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ALTERNATING-CURRENT CIRCUITS

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PREFACE TO SECOND EDITION

In the revision, many changes and rearrangements have been made to improve the presentation of electric-circuit theory. Wherever it was necessary and desirable, materials and examples have been added to assist the student in obtaining a better understanding of the basic principles of circuit analysis. Since there has been a tendency to use the concept of equivalent circuits not only in communication and electronic work but also in the study of power transformers and machinery, more emphasis has been placed on what is meant by the equivalence of two networks. Also, the operation of many electric networks has been treated to apply to both power and communication circuits.

The rationalized meter-kilogram-second (MKS) system of units has been used throughout the text. However, for practice in conversion from one system of units to another, the data of some problems and examples have been given in other units, such as the English units.

It was deemed advisable to omit the chapter on "Voltage and Current Loci," which appeared in the First Edition, and to include a substantial part of the material in Chapters 5 and 7. The consideration of current loci for *RL* and *RC* circuits has been included at the proper places in the study of series circuits. The treatment of current loci for two branches in parallel has been presented along with the associated topics of parallel circuits. The author and other staff members are convinced that the new arrangement is more satisfactory than grouping the materials on voltage and current loci into one chapter.

The Norton, Reciprocity, and Compensation Theorems have been added to Chapter 10, "Network Theorems." The concept of a Norton generator or a constant-current generator has been very useful in the analysis of communication and electronic problems.

The material in the old Chapter 12, "Mutual Induction and Coupled Circuits," has been rewritten and rearranged into Chapters 11 and 12. Great improvement has been made in the treatment of coupled circuits on the basis of the so-called classical coupled circuit theory by adding explanatory material and numerical examples. The material on equivalent coupled circuits referring to the primary side or to the secondary side has been reorganized and included in the new Chapter 12, "T-Sections of Modified or Equivalent Coupled Circuits." In this new chapter, an attempt was made to correct the wrong impression of some students that the transformer theory presented in many books on alternating-current machinery is entirely different from the coupled circuit theory in books on communication circuits.

A new chapter, "Impedance Transformation," has been included because the problem of impedance matching for maximum power transfer plays an important role in communication and electronic circuits.

An effort has been made to improve the chapters on "Differential Equations," "Direct-Current Transients," and "Alternating-Current Transients" by rewriting parts of the materials on these three associated topics.

The chapter on "Transmission-Line Calculations" has been rewritten and expanded to give a better physical interpretation of the transmission-line formulas.

There has been some indication that it is desirable to introduce the *symbolic method* of solving alternating-current problems earlier than the arrangement given in the book. As has been suggested, this may be done by considering Complex Quantities before the article on Resonance in Series Circuits in Chapter 5.

Again, the author is indebted both to the faculty members and to students of the Department of Electrical Engineering, Ohio State University, for their many valuable criticisms and suggested improvements in the revision of this book. Acknowledgment and thanks are due to staff members of other institutions for sending to me, either directly or indirectly through the representatives of the International Textbook Company, their constructive criticisms of the arrangement and presentation of the text material.

The author is especially grateful to Professor Frederic P. Fischer of the Department of Electrical Engineering at the University of Buffalo and Dr. Laurel J. Lewis of the Department of Electrical Engineering at the University of Washington for their excellent reviews of the revised manuscript.

K. Y. TANG

Columbus, Ohio
May, 1951

PREFACE TO FIRST EDITION

The material presented in this book has been developed from notes on alternating-current circuits used with juniors in Electrical Engineering at The Ohio State University for the past five years. It is intended as an introductory course in the study of circuit analysis. This course should precede those in alternating-current machinery, power, and communication networks.

Progressive work in both the field of power and the field of communication demands a more thorough grasp of the physical nature of circuit elements and a fuller understanding of the principles and laws of electric circuits. For this reason, much time is devoted to the study of the effects of the circuit elements on the flow of current in an electric circuit. Also, the relations of instantaneous values are emphasized throughout the text.

The book is designed to be useful to both power and communication engineers. The chapter on Network Theorems includes the Superposition Theorem, Thévenin's Theorem, the Maximum Power Transfer Theorem and the Y-Delta Transformation. These are used quite extensively, especially by communication engineers. The material presented in Chapter 10, "Voltage and Current Loci," is a good introduction to circle diagram work in alternating-current machinery. The approach in the study of Symmetrical Components is a sympathetic one. Various balanced systems are first added together to give balanced and unbalanced systems of voltages and currents. Later the process is reversed and any unbalanced system is shown to be resolvable into the usual balanced positive, negative, and zero sequence systems. It has been felt by the author that in present-day teaching of transients the student has been given the solution of the differential equation without knowing where the solution came from. Certain inquiring types of students and instructors will desire to use the chapter entitled "Differential Equations" which is a partial fulfilment of this need.

It is the thought of the author that the material should be so presented that chapters can be selected at will in building up a course to suit the particular needs of any individual school or curriculum. With this intent in mind, the various chapters are made as independent of one another as possible. It is felt, however, that all curricula will demand the first eight chapters, and after that different instructors will select different chapters according to their desires and purposes. When the topic of transient phenomena is taken up in class, it is suggested that Chapters 18, 19, and 20 be considered together.

It is a common practice for electrical engineers to use the single-subscript notation for simple circuits, and the double-subscript notation for polyphase circuits, and the single-subscript with arrows in meshes denoting positive sense in communication networks. This practice is followed throughout the various chapters.

The author is indebted both to the faculty members and to the students of the Department of Electrical Engineering, Ohio State University, for their helpful suggestions. Acknowledgment and thanks are also due to Dr. Hugh H. Skilling of the Department of Electrical Engineering at Stanford University for his constructive criticisms of the text material.

K. Y. TANG

Columbus, Ohio
January, 1940

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CHAPTER 1

ALTERNATING CURRENT AND VOLTAGE

1–1. The Use of Alternating Currents.—The use of electricity as a source of power is almost universal. Up to the present time, electric service has grown faster in America than any other place in the world. For commercial purposes, electric power is derived chiefly either from coal (steam plants) or from water power (hydroelectric plants). Of the total kilowatt* capacity available, hydro accounts for approximately 28 per cent, internal combustion for about 2 per cent, and steam for nearly 70 per cent.

In the early days, plants were all direct current. But direct current had the unfortunate limitations of distance and of efficiency in generation and distribution. Then came alternating current, which by now has almost completely superseded direct current for public power supply. Of course, there are many instances where direct current is absolutely necessary for industrial purposes such as electrolytic processes. Some of the reasons for generating electrical energy in the alternating current form are:

When it is desired to transmit power over a considerable distance, practically the only way in which it can be done effectively is by means of electricity. The efficiency of electrical power transmission is higher for systems with higher transmission voltages. For the same kva† transmitted, the higher the voltage the lower the current. A lower current means a smaller I^2R or power loss. Alternating currents can be generated at higher voltages than direct currents. With the transformer, a comparatively cheap and reliable stationary piece of apparatus, alternating voltages can be raised or lowered. Thus, electric power can be generated at voltages as high as 16,500 volts; current can be transmitted at higher voltages such as 220,000 volts by means of step-up transformers; and, by the use of step-down transformers, current can be delivered at the customers' premises at 110, 220, or 440 volts as desired.

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* For definition, see pages 185 and 186.

† For definition, see page 185.

The high transmission efficiencies obtainable with alternating current make it economical to generate electrical energy in large quantities in a single station and distribute it over a large territory. Moreover, the larger the generating unit, the more efficient it is and the lower are its cost and its weight per kilowatt of capacity. Alternating-current generators can be built with capacities of 200,000 kva at 16,500 volts, while direct-current generators may have ratings of 5000 kw at 500 volts.

For most purposes, alternating-current motors and other alternating-current devices are quite satisfactory. In applications where flexibility of speed control is not necessary, alternating-current motors are usually preferable to direct-current motors. This preference is due to the fact that they are cheaper in first cost and require less care and maintenance than direct-current motors. The absence of the commutator and brushes is an advantage for the induction motor in installations where sparking must be avoided.

It may thus be said that alternating current owes its importance partly to the following characteristics:

(*a*) It can be generated economically with large units.

(*b*) Its voltage can be readily raised and lowered by transformers so that energy can be transmitted economically over great distances.

(*c*) Alternating-current motors for constant-speed work are usually preferable to direct-current motors.

1–2. The Electric Field.—In 1820, Oersted showed that a conductor carrying an electric current is surrounded by a magnetic field. Any variation of the current in the conductor produces a corresponding change in its magnetic field. While the current is increasing, energy is stored in the magnetic field; and, while the current decreases, the magnetically stored energy is returned to the electric circuit.

Similarly, energy is stored dielectrically (electrostatically) in the space between conductors which differ in electric potential. With increasing difference of potential, energy is stored in the dielectric; and, while the potential decreases, the dielectrically stored energy is returned to the electric circuit. These effects of electric current and voltage on the surrounding space are termed magnetic induction and electrostatic induction. The laws of the energy changes causing these magnetic and dielectric phenomena are of fundamental importance in the study of alternating currents.

1-3. Lines of Force.—In the description of his experiments, Faraday developed the idea of "lines of magnetic force." Just as it is convenient for us to map out the manner in which water flows through an orifice by drawing lines to indicate the direction which the water takes at every point, so it was convenient for Faraday to map, by means of a system of lines of force, the properties of the magnetic and dielectric fields. This method has proved extremely useful for both practical and theoretical purposes.

It has been shown by experiment that changing a magnetic field with respect to a conductor will generate a voltage in the conductor. The student, after having a course or courses in electrical physics, is familiar with this concept. In practical applications, the magnitude of a magnetic field may be varied with respect to time (as in the case of a transformer) or the position of the field may be varied with respect to the conductor as time progresses (as in the case of an alternator).

1-4. Faraday's Law.—We have learned from experiments that whenever the flux linkages with a coil are changing, an electromotive force (abbreviated emf) is induced in the coil. This is true whether there is a current in the coil or not. The value of the induced emf is dependent on: (1) the number of turns in the coil and (2) the rate at which the flux linking the coil is changed. The fundamental law that governs quantitatively the relationship between the induced emf and the magnetic flux linkage is known as *Faraday's Law*. This law states that *in any coil of conductors where the flux linking the coil is changing, there is induced in the coil an emf which is proportional to the rate of change of the flux linkages.* Mathematically, it is

$$e_i = -N\frac{d\Phi}{dt} \text{ volts} \qquad (1\text{-}1)$$

or

$$e_i = -\frac{d}{dt}(N\Phi) \text{ volts} \qquad (1\text{-}2)$$

where e_i = induced emf, in volts;
N = number of turns in coil;
Φ = flux linking coil, in webers;
$N\Phi$ = flux linkages, in weber-turns;
t = time, in seconds.

In the rationalized MKS system, 1 weber = 10^8 lines or maxwells. The quantity $\frac{d\Phi}{dt}$ is the time rate of change of the flux linking the turns of the coil. It may vary from instant to instant. If the number of turns of a coil remains constant, the instantaneous value of the induced emf in the coil is directly proportional to the instantaneous rate of change of the flux. Thus, *if the flux threading a coil of one turn is changing at the rate of 1 weber or 10^8 lines per second, the emf induced in that one turn is 1 volt.*

The negative sign* is used in equations (1–1) and (1–2) to indicate that the induced emf will tend to produce a current in such a direction as to prevent any change of the original field. If the circuit is closed, this *induced* current will itself have a magneto-

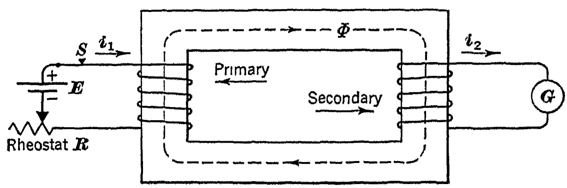


FIG. 1–1

motive force (mmf), which will tend to be equal and opposite to the change of magnetomotive force that induces the current. Thus, the induced emf may be considered as *reactive* in character.

Suppose, for example, that a galvanometer is connected to the secondary coil or winding of a transformer,† as shown in Fig. 1·1. When the switch S in the primary circuit is closed, it will be observed that the needle of the instrument is deflected while the primary current is changing, this deflection indicating that an emf is induced in the coil because of the change in flux which threads the coil. If the primary current i_1 is flowing in the direction shown by the arrow, it will produce a flux in the iron core in the direction shown by the dotted arrow. By adjusting the rheostat R the current i_1 may be increased. Saturation effect

* Some authors prefer to consider only magnitudes in equations (1–1) and (1–2); the minus sign is then omitted.

† A simple transformer is an electrical apparatus consisting of two separate coils wound on a common iron core. The coil or winding receiving energy is called the primary, and the coil or winding delivering energy is called the secondary.

being neglected, the flux will be proportional to the current. As the flux Φ is increasing, the direction of the emf produced in the secondary coil will be such as to cause a current i_2 to flow in that coil in the direction indicated. This current i_2 will set up an mmf in the coil opposing the increase in flux. Conversely, if the flux Φ is decreasing, the emf induced in the secondary will be in the reverse direction, causing a secondary current to flow in the direction opposite to that shown in Fig. 1–1. This current flowing in the reverse direction will set up an mmf tending to oppose a decrease in flux.

Example 1–1.—If the flux Φ linking a coil is varied according to the curves shown in Fig. 1–2 for the three intervals of time, draw on the same graph the curves representing the induced emf.

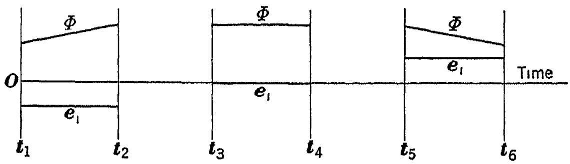


FIG. 1–2

Solution—During the interval between t_1 and t_2, the flux linking the coil is increased at a constant rate. The slope $\frac{d\Phi}{dt}$ of the Φ-time curve is constant and positive. According to Faraday's Law, or equation (1–1), the induced emf in the coil is constant and considered negative. From t_3 to t_4 the flux linking the coil is constant; and the slope $\frac{d\Phi}{dt}$ is constant and equal to zero. Thus, the induced emf is zero.

