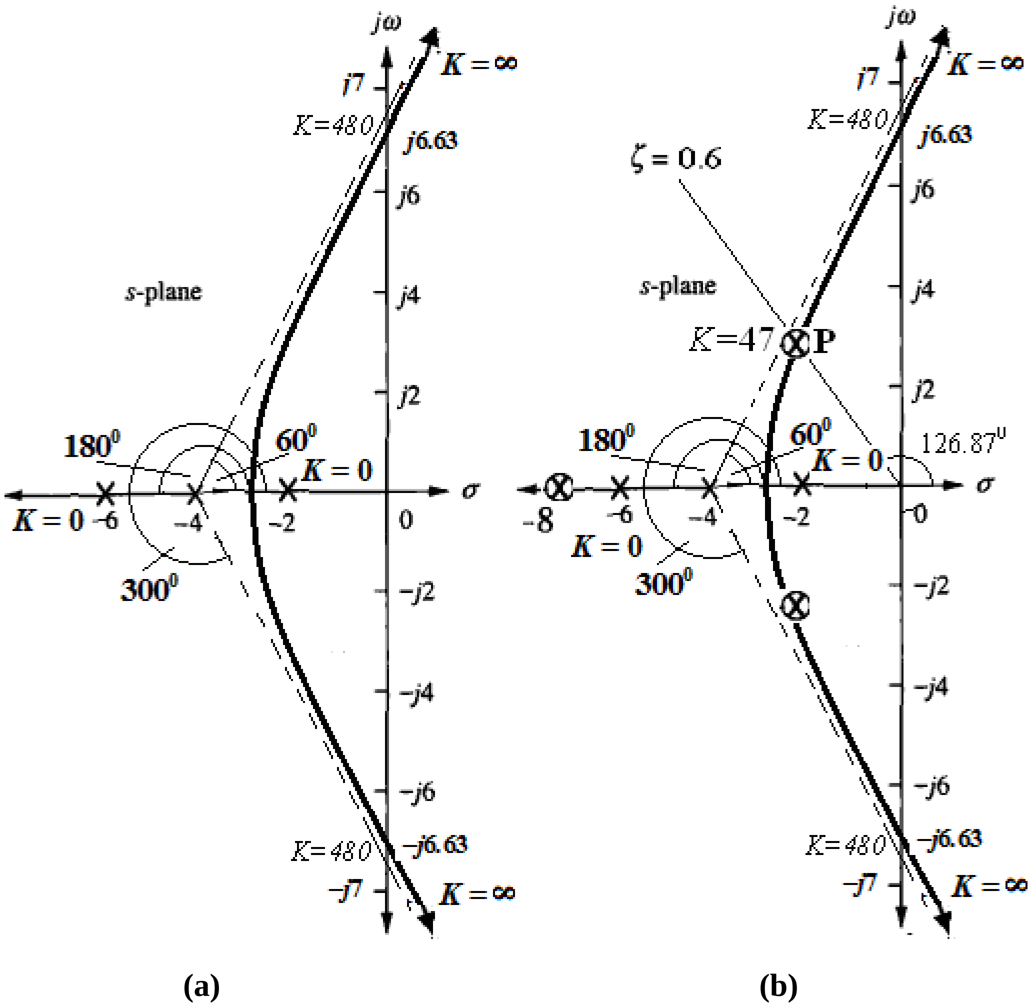

Figure 5: (a) Root Locus Plot (b) Location of the Dominant Closed-loop Pole

The system gain at the dominant pole is determined by using the magnitude criteria that is: $KG(s)H(s) = 1$. Or:

$$K = \frac{1}{|G(s)H(s)|} \text{ at } s = -2 + j2.8$$

Therefore: $K = |(s+2)(s+4)(s+6)|_{s=-2+j2.8}$

Thus: $K = 47$

We can look into the problem in terms of setting the gain of the controller at: $K = 47$, the desired transient performance can be achieved. The situation can be viewed as connecting the controller K in cascade with the plant, whose transfer function is: $G(s) = \frac{1}{(s+2)(s+4)(s+6)}$. This is shown in Figure (6).