When the flux linking the coil is decreased at a constant rate, as during the interval between t_5 and t_6, the slope $\frac{d\Phi}{dt}$ is constant and negative. During this interval the induced emf is constant and positive, since the emf induced during the interval from t_1 to t_2 was considered negative.

This example illustrates the principle that the induced emf in a coil is directly proportional to the time rate of change of flux linking the coil, or directly proportional to the slope of the Φ-time curve.

1–5. Variations of Electromotive Force and Current.—The more common modes of variation of emfs and currents are:

(*a*) *Uni-Directional Current or Voltage:* Any emf which produces an electron movement in one direction only is called uni-directional. All values of a uni-directional voltage or current have the same sign; that is, all are positive or all are negative.

(*b*) *Direct Current or Voltage:* A direct current or voltage is unidirectional. In the solution of ordinary engineering problems, it is considered steady and non-pulsating in magnitude, as shown in Fig. 1–3. Sometimes direct current is called continuous current.

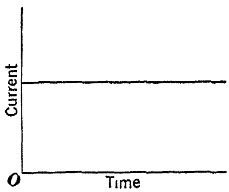


FIG. 1–3

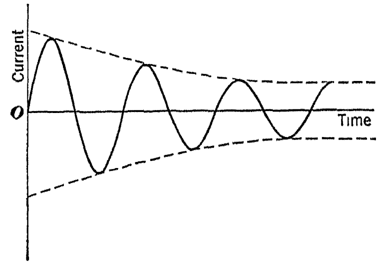


FIG. 1–4

(*c*) *Oscillating Current or Voltage:* An oscillating current or voltage is one which alternately increases and decreases in value, always remaining within finite limits. An example is shown in Fig. 1–4.

(*d*) *Periodic Current or Voltage:* A periodic current or voltage is one which oscillates and the values of which recur for equal increments of time.

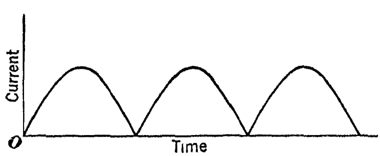


FIG. 1–5

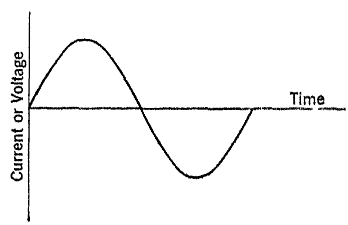


FIG. 1–6

(*e*) *Pulsating Current or Voltage:* A pulsating current or voltage is a periodic quantity of which the values are always positive (or always negative). An example with values varying between well-defined maxima and minima is given in Fig. 1–5.

(*f*) *Alternating Current or Voltage:* An alternating current or voltage is a periodic quantity which has alternately positive and negative values.

(*g*) *Sinusoidal Current or Voltage:* A sinusoidal current or voltage varies harmonically. That is, its values vary along sine functions of time as shown in Fig. 1–6.

(*h*) *Non-Sinusoidal Current or Voltage:* Any periodic current or voltage which does not vary harmonically is said to be non-sinusoidal. Such a wave form is represented in Fig. 1–7.

1–6. Reason for Using Sine Waves for A-C Computation.—In alternating-current problems it will be found necessary to deal with time derivatives of current and voltage, as $\frac{di}{dt}$; with integral expressions involving current or voltage, as $\frac{1}{C}\int i\,dt$; and with

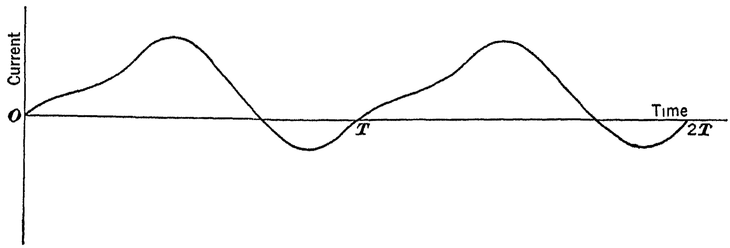


FIG. 1–7

addition of currents or voltages. When adopting a fundamental wave form, it is very convenient to choose one which has the following properties:

(*a*) The addition of two or more of such waves will result in a wave of the same form.

(*b*) The subtraction of one wave from another will result in a wave of the same form.

(*c*) The integration of such a wave will result in a wave of the same form (the constant of integration being neglected).

(*d*) The differentiation of such a wave will give a wave of the same form.

It has been shown that all these properties occur only in the case of a sine wave. For this reason, the sine wave has been adopted universally as a fundamental wave for all alternating-current problems.

1-7. Generation of a Sine Wave of Electromotive Force. Assume that a single turn of wire with its ends connected to slip rings, as turn AA' in Fig. 1-8(a), is rotated by some outside source of power at a uniform angular velocity in a uniform magnetic field. When the coil is rotated in the counter-clockwise direction, as in Fig. 1-8(b), the flux threading the coil is changed; hence, by Faraday's Law, an emf is induced in it. The *wave shape* or *wave form* of this emf is the shape of the curve obtained when the values of the emf at different instants of time (these are called *instantaneous values* and generally represented by lower-case letters) are plotted against time in rectangular coordinates. The coil in the position shown in Fig. 1-8(a) has a maximum magnetic flux passing through it.

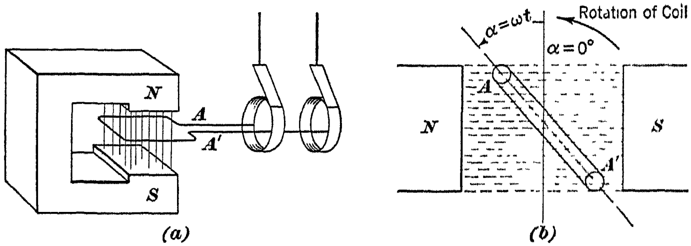


Fig. 1-8

If the flux distribution is *uniform*, as shown in Fig. 1-8, the flux enclosed in the coil AA' as it rotates at *a uniform angular velocity* follows the law*

$$\Phi = \Phi_m \cos \alpha = \Phi_m \cos \omega t \text{ webers} \qquad (1\text{-}3)$$

where Φ_m = maximum flux enclosed by coil, in webers;
$\alpha = \omega t$ = angle between plane of coil and vertical reference line, in radians (which may be changed to degrees);
ω = angular velocity, in radians per second;
t = time, in seconds.

The variation of the flux enclosed in the coil is shown by the full-line curve in Fig. 1-9. Since the emf induced in the coil AA', Fig. 1-8, at any instant is proportional to the slope of the flux

* To check this statement it is only necessary to draw a diagram similar to Fig. 1-8(b) with 50 lines per inch and plot a curve with the lines enclosed in the coil against the angle α between the plane of the coil and a vertical reference line.

curve at the same instant, the variation of the induced emf is shown by the broken-line curve in Fig. 1–9. When $\omega t = 0$, the slope of the flux curve, or the rate of change of flux linking the coil, is zero; hence, the induced emf at that instant is zero. As ωt increases from 0, the slope of the flux curve is negative and increasing. It reaches a maximum at the instant when $\omega t = \frac{\pi}{2}$. The induced emf is a maximum when the slope is a maximum. From $\omega t = \frac{\pi}{2}$ to $\omega t = \pi$, the slope is still negative; and it becomes zero when $\omega t = \pi$. A similar method of analysis may be applied from $\omega t = \pi$ to $\omega t = 2\pi$.

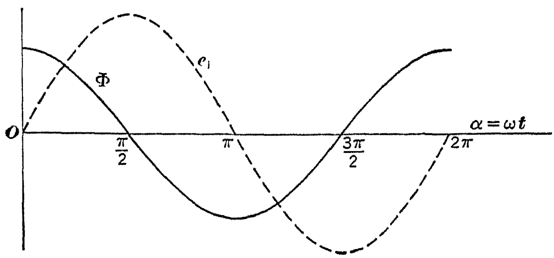


FIG. 1–9

From equation (1–1) the emf induced or generated in the coil at the end of t seconds, or after the coil has moved through an angle $\alpha = \omega t$ radians, is

$$e = -N \frac{d\Phi}{dt} \text{ volts}$$

Thus,

$$e = -N \frac{d}{dt} (\Phi_m \cos \omega t) \text{ volts} \tag{1–4}$$

or

$$e = \omega N \Phi_m \sin \omega t \text{ volts} \tag{1–5}$$

When $\alpha = \omega t = \frac{\pi}{2}$, coil side A, Fig. 1–8, is under the middle of the pole N and the emf wave has its maximum value or amplitude, which is

$$E_m = \omega N \Phi_m \text{ volts} \tag{1–6}$$

By combining equations (1–5) and (1–6), we find the equation of the generated voltage to be

$$e = E_m \sin \omega t = E_m \sin \alpha \tag{1–7}$$

If the potential circuit of an oscillograph* were connected to the slip rings in Fig. 1–8(a), it would record a sine wave as shown in Fig. 1–10. The value of the ordinate at any time may be computed by multiplying E_m by the sine of the corresponding angle $\alpha = \omega t$. For example,

$$\left.\begin{array}{l} \text{at } t = t_1 \; . \; . \; . \; . \; . \; e_1 = E_m \sin \alpha_1 = E_m \sin \omega t_1 \\ \text{at } t = t_2 \; . \; . \; . \; . \; . \; e_2 = E_m \sin \alpha_2 = E_m \sin \omega t_2 \end{array}\right\} \qquad (1\text{–}8)$$

However, a simple method of constructing a sine wave is given in Chapter 2.

Many dynamos in actual use are built to generate emfs that are very nearly of sinusoidal form. Because of its mathematical simplicity and practical advantages the sine wave is

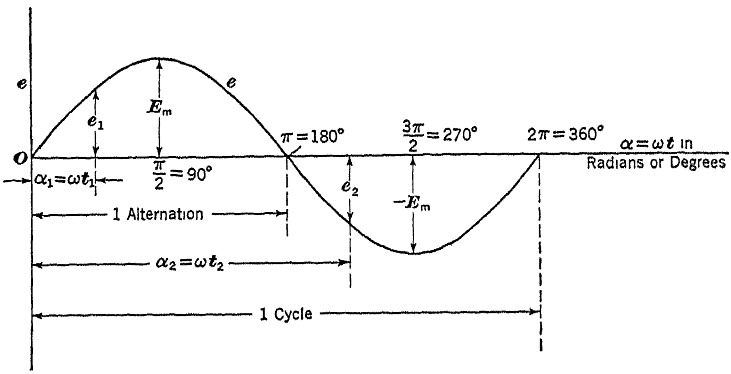


FIG. 1–10

assumed in most problems on alternating-current theory. It should not be forgotten, however, that the sine-wave theory is in many practical cases only an approximation because the emf is not rigorously of sine-wave form.

Although Fig. 1–8(a) may be considered as a sketch of a very simple dynamo, such machines are never actually built. The long path of magnetic flux through air is impractical. The moving coil is actually placed in slots in a cylinder of iron that rotates between the north and south poles, and the air-gaps between this cylindrical armature and the pole faces are made quite short. This design has a tendency to alter the voltage from true sinusoidal form, but the principle of operation is as just described.

* See Appendix A.

1-8. Cycle, Frequency, and Period.—When one side of the coil AA', Fig. 1-8, has passed one pair of poles (a north pole and a south pole), it has traversed 360 electrical degrees* and the voltage wave has gone through one complete cycle of values, as indicated in Fig. 1-10. After this the wave is ready to repeat itself. A cycle of an alternating emf is the complete series of positive and negative values of the emf. In a two-pole machine, one revolution gives one cycle and an electrical degree equals a mechanical space degree. In a four-pole machine, one cycle is obtained in one-half of a revolution and 2 electrical degrees equal 1 mechanical space degree. In a six-pole machine, one cycle is completed in one-third of a revolution and 3 electrical degrees equal 1 mechanical space degree. In general,

$$f = \frac{P}{2} \times \frac{\text{rpm}}{60} \tag{1-9}$$

where f = frequency, in cycles per second;
P = number of poles;
rpm = speed of rotation, in revolutions per minute.

The number of cycles through which the emf passes in one second is called its frequency; that is, a 60-cycle source would have 60 complete cycles in each second. If one alternation is a half-cycle, then a 60-cycle source would have 120 alternations per second. The angular velocity ω of the armature of an alternator is measured in electrical radians per second. One complete cycle is 360 electrical degrees or 2π radians, and the frequency f may be expressed as

$$f = \frac{\omega}{2\pi} \text{ cycles per second}$$

or

$$\omega = 2\pi f \text{ electrical radians per second} \tag{1-10}$$

The frequency of an alternating emf may be determined by a frequency meter.†

Substituting the value of ω from equation (1-10) in equation (1-5) gives

$$e = 2\pi f N \Phi_m \sin 2\pi f t \text{ volts} \tag{1-11}$$

or

$$e = E_m \sin 2\pi f t \text{ volts} \tag{1-12}$$

* The A.I.E.E. Standards give an electrical degree as the 360th part of the angle subtended, at the axis of a machine, by two consecutive field poles of like polarity. Thus, 1 mechanical degree is equal to as many electrical degrees as there are pairs of poles in the machine.

† See Appendix A.

The abscissas in Fig. 1–10 may be given as a function of time, as shown in Fig. 1–11.

The period T of a sine wave is the time required for the wave to pass through one complete cycle of values, or

$$T = \frac{1}{f} \text{ second} \tag{1-13}$$

Thus, for a 60-cycle source, the period is $T = \frac{1}{60} = 0.01667$ second, and the time required for one alternation or a half-cycle is $\frac{T}{2} = \frac{1}{120} = 0.00833$ second.

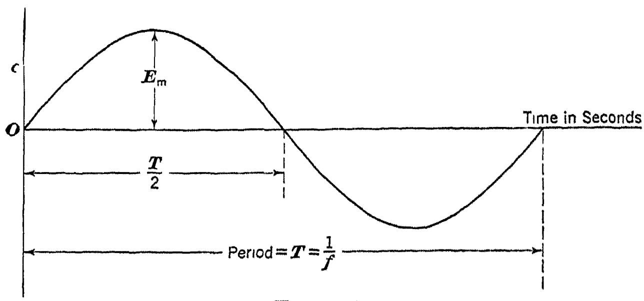


FIG. 1–11

Example 1–2.—The armature of a six-pole machine is revolving at 1200 rpm. What is the frequency of the generated emf?

Solution.—By equation (1–9),

$$f = \frac{P}{2} \times \frac{\text{rpm}}{60}$$

$$= \frac{6}{2} \times \frac{1200}{60} = 60 \text{ cycles per second}$$

Example 1–3.—If the armature of the machine in Example 1–2 has 50 turns in series and is generating a maximum emf of 300 volts, what is the value of the flux per pole? Assume that the flux linking the turns varies sinusoidally.

Solution.—From equation (1–6),

$$\Phi_m = \frac{E_m}{2\pi f N}$$

$$= \frac{300}{2\pi \times 60 \times 50} = 0.0159 \text{ weber} = 1.59 \times 10^6 \text{ maxwells}$$

1–9. Commercial Frequencies.—In power generation and distribution a frequency of 60 cps (cycles per second) is generally employed, although frequencies of 50 and 25 cps are also used.

The principal advantage of higher frequencies is that transformers require less iron and copper and therefore are lighter in construction and may be manufactured more cheaply. On the other hand, the voltage drop in transmission lines and alternating-current apparatus is almost directly proportional to the frequency, and better voltage regulation throughout the system is obtained with lower frequencies. A frequency of 25 cps was formerly used because power apparatus, such as synchronous converters and alternating-current commutator motors, operates better at this frequency than at higher frequencies. However, at 25 cps the flicker of lamps is perceptible and objectionable. Hence, with a better understanding of the design of power apparatus and long distance power transmission, a power and lighting company would ordinarily operate at 60 cps. Outside the United States a frequency of 50 cps is common and frequencies such as $16\frac{2}{3}$, 15, and even 12.5 cps are used. In engineering practice, it is customary to say, "a 60-cycle source," since the frequency is always understood to be cycles per second.

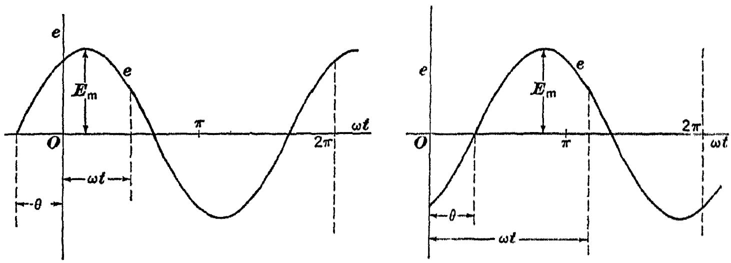
In telephone communication the frequency band extends from approximately 20 cps to about 4,000,000 cps. Radio communication occupies a very wide frequency band, extending from about 15,000 cps to 30,000 or more megacycles per second (1 megacycle = 10^6 cycles). In railroad operation clear, caution, and stop signals are transmitted through the rails to the cab of the locomotive by the use of code frequencies, such as 180 and 75 or 80 cycles per minute.

1–10. Phase and Phase Difference.—It is not necessary to start counting time from the instant at which an emf wave passes through zero. In many cases, time is not so reckoned, as shown by the simple harmonic waves in Fig. 1–12. The general equation for the waves shown is

$$e = E_m \sin(\omega t + \theta) \tag{1–14}$$

The quantity $(\omega t + \theta)$ is called the phase of the wave. The phase of a periodic quantity is the fractional part of a period which has elapsed since the periodic function passed through zero. The angle θ is called the initial phase. It indicates the number of radians between the point where the wave passes through zero while increasing in the positive direction and the point from which time is reckoned.

The initial phase gives the ordinate of the wave at $t=0$; that is, at $t=0$, $e=E_m \sin \theta$. Because of the way in which equation (1-14) is written (with the sign + before θ), the angle θ in Fig. 1-12(a) is a *positive* angle, as $+\frac{\pi}{3}$, while the angle θ in Fig. 1-12(b) is a *negative* angle, as $-\frac{\pi}{3}$. If equation (1-14) were written in the form $e=E_m \sin(\omega t-\theta)$, then θ would be negative in Fig. 1-12(a) and positive in Fig. 1-12(b).*



(a) θ = Positive Angle (b) θ = Negative Angle

FIG. 1-12

To be mathematically correct, θ in equation (1-14) should be given in radians; but it is quite common to find the expression $e=E_m \sin\left(\omega t+\frac{\pi}{3}\right)$ written as $e=E_m \sin(\omega t+60°)$. It must be remembered, however, that both ωt and θ should be reduced to the same units, in radians or degrees, before they can be added.

If two sinusoidal quantities have the same frequency, as $e_1=E_{m1} \sin(\omega t+\theta_1)$ and $e_2=E_{m2} \sin(\omega t+\theta_2)$, the difference between their phases at any instant is called their phase difference. That is, the phase difference between e_1 and e_2 is

$$(\omega t+\theta_1)-(\omega t+\theta_2)=\theta_1-\theta_2$$

Whether e_1 is *leading* e_2 or *lagging* behind e_2 can be determined from the phase difference. The terms leading and lagging will be explained in the following article.

* This consideration of θ as a positive angle or a negative angle is quite arbitrary. Some prefer to consider θ simply as an angle and to write $e=E_m \sin(\omega t+\theta)$ and $e=E_m \sin(\omega t-\theta)$ for the waves in Fig. 1-12(a) and Fig. 1-12(b), respectively.

Example 1–4.—Two sinusoidal emfs are shown in Fig. 1–13. What are the expressions for the emfs? What are their ordinates at $t=0$? What is their phase difference?

Solution.—The expressions for the emfs are:

$$e_1 = 100 \sin (\omega t + 30°)$$
$$e_2 = 75 \sin (\omega t - 30°)$$

or

$$e_1 = 100 \sin \left(\omega t + \frac{\pi}{6}\right)$$
$$e_2 = 75 \sin \left(\omega t - \frac{\pi}{6}\right)$$

The ordinates at $t=0$ are:

$$e_1(\text{at } t=0) = 100 \sin 30° = 50 \text{ volts}$$
$$e_2(\text{at } t=0) = 75 \sin(-30°) = -37.5 \text{ volts}$$

The phase difference of the emfs is

$$\theta_1 - \theta_2 = 30° - (-30°) = 60°$$

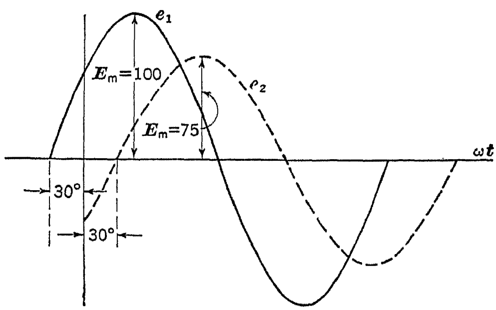


Fig. 1–13

1–11. Alternating Currents.—When an alternating emf is applied to a circuit, an alternating current will flow in the circuit. Let the form of the emf be

$$e = E_m \sin \omega t$$

Then the form of the current will be

$$i = I_m \sin (\omega t + \theta) \tag{1–15}$$

The current changes its direction of flow periodically, assuming alternately positive and negative values. The sequence of values assumed during a negative half of the cycle are exactly similar to the sequence of values for the positive half. Whatever has been said about the frequency, phase, and other characteristics of an alternating emf may be applied to an alternating current.

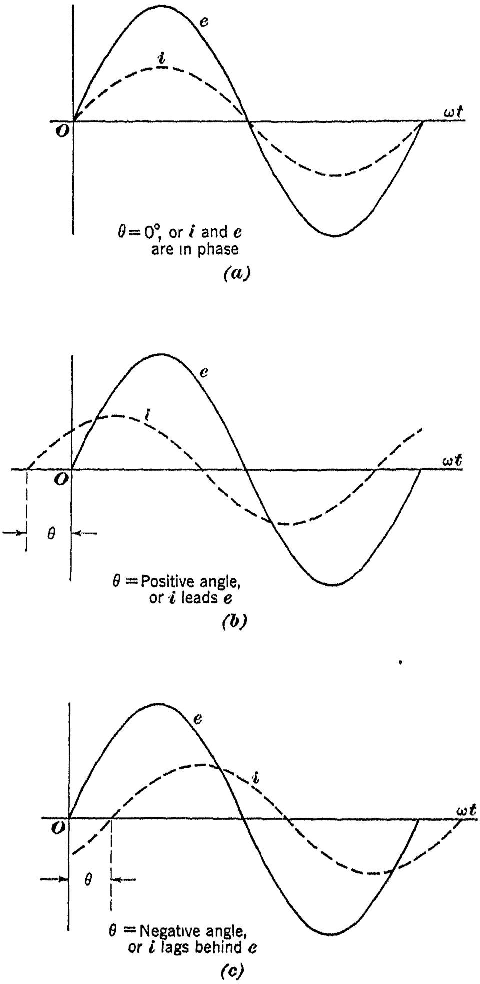


Fig. 1-14

As shown in Fig. 1–14, the current and the emf are of the same frequency because the distance, along the ωt-axis, for one complete cycle of the current wave is the same as the distance for one complete cycle of the emf wave; in other words, the period of the emf wave is equal to the period of the current wave.

If the current i reaches its maximum positive value at the same instant at which the emf e reaches its maximum positive value, as in Fig. 1–14(*a*), the emf and the current are said to be *in phase.* However, there may be a phase difference between the emf and the current because of the presence of inductance or capacitance in the circuit, as will be explained in Chapters 4 and 5. If the current reaches its maximum positive value some time before the emf wave reaches its maximum positive value, as shown in Fig. 1–14(*b*), the current is said to *lead* the emf. If the maximum positive value of the current wave is reached some time after the emf has reached its maximum positive value, as indicated in Fig. 1–14(*c*), the current is said to *lag* behind the emf. In this consideration, θ is equal to or less than 90°. It can be seen from the figure that the current leads the emf when θ in equation (1–15) is positive; and the current lags behind the emf when θ is negative.*

1–12. Actual Amount of Motion of the Electrons.—It must not be thought that the electrons in a wire carrying an alternating current move back and forth from one end of the circuit to the other; such is far from being the case. The electron drift, which constitutes the current, is very slow. The electrons on the average move along the conductor only a very small fraction of a centimeter in a second. In fact, even if a wire is carrying so much current that the resultant I^2R loss causes the wire to become red hot, the electrons are drifting along the conductor at an average velocity of less than 1 cm per second.

It appears that, with a 60-cycle current reversing 120 times per second and with consequent to-and-fro drift of the electrons this number of times per second, the average to-and-fro motion of an electron in a copper wire is perhaps 0.0001 cm. The motion would be less than 0.0001 cm for small current densities, and possibly 100 times as much for a wire carrying sufficient current to bring it to the melting point. Although these distances are extremely small, there are plenty of opportunities for collisions of electrons and molecules and consequent production of heat; because, even when an electron moves only 0.0001 cm it passes by 10,000 molecules of the copper.

* Some authorities prefer not to give the angle θ any sign but simply to consider it as a phase angle between the voltage and the current. If the form of the applied emf is $e = E_m \sin \omega t$, leading current is written as $i = I_m \sin (\omega t + \theta)$ and lagging current is written as $i = I_m \sin (\omega t - \theta)$.

1–13. Effective or rms Value of a Sine Wave.—Since the value of the ordinate to a sine wave varies from instant to instant, it is apparent that we must establish some method of determining the effective numerical value of such a wave. The heating effect of an alternating current, being independent of the direction of flow, forms a convenient basis of comparison of alternating and continuous currents. Hence, *an alternating current is said to have an effective value of 1 ampere when it produces heat in a certain number of ohms of resistance at the same average rate as heat is produced in the same number of ohms of resistance by 1 ampere of continuous current.*

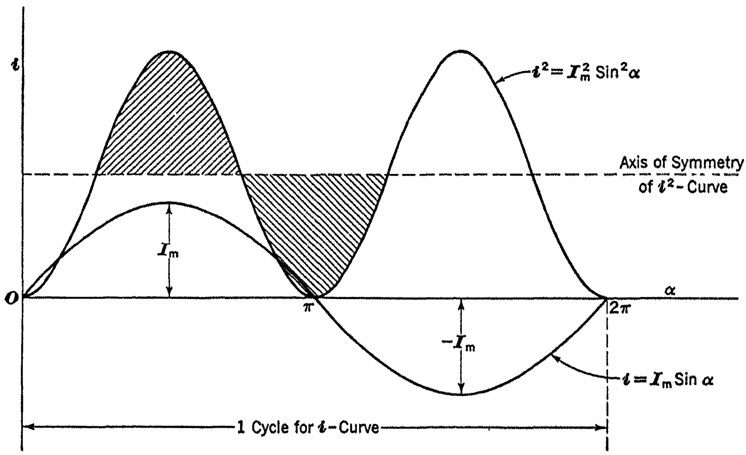


FIG. 1–15

Since the heating effect of a continuous current i is equal to i^2R, the heating effect of an alternating current at any instant is proportional to the square of its value at that instant. The effective value of the current wave $i = I_m \sin \alpha$ in Fig. 1–15, therefore, must be based upon the values of i^2. The i^2-curve in Fig. 1–15 is obtained by squaring each ordinate of the i-wave. The squared wave lies entirely above the zero axis, since the square of a negative value is positive.