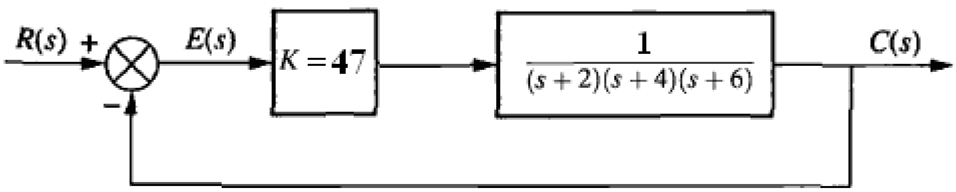

Figure 6: Plant with simple gain controller

Thus to obtain the desired transient performance in terms of limiting the overshoot to 10%, the controller gain should be adjusted to 47. This will result in the peak time of 1.17 sec and settling time of 2 sec. By altering the value of K from the controller will change the location of system dominant pole and the value of damping ratio and natural frequency will thus change. This will vary the system transient performance specifications such as; percent overshoot, time to peak, rise time and settling time. But remember that any operating point must lie on the root locus. Thus by adjusting the gain of controller within which the stability of the system is maintained an optimum desired transient response can be achieved. The gain margin GM (ratio of gain at $j\omega$-crossing to the gain calculated at the desired operating point) can be determined by using:

$$GM = \frac{K(j\omega)}{K} = \frac{480}{47} = 10.21$$

Thus the system has sufficient gain margin, so the desired overshoot of 10% is quite reasonable. It is advisable to allow a gain margin of at-least 5 with many systems.

Example 2: Let us now consider the unity feedback system of the type shown in Figure (4), but in this case the plant transfer function is: $G(s) = \frac{(s+2)}{s(s+4)(s+6)}$ so that the open-loop transfer function of the system will be: $KG(s)H(s) = \frac{K(s+2)}{s(s+4)(s+6)}$. This system is still a third order but differs from the system in example 1 in terms of presence of a zero and that the system is Type-1. However, the system in example 1 is third order with no zero and is Type-0.

Let us draw the root locus for which the requirements are as follows:

Poles and Zeros: The system poles are: 0, –4 and –6, and a zero at –2. There are two zeros are lying at infinity.

Real-axis loci: From the location of open-loop poles in the *s*-plane, the real-axis loci is between 0 and –2 and between –4 and –6.

Breakaway point: $\frac{1}{\sigma_b+2}=\frac{1}{\sigma_b}+\frac{1}{\sigma_b+4}+\frac{1}{\sigma_b+6}$

From which, on simplification:

$$\sigma_b^{\,3}+8\sigma_b^{\,2}+20\sigma_b+24=0$$

Which gives: $\sigma_b=-4.93$ and two other roots are complex and cannot be regarded as breakaway points. Thus the breakaway point is –4.93 since it lie on the real-axis loci.

Center of asymptotes: $\sigma_C=\frac{\sum p_i-\sum z_i}{p-z}=\frac{0-4-6-(-2)}{3-1}=-4$

Angles of asymptotes: $\theta_C=\frac{(2m+1)180^0}{p-z}=\frac{(2m+1)180^0}{3-1}$

For m = 0; $\theta_C=90^0$, for m = 1; $\theta_C=270^0$. The angles are repeated in the same sequence for m = 2, 3, 4, …

There is no $j\omega$-crossing as the angles of asymptotes that guides the root locus are 90 and 270, so that the asymptotes, and hence the root locus will be a straight vertical line in the *s*-plane as shown in Figure (7). Let us set two operating points on the root locus such that one gives 10% overshoot corresponding to ζ = 0.6 and the other giving 5% overshoot corresponding to ζ = 0.45. In order to set the points on the root locus, damping lines have to be drawn; one at angle 127° corresponding to ζ = 0.6 and the other at 135° corresponding to ζ = 0.45 as shown in Figure (7). These lines can be seen to intersect the root locus at P_1 and P_2 respectively.

Consider point P_1, where the closed-loop pole is: $s=-4.93+j6.35$, which when substituted for the value of *s* in the open-loop transfer function of the system and using the magnitude criteria will give us the gain *K* of the controller that is:

$$KG(s)H(s)=\frac{K(s+2)}{s(s+4)(s+6)}=1$$

Or $$\frac{K\,|(-4.93+j6.35+2)|}{|(-4.93+j6.35)(-4.93+j6.35+4)(-4.93+j6.35+6)|}=1$$

Or K = 47.55

Comparing the closed-loop pole on the root locus at P_1 with the general form of the closed-loop pole: $s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$, the value of $\zeta = 0.61$. This gives the percent overshoot amounting to 8.9%. Let us also compute the steady-state error at P_1 for the system which is Type-1. For this purpose, the static error constant K_v (due to constant velocity) is to be evaluated using:

$$K_v = \lim_{s\to 0}[sG(s)] = \lim_{S\to 0}\left[s\left(\frac{47.55(s+2)}{s(s+4)(s+6)}\right)\right] = 3.96$$

The steady-state error is:

$$e(\infty) = \frac{1}{K_V} = \frac{1}{3.96} = 0.2523 \text{or } 25.23\%$$

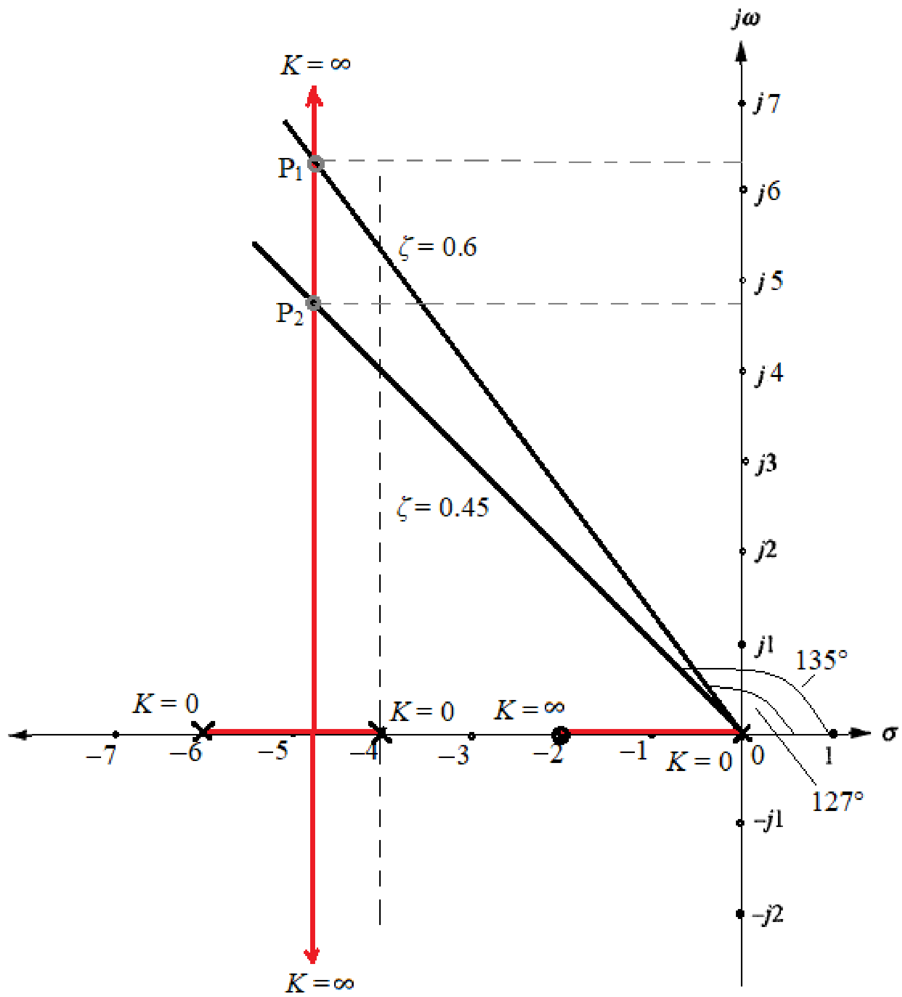

Figure 7: Root locus of Example 2

Consider point P_2, where the closed-loop pole is: $s = -4.93 + j4.78$, which when substituted for the value of s in the open-loop transfer function of the system and using the magnitude criteria will give us the gain K of the controller that is:

$$KG(s)H(s) = \frac{K(s+2)}{s(s+4)(s+6)} = 1$$

Or $$\frac{K\,|(-4.93+j4.78+2)|}{|(-4.93+j4.78)(-4.93+j4.78+4)(-4.93+j4.78+6)|} = 1$$

Or $K = 29.16$

Likewise comparing the closed-loop pole on the root locus at P_2 with the general form of the closed-loop pole: $s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$, the value of $\zeta = 0.71$. This gives the percent overshoot amounting to 4.2%. Likewise let us also compute the steady-state error at P_2 for the system which is Type-1. For this purpose, the static error constant K_v is:

$$K_v = \lim_{s\to 0}[sG(s)] = \lim_{S\to 0}\left[s\left(\frac{29.16(s+2)}{s(s+4)(s+6)}\right)\right] = 2.43$$

The corresponding steady-state error is:

$$e(\infty) = \frac{1}{K_V} = \frac{1}{2.43} = 0.4115 \text{ or } 41.15\%$$

We note that as the operating point move up on the root locus, the percentage overshoot will increase and the steady-state error will reduce. All we need to do is to make a compromise between overshoot and steady-state error to fix the operating point on the root locus. In this particular case we need not worry about gain margin. Since the root locus does not cross the $j\omega$-axis, the system has very large gain margin for any point all along the root locus and will always be stable.

We can conclude that simple gain controller will improve the transient performance by reducing the percentage overshoot but at the expense of steady-state performance. Thus as we increase the controller gain, the percentage overshoot will increase but the steady-state error will reduce. On the other hand decreasing the controller gain will reduce the percentage overshoot but will increase the steady-state error. Thus a suitable gain from the controller can make a compromise between overshoot and steady-state error for optimization.