It can be seen that the i^2-wave has a frequency twice that of the original i-wave. Its horizontal axis of symmetry is above the zero axis, and the displacement of that axis is equal to the average of the values of i^2. This is so because the total area above the dotted line representing the axis of symmetry is equal to the area below the dotted line. During a cycle, the average heating of an alternating current will depend on the average of the squares of all instantaneous values of current during the cycle. The squared current represented by the straight dotted line is therefore equivalent to the square of a direct current which would produce the same heating effect as the alternating current.

The procedure in obtaining the effective value of any wave (not necessarily a sine wave) is:

1. Plot a wave whose ordinates are equal to the squares of the ordinates of the original wave.
2. Find the average ordinate (equal to the area under the squared wave divided by the base).
3. Determine the effective value by extracting the square root of the average value found in step 2.

The name "root-mean-square" value, or "rms" value, derived from the method by which it is found, is often used in place of "effective" value. The effective value of an alternating current is usually considered to be the number of amperes which will be indicated by an ammeter* capable of measuring both continuous and alternating currents, such as a meter of the movable-iron, thermocouple, or hot-wire type.

It is now desirable to find a mathematical expression for the effective value of a sine wave. This procedure is followed for non-sinusoidal waves also. Let an alternating current flowing through a resistance R be

$$i = I_m \sin \alpha$$

Then energy will be transformed into heat at the instantaneous rate of Ri^2 watts. The average rate of transformation of energy is

$$P = \frac{1}{2\pi}\int_0^{2\pi} Ri^2\, d\alpha = \frac{1}{2\pi}\int_0^{2\pi} RI_m^2 \sin^2 \alpha\, d\alpha \qquad \textbf{(1–16)}$$

Since it is obvious from Fig. 1–15 that all the ordinates of the i^2 curve are positive and the lobes or half-cycles are identical, the limits of integration may be changed to 0 and π. Thus,

* See Appendix A.

$$P = \frac{1}{\pi}\int_0^{\pi} RI_m^2 \sin^2 \alpha \, d\alpha \tag{1-17}$$

or

$$P = \frac{RI_m^2}{2\pi}\int_0^{\pi} (1 - \cos 2\alpha)\, d\alpha = \frac{RI_m^2}{2} = RI^2 \tag{1-18}$$

where I is the effective or rms value of the current and is written without a subscript.

The effective current is

$$I = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} i^2\, d\alpha} = \sqrt{\frac{1}{\pi}\int_0^{\pi} I_m^2 \sin^2 \alpha \, d\alpha} \tag{1-19}$$

This equation states that the effective value of a periodic wave may be obtained by squaring the ordinates of the wave, as indicated by $I_m^2 \sin^2 \alpha$, finding the average value of the square curve, and finally taking the square root of this average value.

It should be noted from equation (1–18) that

$$I = \frac{I_m}{\sqrt{2}} = 0.707 I_m \tag{1-20}$$

and

$$I_m = \sqrt{2} I = 1.414 I \tag{1-21}$$

Similarly, if a voltage wave is given as $e = E_m \sin \alpha$, its effective value—or the value read by a voltmeter*—is

$$E = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} e^2\, d\alpha} = \sqrt{\frac{1}{\pi}\int_0^{\pi} E_m^2 \sin^2 \alpha \, d\alpha} \tag{1-22}$$

Also,

$$E = \frac{E_m}{\sqrt{2}} = 0.707 E_m \tag{1-23}$$

and

$$E_m = \sqrt{2} E = 1.414 E \tag{1-24}$$

Example 1–5.—If an alternating current is represented by $i = 14.14 \sin \omega t$, what is its effective or meter value?

Solution.—Substituting values in equation (1–20) gives

$$I = \frac{I_m}{\sqrt{2}} = \frac{14.14}{\sqrt{2}} = 10 \text{ amp}$$

Example 1–6.—See Appendix I.

* See Appendix A.

1–14. Average Value of a Sine Wave.—In general, the average value of any function $f(\alpha)$ between two limits γ and β is found by averaging the ordinates between these limits. A mathematical expedient is to obtain the area under the curve by integration and divide it by the base. Thus,

$$\text{Average ordinate} = \frac{1}{\beta - \gamma} \int_{\gamma}^{\beta} f(\alpha)\, d\alpha \qquad \textbf{(1–25)}$$

For a sine wave of the form $E_m \sin(\alpha + \theta)$,

$$\text{Average ordinate} = \frac{1}{\beta - \gamma} \int_{\gamma}^{\beta} E_m \sin(\alpha + \theta)\, d\alpha$$

or

$$\text{Average ordinate} = \frac{E_m}{\beta - \gamma} [\cos(\gamma + \theta) - \cos(\beta + \theta)] \qquad \textbf{(1–26)}$$

This equation indicates that the average value of a sine wave varies, the result depending on the limits γ and β.

It can be seen from Fig. 1–15 that the average ordinates of the wave $i = I_m \sin \alpha$ between the limits 0 and 2π would be zero because the two lobes are identical. The ordinates of the lower lobe, being opposite in sign, will cancel the corresponding ordinates of the upper lobe. Thus, the average value (or, more accurately, the half-period average value) of a sine wave $i = I_m \sin \alpha$ should be taken as the algebraic average of the instantaneous values of the current taken throughout a half-period beginning with a zero value of the current. If $t = 0$ when $i = 0$, then

$$I_{\text{avg}} = \frac{1}{\pi} \int_0^{\pi} I_m \sin \alpha\, d\alpha = \frac{2}{\pi} I_m = 0.637 I_m \qquad \textbf{(1–27)}$$

Similarly, for an emf wave of the form $e = E_m \sin \alpha$, the average value is

$$E_{\text{avg}} = \frac{1}{\pi} \int_0^{\pi} E_m \sin \alpha\, d\alpha = \frac{2}{\pi} E_m = 0.637 E_m \qquad \textbf{(1–28)}$$

1–15. Relation Between Effective and Average Values of a Sine Wave; Form Factor.—The form factor of any wave is defined as follows:

$$\text{Form factor} = \frac{\text{Effective value}}{\text{Average value}} \qquad \textbf{(1–29)}$$

For a sine wave, the form factor is obtained in the following way. From equation (1–28),

$$E_m = 1.5708 E_{avg} \tag{1–30}$$

Equating the values of E_m from equations (1–24) and (1–30), we obtain the relation

$$1.414E = 1.5708 E_{avg} \tag{1–31}$$

from which

$$\text{Form factor} = \frac{E}{E_{avg}} = 1.11 \tag{1–32}$$

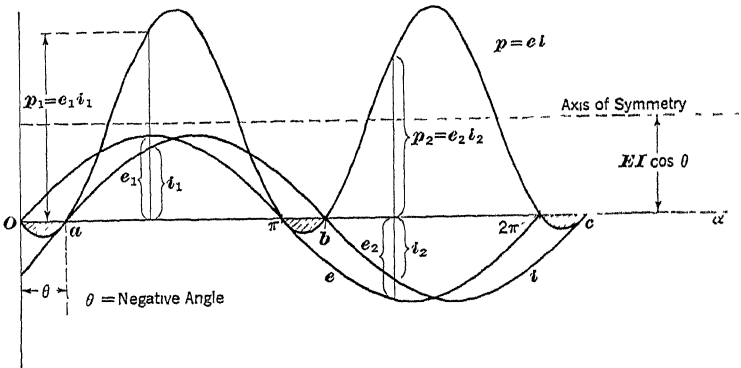


FIG. 1 16

1–16. Instantaneous Power.—In a direct-current circuit the power delivered by the source is equal to the product of the voltage across the source and the current supplied by the source. This rule applies to alternating-current circuits, provided that the instantaneous values of the generator voltage and current are considered. In other words, the power delivered by an a-c generator at a given instant is equal to the product of the voltage at that instant and the current flowing at the same instant, or

$$p = ei \text{ watts} \tag{1–33}$$

where e and i are instantaneous values in volts and amperes, respectively. Thus, if the ordinates of a curve are obtained by multiplying together the ordinates of the emf wave and the corresponding ordinates of the current wave, the result will be a power curve as shown in Fig. 1–16, where the ordinate of the p-curve gives the power in the circuit at any instant.

The expression for instantaneous power in a circuit may be obtained in the following manner. Let the generator emf applied to the circuit be of the form

$$e = E_m \sin \alpha$$

Also, let the generator current or the current flowing in the circuit be of the form

$$i = I_m \sin (\alpha + \theta)$$

When these values are substituted in equation (1–33), the instantaneous power equation becomes

$$p = E_m \sin \alpha \times I_m \sin (\alpha + \theta) \tag{1–34}$$

Since $\sin A \sin B = \frac{1}{2}[\cos (A - B) - \cos (A + B)]$, then

$$p = \frac{E_m I_m}{2} [\cos (-\theta) - \cos (2\alpha + \theta)]$$

$$= \left(\frac{E_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right) \cos \theta - \left(\frac{E_m}{\sqrt{2}}\right)\left(\frac{I_m}{\sqrt{2}}\right) \cos (2\alpha + \theta)$$

or

$$p = EI \cos \theta - EI \cos (2\alpha + \theta) \tag{1–35}$$

where E and I are meter or effective values.

As shown in Fig. 1–16, the power curve is sinusoidal in form and has double frequency as compared with the voltage and current. The first term of equation (1–35), or $EI \cos \theta$, is a constant for a given circuit because θ is a fixed angle. This constant, $EI \cos \theta$, is the offset of the axis of symmetry of the power wave from the α-axis. The second term of equation (1–35) is a cosine wave of double frequency since $\alpha = 2\pi f t$ and $2\alpha = 2\pi(2f)t$.

Graphically, the power curve is obtained by plotting points with ordinates $p_1 = e_1 i_1$, $p_2 = e_2 i_2$, etc., as in Fig. 1–16, and drawing a smooth curve through them. The instantaneous power p_1 is positive because e_1 and i_1 are positive at that instant; the instantaneous power p_2 is positive because e_2 and i_2 are both negative and we have a product of two negative quantities; when $\alpha = \pi$ the instantaneous power is zero because the voltage at that instant is zero; and so on. The ordinates of the power curve between the limits 0 and a, π and b, 2π and c, etc. are negative, because in each case we have the product of a positive quantity and a negative quantity. The instantaneous value of the power in the circuit is at all times equal to the product of the corresponding instantaneous current and voltage. The direction of energy flow for sinusoidal voltage and current waves reverses four times in each cycle. The areas between the p-curve and the α-axis indicate energy. The positive areas above the α-axis give the energy delivered by the source to the circuit, and the negative areas give the energy returned by the circuit to the source.

1–17. Average Power.—The difference between the positive and negative areas of the power curve in Fig. 1-16 is the net energy delivered to the circuit during one cycle for values of α from 0 to 2π. If this quantity is divided by the time for one cycle, the quotient is an average ordinate of the power curve, or the average power delivered to the circuit. That is, even though the power may vary over wide limits during the cycle, its effect will be determined by its average value. The average power is the value of power in which the customer is interested. The product of the average power and the time required to complete a cycle is the energy over a whole cycle.

For the general case where the voltage and the current differ in phase by an angle θ, the instantaneous power equation is given by equation (1–35) and the average power is

$$P=\frac{1}{2\pi}\int_0^{2\pi} p\,d\alpha=\frac{EI}{2\pi}\int_0^{2\pi}[\cos\theta-\cos(2\alpha+\theta)]\,d\alpha$$

$$=\frac{EI}{2\pi}\int_0^{2\pi}[\cos\theta-\cos 2\alpha\cos\theta+\sin 2\alpha\sin\theta]\,d\alpha$$

or

$$P=EI\cos\theta \text{ watts} \tag{1–36}$$

Equation (1–36) states that the average power is equal to the product of the effective values of voltage and current multiplied by the cosine of the phase angle θ between the voltage and the current. This can be represented by the average height of the integrated net area between the power curve and the α-axis in Fig. 1–16. Average power is measured by a wattmeter.*

The quantity $\cos\theta$ is called the *power factor* (abbreviated P.F.) of the circuit. From equation (1–36),

$$\cos\theta=\frac{\text{Wattmeter reading}}{\text{Voltmeter reading}\times\text{Ammeter reading}}=\frac{P}{EI} \tag{1–37}$$

The power factor of a circuit can never be greater than unity.

* See Appendix A.

1–18. Summary.—(*a*) *Faraday's Law:* Faraday's Law may be given by the equation

$$e_i = -\frac{d}{dt}(N\Phi) \text{ volts}$$

It may also be given by the statement that the induced emf in a coil is proportional to the time rate of change of the flux linkages.

(*b*) *Frequency:* The frequency of an alternating emf generated in a coil rotating at a constant angular velocity in a P-pole machine is

$$f = \frac{P}{2} \times \frac{\text{rpm}}{60} \text{ cps}$$

(*c*) *Period:* A cycle of a periodic quantity is a complete series of positive and negative values of the periodic quantity. The time required for a periodic wave to pass through one complete cycle of values is called the period of the wave. Thus,

$$T = \frac{1}{f} \text{ second}$$

(*d*) *Phase and Phase Difference:* The expressions for an emf and a current in a circuit are:

$$e = E_m \sin \omega t$$
$$i = I_m \sin (\omega t + \theta)$$

The quantities ωt and $(\omega t + \theta)$ are called the phase of the emf and the phase of the current, respectively. The phase difference between the emf and the current waves is θ. The angle θ in the current expression is called the initial phase because it gives the ordinate of the current wave when $t = 0$. When θ is positive, the current leads the emf; and, when θ is negative, the current lags behind the emf.

(*e*) *Average and Effective Values:* The half-cycle average value of a sine wave $e = E_m \sin \alpha$ is

$$E_{\text{avg}} = \frac{2}{\pi} E_m \text{ volts}$$

The effective or meter value is

$$E = \frac{E_m}{\sqrt{2}} \text{ volts}$$

(*f*) *Power:* Let the emf and current in a circuit be

$$e = E_m \sin \alpha \text{ volts}$$
$$i = I_m \sin (\alpha + \theta) \text{ amperes}$$

Then, the instantaneous power is defined as

$$p = ei$$

and
$$p = EI \cos \theta - EI \cos (2\alpha + \theta) \text{ watts}$$

This is a double frequency quantity.

The average power is

$$P = \frac{1}{2\pi} \int_0^{2\pi} p \, d\alpha = EI \cos \theta \text{ watts}$$

(*g*) *Power Factor:* The power factor of a circuit is

$$\text{P.F.} = \cos \theta = \frac{\text{Wattmeter reading}}{\text{Voltmeter reading} \times \text{Ammeter reading}} = \frac{P}{EI}$$

PROBLEMS

1–1. When the flux Φ in Fig. 1–1 is increasing in the direction of the arrow, determine whether the upper or lower wire of the secondary is positive; and explain how you arrived at your answer. Also, determine the positive wire and explain for the case when Φ is decreasing in value.

1–2. A coil has 1000 concentrated turns. A current of 5 amp flowing in the coil produces a flux of 5,000,000 lines. Assuming that the lines of force link every turn of the coil, determine the number of flux linkages. If the flux is reduced to 3,000,000 lines in 0.01 second by decreasing the current, what is the rate of change of flux during this time?

1–3. There are 1000 turns in each field coil of a two-pole, d-c dynamo. The two field coils are connected in series. When an exciting current is flowing, 2×10^8 lines link each coil. If the current is interrupted in 0.2 second, what average voltage is induced?

1–4. Draw a diagram similar to Fig. 1–8(*b*) with 25 lines per inch. Rotate the coil at uniform velocity and plot the wave shape of the flux enclosed and the emf induced in the coil against values of α as abscissas. Assume $N = 1$.

1–5. Repeat Problem 1–4 with the middle third of the field having 50 lines per inch.

1–6. Repeat Problem 1–4 with the middle third of the field having 10 lines per inch.

1–7. (*a*) Given an emf wave of the form $e = 100 \sin \alpha$. Assume different values of α and plot the curve of e. (*b*) If an emf wave has the form $e = 100 \sin 377t$, what is its frequency? Assume different values of t and plot the emf wave.

1–8. Given $e = 141.4 \sin (\alpha + \theta)$. Assume different values of α and plot the curve of e for: (*a*) $\theta = 30°$; (*b*) $\theta = -60°$; (*c*) $\theta = 90°$; (*d*) $\theta = -90°$; (*e*) $\theta = 180°$.

1–9. Given $i = 141.4 \sin (377t - 60°)$. Determine the value of i at: (*a*) $t = \frac{1}{480}$ sec; (*b*) $t = \frac{1}{240}$ sec; (*c*) $t = \frac{1}{120}$ sec; (*d*) $t = \frac{1}{60}$ sec. What is the effective value of the current?

1–10. Determine the frequency of each of the following waves:

(*a*) $e = 100 \sin (377t + 10°)$
(*b*) $e = 100 \sin 157t$
(*c*) $e = 100 \sin (754t - 20°)$
(*d*) $e = 100 \sin 1131t$

1–11. There is a time interval of 0 01 sec between two successive zero values of a sinusoidal current in a certain circuit. What is the frequency of the current?

1–12. A man's body lies across a 60-cycle, 115-volt line carrying simple harmonic current and emf. What is the maximum voltage to which he is subjected, and how many times per second is he subjected to maximum voltage?

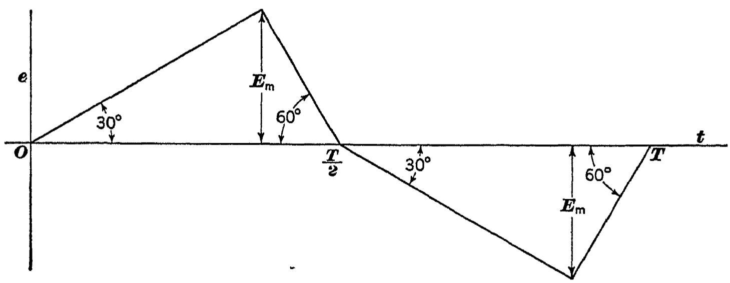


FIG. 1–17

1–13. Two sinusoidal emfs with the same period have amplitudes of 10 and 8, respectively. At a certain instant the values of the two emfs are +5 and −4, respectively; and 0.0001 sec later their values are +10 and +4, respectively. Find: (*a*) the angle of phase difference; (*b*) the frequency, assuming it to be less than 3,000 cps.

1–14. At what points in its cycle does a simple harmonic current or emf attain an instantaneous value equal in magnitude to its average value? At what points is the instantaneous value equal to its effective value?

1–15. What is the average value of a semicircular wave of emf having its diameter coinciding with the axis of abscissas?

1–16. Sketch a curve showing the variation of accumulated average values of a sine wave over several cycles. That is, find the accumulated average value from 0 to $\frac{\pi}{2}$, from 0 to π, from 0 to $\frac{3\pi}{2}$, from 0 to 2π, and so on. Would this be a good definition for the average value of a sine wave?

1–17. What is the effective value of a semicircular wave of current having its diameter coinciding with the axis of abscissas? What is the form factor of this wave?

1–18. Given an emf wave as shown in Fig. 1–17. Find its average and effective values mathematically. Also, find the form factor of this wave.

1-19. The emf and current expressions for a circuit are

$$e = 141.4 \sin (157t + 30°) \quad \text{and} \quad i = 7.07 \sin (157t - 30°)$$

Determine the equation of instantaneous power. Sketch curves of e, i, and p. Indicate the intervals during which the circuit is receiving power from the source of supply.

1-20. Given $e = 110\sqrt{2} \sin (377t - 10°)$ and $i = 10\sqrt{2} \sin (377t + 40°)$. Determine the equation of instantaneous power. Sketch curves of e, i, and p. What is the average power? What is the energy over one complete cycle?

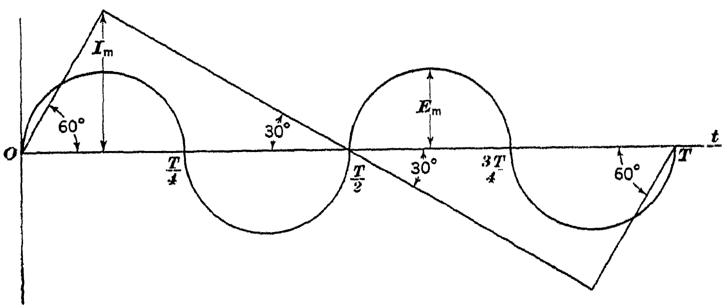


FIG. 1-18

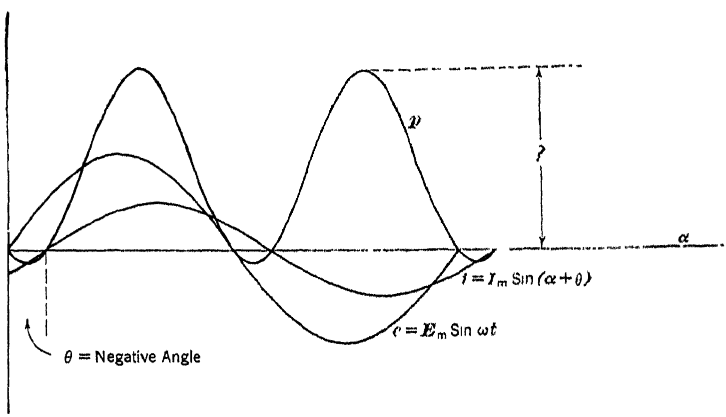


FIG. 1-19

1-21. The waves of semicircular emf and triangular current in a circuit are shown in Fig. 1-18. Sketch the power curve, and find the average power by determining the area under the power curve. Indicate the intervals during which power is delivered by the source to the circuit. Also indicate the intervals when power is returned by the circuit to the source.

1-22. Derive the expression for the maximum value of the power curve in terms of E, I, and θ for the curves shown in Fig. 1-19.

1–23. The effective value of an alternating current is

$$I = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} I_m^2 \sin^2 \alpha \, d\alpha}$$

Letting $\alpha = \omega t$, give the steps in changing the equation to

$$I = \sqrt{\frac{1}{T}\int_0^{T} I_m^2 \sin^2 \omega t \, dt}$$

1–24. Letting $\alpha = \omega t$, show that the following equation is true:

$$E_{avg} = \frac{1}{\pi}\int_0^{\pi} E_m \sin \alpha \, d\alpha = \frac{2}{T}\int_0^{\frac{T}{2}} E_m \sin \omega t \, dt$$

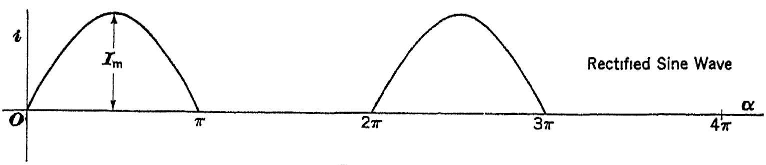


FIG. 1–20

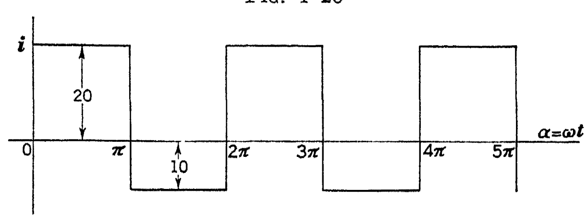


FIG 1–21

1–25. Let

$$e = E_m \sin \omega t$$
$$i = I_m \sin (\omega t + \theta)$$

Derive the power equation

$$p = EI \cos \theta - EI \cos (2\omega t + \theta)$$

1–26. Let $i = I_m \sin \omega t$. Explain that the effective and average values can be written in the following forms:

$$I = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} I_m^2 \sin^2 \omega t \, d(\omega t)}$$

$$I_{avg} = \frac{1}{\pi}\int_0^{\pi} I_m \sin \omega t \, d(\omega t)$$

1–27. A direct-current meter reads the average value of any periodic wave over a complete cycle. Find the reading of a direct-current ammeter connected in a circuit in which the current is: (*a*) of the shape $i = 10 + 100 \sin 377t$; (*b*) of the form shown in Fig. 1–20.

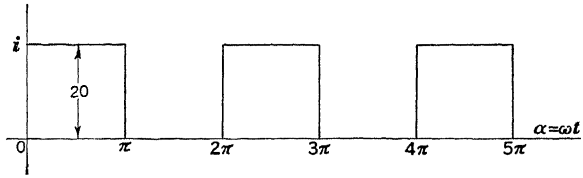


FIG. 1–22

1–28. Find the readings of a d-c ammeter (which reads the average value of any periodic wave over a complete cycle) and an a-c ammeter (which reads $I = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} i^2\, d\alpha}$ where i is any periodic wave) connected in a circuit in which the current has: (*a*) the wave form shown in Fig. 1–21; (*b*) the wave form shown in Fig 1–22.

CHAPTER 2

VECTOR REPRESENTATION OF SINE WAVES

In the solution of ordinary direct-current problems all quantities involved are simple algebraic quantities. The equations are usually of the simple algebraic type and may be handled by ordinary algebraic methods. These statements are true when applied to the emfs, currents, and power existing at *any instant of time* in an alternating-current circuit. That is, voltage drops around a closed circuit may be added to obtain the applied emf by going through a somewhat lengthy process of plotting the sine waves representing the drops and adding their ordinates. The resultant wave representing the applied emf will be another sine wave.

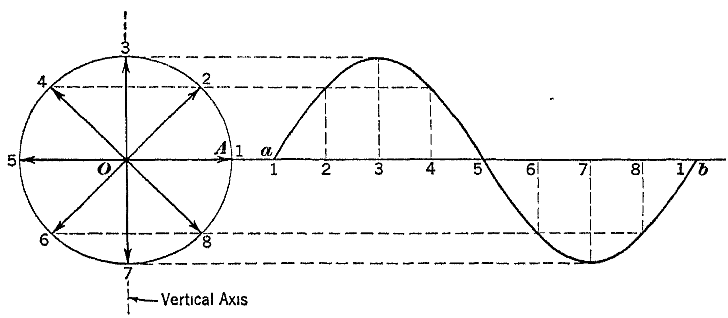


FIG. 2–1

Meter values, rather than instantaneous values, are usually desired in practical problems. Unfortunately, effective values of emfs and currents cannot be handled by ordinary algebraic methods. It is necessary to develop a simple method for handling meter values in alternating-current circuits. A simplification in the expressions and diagrams of alternating-current circuits has been accomplished by representing effective values in terms of *vectors* and *vector quantities*.

2–1. A Sine Wave Produced Graphically.—The actual shape of a sine wave may be obtained by construction, as shown in Fig. 2–1. Draw a circle whose radius OA is equal to the maximum value of the sine wave, and divide its circumference into n equal parts. Through the center of the circle draw a horizontal line and divide a portion of it, as ab, also into n equal parts. The divisions from 1 to 2, 2 to 3, 3 to 4, etc. will represent the same intervals of time or angle as the arcs 1–2, 2–3, 3–4, etc., respectively. Erect an ordinate at each point on ab equal to the vertical projection of the radius OA on the vertical axis for the corresponding position of OA. A smooth curve drawn through the ends of these ordinates will be a sine wave.

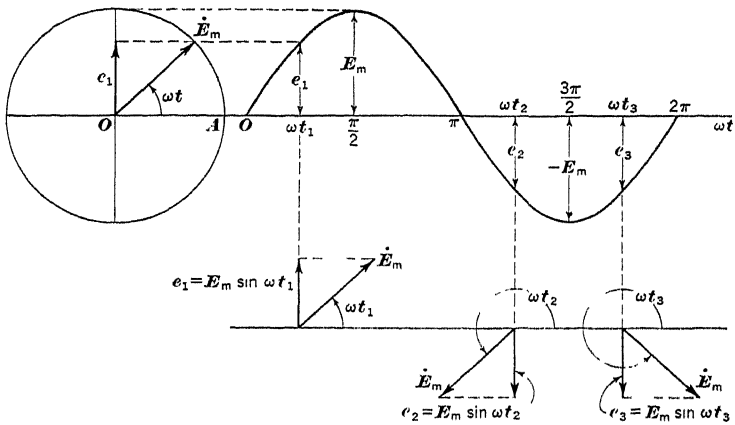


FIG. 2–2

It is to be noted that, since the resulting sine wave has a maximum value equal to the radius of the circle, any ordinate of the sine wave may be determined by projecting, upon the vertical axis, the radius in the corresponding position. When a cosine wave is desired, the projections of the radius on the horizontal axis are used.

The sine wave may also be plotted from a table of sines. Mark a horizontal axis, such as ab in Fig. 2–1, in degrees. At each point erect an ordinate equal to the sine of the corresponding angle. Thus, at 30° the ordinate is 0.5; at 60° the ordinate is 0.866; at 90° it is 1.0; and so on. When the angle is greater than 180°, the sine becomes negative and the wave falls below the horizontal line.

2–2. Representation of Sine Waves by Revolving Vectors.—It has been shown in Chapter 1 that a single conducting coil, rotating at a constant speed in a uniform magnetic field, will generate a sinusoidal emf like that shown in Fig. 2–2. The wave may be represented by the equation

$$e = E_m \sin 2\pi ft = E_m \sin \omega t$$

A harmonic voltage may also be represented by a revolving vector* $\dot{E}_m$. A dot over a quantity indicates that it is a vector. This vector has a length equal to the maximum value E_m of the sine wave and is rotating at a constant angular velocity of $\omega = 2\pi f$ radians per second. If the frequency f is 60 cps, the rotating vector must make 60 complete revolutions per second. The instantaneous values will be represented by the projections of the vector upon the vertical axis, Fig. 2–2. *Counter-clockwise rotation has been adopted internationally as the positive direction of rotation.*

Time is usually reckoned as zero when the vector $\dot{E}_m$ lies along OA or the *reference axis*. At the end of time t the vector has swept through an angle ωt. When the revolving vector makes an angle ωt_1 with the reference axis, its vertical projection is e_1; and the magnitude of e_1 is the instantaneous value of the emf wave at the instant t_1. It is considered positive because it is measured above the ωt-axis. When the vector has advanced to a position which makes an angle ωt_2 with the reference axis, its projection e_2 on the vertical represents the instantaneous value of the emf wave at time t_2. This value is considered negative because it is measured below the ωt-axis.

It should be noted that the use of $\dot{E}_m$ instead of $e = E_m \sin \omega t$ and the use of a vector instead of a sine wave simplify the expressions and diagrams in alternating-current problems.

2–3. Notation of Voltage Vectors.—In order to designate clearly the vector by which a voltage wave is generated, some convention must be adopted. The full-line curve in Fig. 2–3(c) is assumed for the voltage acting through the coil from A to A'. The vector representing this particular voltage is designated by

* A scalar is a quantity possessing magnitude only. Examples of scalar quantities are energy, power, mass, and temperature. Such quantities are added algebraically. A vector is a quantity possessing both magnitude and direction. A common example of a vector is a force. Vectors are not necessarily added algebraically, but must be combined in such a way as to take into consideration their directions as well as their magnitudes. A vector which is constant in magnitude and revolves in a plane with constant angular velocity is called a time vector. Time vectors are treated like other vectors.

$\dot{E}_{AA'}$. The idea to be conveyed by the subscript AA' (as contrasted to the subscript $A'A$) is that the projections of the vector $\dot{E}_{AA'}$ give a sine wave of voltage whose *instantaneous positive values* act through the coil in a direction from A to A'.

To be consistent with the convention adopted, the projections of the voltage vector $\dot{E}_{A'A}$ give a sine wave of voltage whose *instantaneous positive values* act through the coil in a direction from A' to A. This sine wave is given by the broken-line curve in Fig. 2–3(c). The vectors $\dot{E}_{AA'}$ and $\dot{E}_{A'A}$ are equal in magnitude and opposite to each other, as shown in Fig. 2–3(b); thus,

$$\dot{E}_{A'A} = -\dot{E}_{AA'}$$

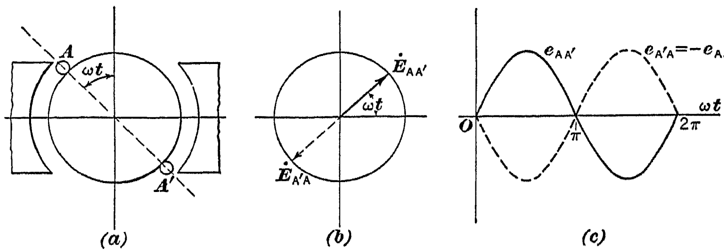


FIG. 2–3

2–4. Addition of Emf Waves.—If, as shown in Fig. 2–4(a), two coils AA' and BB' are placed 60° apart on the same armature, they rotate at the same angular velocity and cut the same flux. When A and B are considered as corresponding terminals or ends of the coils, the emf in coil AA' will reach its maximum value 60° ahead of the emf in coil BB' for the indicated direction of rotation of the armature. The two emfs are said to differ in phase by 60°. The expressions for the emf waves are:

$$e_{AA'} = E_{AA'} \sin \omega t \qquad (2\text{–}1)$$

$$e_{BB'} = E_{BB'} \sin (\omega t - 60°) \qquad (2\text{–}2)$$

where $E_{AA'}$ and $E_{BB'}$ are the maximum values of the waves. If the numbers of turns in the coils are different, then $E_{AA'} \neq E_{BB'}$.

If the coil end A' is connected to the coil end B, the emf from A to B' is the sum of the two sinusoidal emfs. Thus,

$$e_{AB'} = e_{AA'} + e_{BB'} \text{ (algebraic addition)} \qquad (2\text{–}3)$$

That is, the voltage from A to B' is equal to the voltage from A to A' plus the voltage from B to B'. As shown in Fig. 2–4(c), the resultant wave $e_{AB'}$ is obtained by adding the ordinates of the two emf waves at each instant. Both the maximum value of $e_{AB'}$ and its phase angle α depend on the maximum values of $e_{AA'}$ and $e_{BB'}$ and on the phase difference between the two waves.

If the sinusoidal emfs $e_{AA'}$ and $e_{BB'}$ are to be represented by revolving time vectors, as shown in Fig. 2–4(*b*), the magnitudes of the vectors must be equal to the maximum values $E_{AA'}$ and $E_{BB'}$

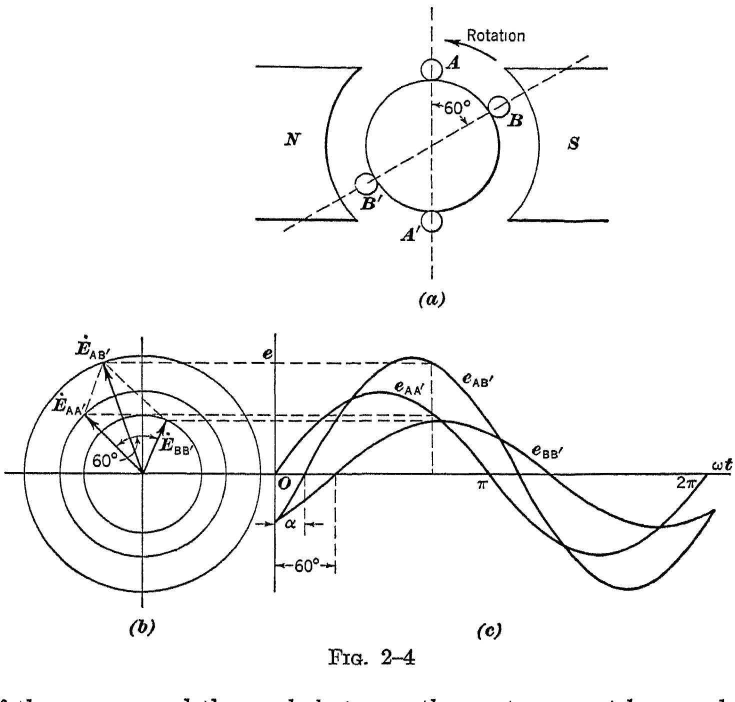


FIG. 2–4

of the waves, and the angle between the vectors must be equal to their phase difference, or 60°. Their relative positions in the vector diagram must show that $e_{AA'}$ is leading $e_{BB'}$ with the *conventional counter-clockwise direction of rotation of vectors.* By completing the parallelogram, the vector $\dot{E}_{AB'}$ with a maximum value of $E_{AB'}$ represents the vector sum of $\dot{E}_{AA'}$ and $\dot{E}_{BB'}$. Thus,

$$\dot{E}_{AB'} = \dot{E}_{AA'} + \dot{E}_{BB'} \text{ (vector addition)} \qquad (2\text{–}4)$$

The vector $\dot{E}_{AB'}$ obtained by vector addition of $\dot{E}_{AA'}$ and $\dot{E}_{BB'}$ is a correct representation of the voltage wave $e_{AB'}$. To see that this is true, consider that the projection of the vector $\dot{E}_{AB'}$ on the vertical axis is always equal to the sum of the projections of the vectors $\dot{E}_{AA'}$ and $\dot{E}_{BB'}$. Consider further that the projections of these last two vectors are equal to the instantaneous values of $e_{AA'}$ and $e_{BB'}$, as seen in Fig. 2–4(*c*). It follows, then, that equation (2–3) is satisfied when $\dot{E}_{AB'}$ is used to represent the voltage wave $e_{AB'}$.

It must be noted that only vectors representing sinusoidal waves of the *same frequency* can be added in a vector diagram like that shown in Fig. 2–4(*b*). The phase relation of these vectors remains the same irrespective of the instant considered (or when the vectors are stopped). This is important since the vertical projections of the vectors must give the instantaneous values of the sine waves at all instants. If two vectors of different frequencies are represented on the same vector diagram, the phase angle between them will depend on time. This variation of the phase angle with time makes it very difficult to draw such diagrams.

The effect of the phase difference of the waves $e_{AA'}$ and $e_{BB'}$ on the resultant wave can be seen from the vector diagram. Under all conditions, the vectors representing the two waves are added vectorially to obtain the vector representing the resultant wave. When the two waves are in phase, they pass through zero, increasing positively, at the same instant. The vectors representing these waves coincide and they may be added numerically. When the waves are 180° out of phase, the vectors are in opposite direction and they may be subtracted numerically.

From the work just presented, it is obvious that the representation of sine functions by rotating time vectors simplifies algebraic operations. Much less time is required to draw the vectors of Fig. 2–4(*b*) than the curves of Fig. 2–4(*c*).

2–5. Addition of Emf Waves Using Effective Values.—In the solution of practical problems, effective values rather than maximum values are generally used. This is quite obvious since voltmeters and ammeters are usually designed to read rms values. Effective values are proportional to maximum values, as $E = \frac{E_m}{\sqrt{2}}$, and it is more convenient to use effective values for the lengths of vectors to check meter readings. The vector diagram in Fig. 2–4(*b*) can be changed to a vector diagram with effective values merely by reducing the length of each vector or changing the scale by the factor $\frac{1}{\sqrt{2}}$.

What has been said concerning the representation and addition of emf waves applies to alternating current waves. The result obtained by adding a current wave to an emf wave has no meaning.

Example 2–1.—Each of two identical coils AA' and BB' on the same armature generates an emf of 120 volts effective. The voltage of coil AA' leads that of the coil BB' by 90°, and B is connected to A'. (a) Write the expressions representing the emfs of the two coils. (b) Determine the maximum value of the resultant emf $e_{AB'}$. (c) Determine the reading of a voltmeter connected across the terminals A and B'.

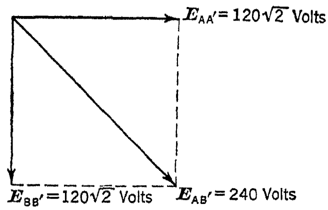


FIG. 2–5

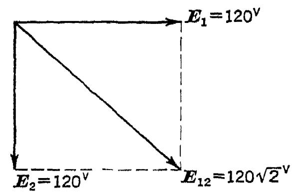


FIG. 2–6

Solution.—(a) The expressions for the emfs are:

$$e_{BB'} = E_{BB'} \sin \omega t = 120\sqrt{2} \sin \omega t$$

$$e_{AA'} = E_{AA'} \sin (\omega t + 90°) = 120\sqrt{2} \sin (\omega t + 90°)$$

or

$$e_{AA'} = 120\sqrt{2} \sin \omega t$$

$$e_{BB'} = 120\sqrt{2} \sin (\omega t - 90°)$$

(b) The vectors representing the maximum values $E_{AA'}$ and $E_{BB'}$ of the emfs are shown in Fig. 2–5. By combining these two vectors vectorially, the vector representing the voltage $E_{AB'}$ is obtained. As $E_{AA'}$ and $E_{BB'}$ are at right angles in this case,

$$E_{AB'} = \sqrt{E^2_{AA'} + E^2_{BB'}} = \sqrt{(120\sqrt{2})^2 + (120\sqrt{2})^2} = 240 \text{ volts}$$

It should be noted that, when $E_{AA'}$ and $E_{BB'}$ are not at right angles to each other, this simple formula for determining $E_{AB'}$ cannot be used.

(c) By changing the lengths of the vectors representing maximum values, a vector diagram representing effective values of the emf waves may be drawn as shown in Fig. 2–6. Let the effective value of $E_{AA'}$ be E_1, the effective value of $E_{BB'}$ be E_2, and that of $E_{AB'}$ be E_{12}. Since the voltmeter reads E_{12}, its reading is

$$E_{12} = \sqrt{E_1^2 + E_2^2} = \sqrt{120^2 + 120^2} = 120\sqrt{2} \text{ volts}$$

2–6. Summary.—(*a*) Any sine wave may be represented by a rotating time vector revolving at a constant angular velocity of $\omega = 2\pi f$ radians per second. The vertical projections of the vector represent the instantaneous values of the sine wave.

(*b*) Theoretically the magnitude of the rotating vector is the maximum value of the sine wave; but, in the solution of problems, effective values should be used for the lengths of vectors to check meter readings.

(*c*) The sum of the instantaneous values of two sine waves depends on their phase relations as well as on their magnitudes. Instantaneous values of emfs or currents are added algebraically.

(*d*) It should be kept in mind constantly that either effective values or maximum values must be combined vectorially.

(*e*) Only vectors representing sinusoidal waves of the same frequency can be added in a vector diagram, such as that shown in Fig. 2–4(*b*).

PROBLEMS

2–1. The expression for an emf wave is $e = 100 \sin \alpha$. By using a table of sines, find the instantaneous values of the voltage for angles of 30, 60, 90, 120, 150, 180, 210, 270, 300, 360, 390, 450, 540, 630, and 720 degrees. Draw a graph carefully to scale, showing two cycles of the emf wave.

2–2. An emf wave has the form $e = 200 \sin 377t$. (*a*) Using a table of sines, find the instantaneous values for angles of 30, 60, 90, 180, 270, and 300 degrees. (*b*) Determine the values of time corresponding to those angles. (*c*) If the wave is to be represented by a revolving time vector, locate the vectors for the different instants of time.

2–3. The expression $e_{AA'} = 100 \sin 157t$ represents the emf generated in coil AA' of an alternator. This is the voltage $e_{AA'}$ through the coil from A to A'. Write a similar expression for the voltage $e_{A'A}$ through the coil from A' to A. Draw diagrams showing vectors and sine waves which represent $e_{AA'}$ and $e_{A'A}$.

2–4. Two coils AA' and BB' of an alternator generate effective values of 50 and 100 volts, respectively. Coil terminal A corresponds to coil terminal B, and the emf of coil AA' leads that of coil BB' by 120 degrees. If B is connected to A', what is the reading of a voltmeter connected across the open ends? Draw two diagrams like those of Fig. 2–4.

2–5. Repeat Problem 2–4 with the direction of rotation of the alternator reversed.

2–6. Repeat Problem 2–4 with B' connected to A'.

2–7. Two alternating-current generators A and B are connected to the same 220-volt bus-bars. Generator A is delivering 75 amp and generator B is delivering 125 amp. If the current in generator A is leading the current in generator B by 60 degrees, what will be the reading of an ammeter connected to record the total current delivered by the two alternators?

2–8. Three coils, having sinusoidal emfs that are equal in magnitude and of 120° phase difference each to each, are connected in series so that the resultant of all three emfs is at all times zero (that is, the coils form a closed delta, three-phase). What is the emf between the junction point of two of the coils and the mid-point of the third coil?

2–9. Write the expressions for the emf's between the mid-points of the three coils in Problem 2–8.

2–10. Two 4-pole alternators have their shafts rigidly coupled. Each machine generates sinusoidal emf between the terminals. A voltmeter connected across one of them indicates 100 volts, a second voltmeter across the other indicates 150 volts, and a third voltmeter across the series combination indicates 200 volts. What must be the mechanical angle between the corresponding points of the two windings to produce the necessary phase relation?

2–11. The resultant of two sinusoidal emfs is

$$e = E_{m1} \sin \omega t + E_{m2} \sin (\omega t - 90°)$$

or

$$e = E_m \sin (\omega t - \alpha)$$

Show that

$$E_m = \sqrt{E_{m1}^2 + E_{m2}^2}$$

and

$$\alpha = \tan^{-1} \frac{E_{m2}}{E_{m1}}$$

2–12. The currents of two alternators connected to a common bus are:

$$i_1 = I_{1m} \sin \omega t$$
$$i_2 = I_{2m} \sin (\omega t - \theta)$$

The combined current is

$$i = I_{1m} \sin \omega t + I_{2m} \sin (\omega t - \theta)$$

or

$$i = I_m \sin (\omega t - \alpha)$$

Find the expressions for I_m and α.

CHAPTER 3

ELEMENTARY CIRCUIT CONCEPTS

The fundamental laws of the electric circuit are Ohm's Law, Kirchhoff's Laws, and Lenz's Law. They are fundamental natural laws applied to the conditions for the flow of electric currents in a circuit. A great many years of experimentation have demonstrated the exactness with which the laws operate. In order to apply these basic laws it is necessary to make use of the concept of circuit elements (sometimes called circuit constants, circuit characteristics, or circuit parameters), such as resistance, inductance, and capacitance. These elements, together with the applied emf, govern the movement of electricity in simple circuits. In direct-current circuits, which are considered in this chapter, the resistance R directly opposes the flow of electricity; the inductance L opposes any change in the flow; and the capacitance C is a measure of the circuit's ability to store electric charge.

The elements of a transmission line are uniformly distributed along the length of the line, or at least along a section of the line. In this course we deal mostly with circuits with concentrated elements (sometimes called "lumped constants") where the elements of the lead wires for completing the circuit are assumed to be negligible. All circuits shall be considered as consisting of concentrated elements unless it is otherwise stated.

Any discussion of a circuit that is purely resistive, purely inductive, or purely capacitive is of course hypothetical, inasmuch as such a circuit is physically unrealizable. It shall be called a *hypothetical* circuit. By minimizing certain effects and neutralizing others, a circuit that closely simulates a pure R circuit, a pure L circuit, or a pure C circuit is obtainable. It is very important that the physical nature of these elements be understood in this chapter before undertaking the more general study of electric circuits.

3–1. Resistance, R.—Two laws, (*a*) Ohm's Law and (*b*) Joule's Law, give us some knowledge of the physical nature of electric resistance.

(*a*) *Ohm's Law:* Ohm, through experimental investigation, found that the ratio of the impressed emf across the terminals of a metallic conductor to the current flowing in the conductor is equal to a factor R which is called the resistance of the conductor. Thus,

$$\frac{E}{I} = R \tag{3-1}$$

If E is given in volts and I is in amperes, then the resistance R is in ohms.

In the application of Ohm's Law to an electric circuit the following considerations must be emphasized:

(1) When Ohm's Law is applied to an entire electric circuit, it is necessary to make sure that *all* the electromotive force of the circuit is used for the value of E, that *all* the current of the circuit is used for the value of I, and that *all* the resistance of the circuit is used for the value of R. That is, when a local emf exists in the circuit, as in the case of a circuit containing two dissimilar metals joined or welded together, the value of the thermal emf at the welded joint must be considered in the value of E.

(2) When Ohm's Law is applied to only a part of an electric circuit, care must be taken that the values which are used for E, I, and R include only the potential, current, and resistance of the particular part of the circuit under consideration.

(3) When Ohm's Law is applied to a conductor, care must be taken that the temperature of the conductor is constant. Experiment shows that the resistance of a metallic conductor generally increases with an increase in temperature. Thus, if R_t is the resistance at a temperature of $t°$ C, and if R_0 is the resistance in ohms at 0 C, which may be obtained from tables, then

$$R_t = R_0(1 + \alpha_0 t) \text{ ohms} \tag{3-2}$$

where α_0 is known as the temperature coefficient of the metal and depends on the kind of metal. The temperature coefficient α_0 of copper on a base temperature of 0 C is 0.00427 per degree centigrade.

(b) *Joule's Law:* It is an experimental fact that under ordinary conditions the flow of current through a resistance is always accompanied by heat. Joule established experimentally the relation that the amount of energy W dissipated in a metallic wire is proportional to the product of the square of the current I and the time t during which the current flows. Thus,

$$W = RI^2t \tag{3-3}$$

Equation (3–3) is known as Joule's Law; and the proportionality factor between W and I^2t is the resistance R of the circuit. If I is in amperes, t in seconds, and W in joules or watt-seconds, then R is in ohms. This equation gives another definition of the resistance R from an energy point of view. It may be said that *resistance is that element of a circuit which accounts for the existence of heat.* An electric circuit is devoid of resistance when the flow of current is not accompanied by a loss of energy through heat. Some metals at a temperature slightly above absolute zero offer practically no resistance to the flow of electricity.

Substituting E for RI from Ohm's Law in equation (3–3) gives

$$W = (RI)It = EIt \tag{3-4}$$

which is the general expression for the energy supplied by a source of emf to a circuit obeying Ohm's Law. If E, I, and t are in volts, amperes, and seconds, respectively, then W is in joules or watt-seconds.

(c) *Resistance as an Intrinsic Property of Metals:* Although the effect of resistance comes into play only when electricity is in motion, resistance may be considered as a function of the material of the conductor and its physical dimensions. Thus,

$$R = \rho \frac{l}{A} \tag{3-5}$$

where ρ is called the resistivity and is defined as the resistance of a conductor per unit length and unit cross-sectional area; l is the length of the conductor; and A is the cross-sectional area of the conductor. In the rationalized MKS system, R is in ohms; l is in meters; A is in square meters; and ρ is in ohm-meters. If R is in ohms, l in feet, and A in circular mils, which units are commonly used, then ρ is in ohms per circular mil-foot; that is, ρ is the resistance of a wire 1 mil (0.001 inch) in diameter and 1 foot long.

One circular mil is defined as the area of a circle 1 mil in diameter. For ordinary "soft drawn" copper of 100 per cent conductivity, at 25 C, the resistivity $\rho = 10.6$ ohms per circular mil-foot. For "hard drawn" copper, the conductivity of which is about 97 per cent, $\rho = 10.9$ ohms per circular mil-foot.

In order that the resistance of a conductor at a given temperature may be calculated by the use of equation (3–5), its length in the direction of current flow must be constant and its cross-section must be uniform and constant.

(*d*) *Effective Resistance:* When the resistance of a circuit is measured with a d-c source, that resistance is called the d-c resistance, or sometimes the ohmic resistance. If the resistance is measured with power frequency, say 60 cycles, the value obtained

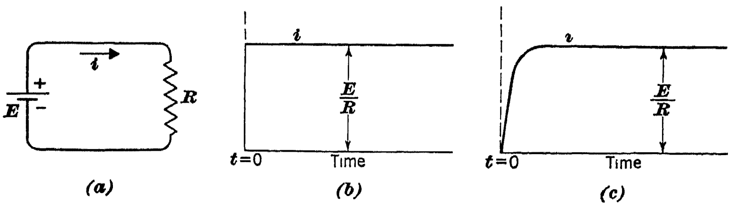


FIG. 3–1

is practically the same as the d-c resistance. In case of high frequencies, such as radio frequencies, the current flows along the outer part of the conductor and, as a result, resistance increases with frequency. This increase in resistance is said to be due to *skin effect.*

If the circuit under consideration contains iron, as in the case of a transformer, the *effective resistance* measured with alternating current is always equal to the loss in power divided by the current squared, or

$$R_{\text{eff}} = \frac{\text{Wattmeter reading}}{(\text{Ammeter reading})^2} = \frac{P}{I^2} \qquad (3\text{–}6)$$

The increase in loss with iron in the circuit is due to the eddy-current and hysteresis losses in the magnetic material. The effective resistance is equal to the d-c resistance plus a fictitious part which accounts for the non-uniform current density and certain local losses produced by the varying magnetic field in the iron and in the conductor itself.

If an unvarying emf E is impressed on a straight metallic conductor, as in Fig. 3–1(a), the current will reach its final value instantly, as in Fig. 3–1(b). It is assumed that the circuit has no inductance. Fig. 3–1(c) is a reproduction of an oscillogram showing the variation of current in a circuit which approximates a purely resistive circuit. That the circuit possesses a small amount of self-inductance is evident from the fact that the current does not reach its final value instantly.

3–2. Kirchhoff's Emf Law.—Fig. 3–2 shows a pure resistance R connected across the terminals of a battery with an emf E. For this simple d-c circuit, it is convenient to choose the direction of current flow as indicated by the arrow in the diagram.

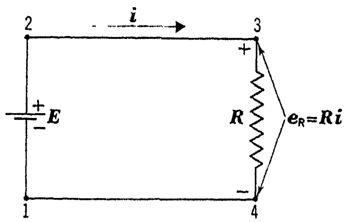


FIG. 3–2

Before Kirchhoff's Emf Law is stated, it is desirable to consider first the significance of: (a) a voltage rise at a source and (b) a voltage drop in relation to the current arrow at the load. If the circuit is traced in the direction of the current arrow in Fig. 3–2, or from 1 to 2 to 3 to 4 to 1, then we have the following:

(a) A voltage rise is encountered across a *source* if, in going from point 1 to point 2, the battery polarity is from minus to plus. Thus, E is a voltage rise from 1 to 2 (or in the direction of a potential rise).

(b) A voltage drop is encountered across a *load element* if, in going with the direction of the current arrow from 3 to 4, point 3 is given a plus polarity and 4 a minus polarity. Thus, e_R or e_{34} is said to be a voltage drop from 3 to 4 in the direction of the current arrow.

When voltage rise and voltage drop are defined in the manner just described, *one way* of stating Kirchhoff's Emf Law is:

At each instant of time, the algebraic sum of the voltage rises is equal to the algebraic sum of the voltage drops, both being taken in the same direction around the closed circuit or loop.

Symbolically, this law may be expressed as follows:

$$\Sigma \text{ rises} = \Sigma \text{ drops} \tag{3–7}$$

To have a better understanding of Kirchhoff's Emf Law, it may be well to consider again the circuit shown in Fig. 3–2. If the circuit is traced in the direction of the current arrow, the voltage drop across R from 3 to 4 is

$$e_R = Ri \tag{3–8}$$

and the battery emf E, with minus to plus from 1 to 2, is a voltage rise. From Kirchhoff's Emf Law,

$$E = e_R \tag{3–9}$$

The student must clearly understand that, in a pure R circuit, the voltage rise E at the source is equal to the voltage drop across R because not only their magnitudes are equal but also their polarities are the same. That is, the potential from the top wire to the bottom wire is from plus to minus for both E and e_R.

It can be seen from equation (3–7) that, when all voltage rises and all voltage drops are taken in the same direction around the closed circuit or loop, Kirchhoff's Emf Law may be stated in other ways. Thus,

$$\Sigma \text{ rises} - \Sigma \text{ drops} = 0 \tag{3–7a}$$

$$-\Sigma \text{ rises} + \Sigma \text{ drops} = 0 \tag{3–7b}$$

Then Kirchhoff's emf equation for the circuit shown in Fig. 3–2 likewise may be written in either of the following forms:

$$E - e_R = 0 \tag{3–9a}$$

$$-E + e_R = 0 \tag{3–9b}$$

In view of equation (3–9), Ohm's Law may be expressed as follows:

$$I = \frac{E}{R} = \frac{e_R}{R} \tag{3–1a}$$

3–3. Inductance, *L*.—Inductance is that element of an electric circuit which opposes any sudden increase or decrease of the current flowing in the circuit. If an unvarying emf E is suddenly applied to a coil of N turns, as in Fig. 3–3(*a*), there will result a current which is zero at the instant $t=0$ and increases steadily with time as shown in Fig. 3–3(*b*). The circuit is assumed to have no resistance. Fig. 3–3(*c*) is a reproduction of an oscillogram showing the rise of current with respect to time in an air-core inductance upon the application of a constant emf. Since the coil contains no magnetic material, the self-inductance is constant. That the circuit does actually possess resistance is evident from the fact that the current does not tend to rise indefinitely.

Viewed from the standpoint of energy, inductance causes an electric circuit to store energy (in the magnetic field) while the current increases and to deliver energy while the current decreases.

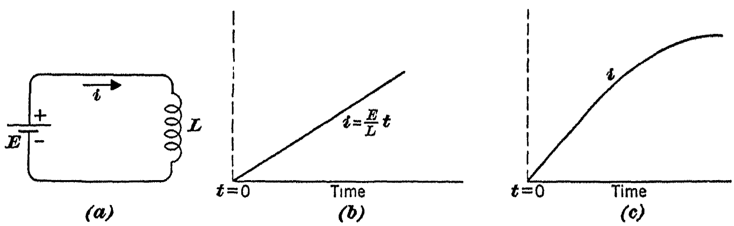


FIG. 3–3

(*a*) *Lenz's Law:* Experiment shows that, in air or in a medium of constant permeability, a change in the current produces a proportional change in the flux encircling the current. This change in the flux linking the circuit induces an emf in the circuit which opposes the change in current. Thus, we have what is known as Lenz's Law, which is

$$e_i = -L\frac{di}{dt} \text{ volts} \tag{3–10}$$

where e_i = emf of self-induction, in volts;

$\frac{di}{dt}$ = time rate of change of current, in amperes per second;

L = coefficient of self-induction in henrys, which is assumed to be a constant.

The negative sign indicates only that, when $\frac{di}{dt}$ is positive, e_i is negative and tends to oppose the impressed emf which causes the increase of current.

From equation (3–10), inductance may be considered as a factor by which the rate of change of current in a coil or circuit is multiplied to obtain the induced emf. When 1 volt is induced in a coil while the current is changing at the rate of 1 ampere per second, the inductance of the coil is 1 henry. The units millihenry (mh) = 10^{-3} henry and microhenry (μh) = 10^{-6} henry are also used.

Consideration of the circuit shown in Fig. 3–4 may help to give a better understanding of the induced emf e_i and the voltage drop e_L across the inductance from 3 to 4. When the current is changing in the pure L circuit, the battery voltage is equal and opposite to the emf of self-induction, or $E = -e_i$. If the current is flowing and increasing in the direction of the current arrow, there is a voltage drop e_L or e_{34} from 3 to 4. From Kirchhoff's Emf Law,

$$E = e_L \tag{3–11}$$

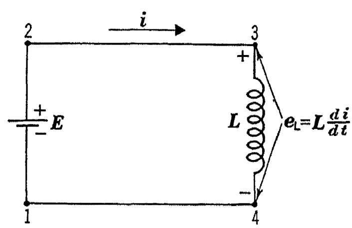


Fig. 3–4

Thus, the voltage drop across L is equal and opposite to the induced emf, or

$$e_L = -e_i = L\frac{di}{dt} \tag{3–12}$$

It follows from equation (3–11) that in a pure L circuit the voltage rise E is equal to the voltage drop e_L because not only their magnitudes are equal but also their polarities are the same. That is, the potential from the top wire to the bottom wire is from plus to minus for both E and e_L.

From equations (3–7*a*) and (3–7*b*), it may be seen that Kirchhoff's emf equation for the circuit shown in Fig. 3–4 may be written in either of the following forms:

$$E - e_L = 0 \tag{3–11a}$$

$$-E + e_L = 0 \tag{3–11b}$$

Example 3–1.—Assume that the current flowing through a resistanceless air-core coil is varied according to the curves shown in Fig. 3–5 for the three intervals indicated. Draw on the same graph the voltage drop e_L across the coil in the direction of current flow.

Solution.—During the interval from t_1 to t_2, the current is increasing at a constant rate, and the slope $\frac{di}{dt}$ of the curve is constant and positive. From equation (3–12), the voltage drop across the resistanceless coil in the direction of current flow is constant and positive.

The current flowing in the coil is held constant from t_3 to t_4. The slope $\frac{di}{dt}$ of the i-curve is constant and zero; therefore, the voltage drop e_L is zero.

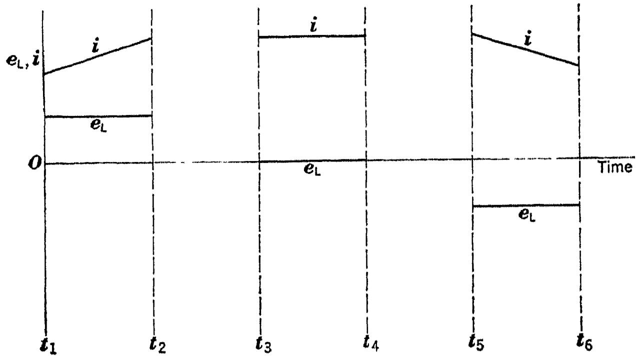


FIG. 3–5

From t_5 to t_6 the current is decreasing at a constant rate. The slope $\frac{di}{dt}$ is constant and negative, giving a negative voltage drop e_L.

Hence, whenever a circuit contains a constant self-inductance, the voltage drop across L in the direction of the current arrow is proportional to the slope of the current-time curve.

(*b*) *Inductance Defined in Terms of Flux Linkages:* When the permeability is constant, inductance of a coil may be defined as the flux per unit current which links with the turns N of a coil. Thus,

$$L = \frac{N\Phi}{i} \text{ henry} \tag{3–13}$$

where Φ is in webers ($= 10^8$ lines) and i is in amperes. Hence, inductance is sometimes defined as flux linkages per unit current.

(*c*) *Inductance as an Intrinsic Property of a Circuit:* Although the effect of inductance comes into play only when the current is changing, the inductance of a circuit may be considered as a function of the dimensions of the circuit. The problem of calculating inductance from the dimensions of a circuit is a difficult one, except for very simple and special cases. For a long air-core solenoid, the coefficient of self-induction in the rationalized MKS system is given by the equation

$$L = \frac{\mu_0 N^2 A}{l} \text{ henry} \tag{3-14}$$

where μ_0 = permeability of air $= 4\pi \times 10^{-7}$ henry/meter;
N = number of turns;
A = cross-sectional area of solenoid, in square meters;
l = length of solenoid, in meters.

In general, it will be assumed that the inductance of a circuit is either known or can be determined experimentally.

(d) General Definition of Inductance: In defining inductance by equation (3–13) it is implied that L is a constant; that is, the conductor is assumed to be made of a material of constant permeability and surrounded by a medium of constant permeability.

For coils wound on iron cores, where the permeability μ varies over a considerable range, there is no constant value of inductance and equation (3–13) must be changed to

$$L = N\frac{d\Phi}{di} \tag{3-13a}$$

It is evident from this equation that, if flux is not proportional to current, the inductance is different for different values of current.

Substituting the value of L from equation (3–13a) in equation (3–10) gives a procedure for deriving Faraday's Law from Lenz's Law and vice versa. Thus,

$$e_i = -\left(N\frac{d\Phi}{di}\right)\frac{di}{dt}$$

or

$$e_i = -N\frac{d\Phi}{dt} \tag{3-15}$$

Since N is constant, being merely the number of turns of the circuit about the magnetic field, equations (3–13a) and (3–15) may also be written as follows:

$$L = \frac{d(N\Phi)}{di} \tag{3-16}$$

$$e_i = -\frac{d(N\Phi)}{dt} \tag{3-17}$$

Equation (3–13a) states that the inductance in a circuit containing magnetic material is proportional to the rate of change of flux with respect to the current causing the change, or $L \propto \frac{d\Phi}{di}$.

It will be of interest to derive an expression for the energy stored in the magnetic field of a coil wound on an iron core, as in Fig. 3–6(a), from the instant $t = t_1 = 0$ when $i = I_1 = 0$ to the instant $t = t_2$ when $i = I_2$. The magnetization curve, showing the relation of flux and current, is given in Fig. 3–6(b).

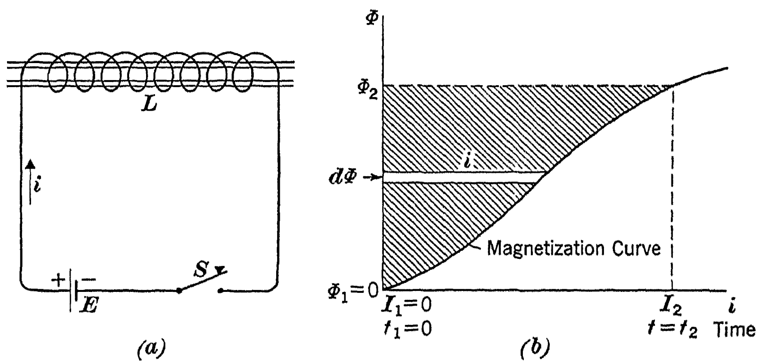


Fig. 3–6

For simplicity, it will be assumed that the resistance of the circuit is negligible. Then the power taken by the inductance after the closing of the switch S is

$$p = e_L i$$

and the energy stored is

$$W = \int_{t_1}^{t_2} e_L i \, dt$$

Since the inductance L is variable, the voltage drop across the inductance is

$$e_L = L \frac{di}{dt}$$

and

$$W = \int_{t_1}^{t_2} L \frac{di}{dt} i \, dt = \int_{I_1}^{I_2} Li \, di \tag{3-18}$$

Substituting the value for L from equation (3–13a) gives

$$W = N \int_{I_1}^{I_2} \frac{d\Phi}{di} i \, di$$

or

$$W = N \int_{\Phi_1}^{\Phi_2} i \, d\Phi \qquad (3\text{–}19)$$

Hence, the inductance of a circuit containing magnetic material is proportional to the slope $\frac{d\Phi}{di}$ of the Φ-i curve; and the energy stored from t_1 to t_2 is proportional to the cross-hatched area in Fig. 3–6(b).

3–4. Capacitance, *C*.—In the earliest study of electricity, scientists were concerned with what is known as "static electricity" or stored electricity, in contrast with "dynamic electricity" or the flow of electric current along a wire. The device in which static electricity is stored is called a condenser or a capacitor.

A simple condenser consists of two metallic conductors separated by a dielectric. A commercial condenser takes the form, in ordinary engineering practice, of sheets of tin foil separated by sheets of paraffined paper, mica, or other good insulating material. For simplicity, the condenser is assumed to be initially uncharged unless otherwise stated.

(a) *Relation Between Charge and Charging Current:* It has been shown experimentally that the charge acquired by a condenser is proportional to the voltage across its plates, or

$$q = Ce_C \qquad (3\text{–}20)$$

where q = quantity of electricity stored, in coulombs;
e_C = voltage drop across the condenser, in volts;
C = capacitance of the condenser, in farads.

When a condenser is charging or discharging, the quantity of stored electricity will be changing by the same amount on each side of the condenser. This tendency will be manifested by a current flowing into one side and an equal current flowing out of the other side. If an electric current is regarded as a transfer of electricity along a conductor, the rate of flow of the quantity of electricity is

$$i = \frac{dq}{dt} = C\frac{de_C}{dt} \text{ amperes} \qquad (3\text{–}21)$$

For a given rate of change of voltage, the capacitance C is a measure of the circuit's ability to permit current flow. A condenser has a capacitance of 1 farad when the voltage across the condenser is varying at the rate of 1 volt per second and the current flowing is 1 ampere. In practice, especially in radio work, the farad is too large and the units microfarad (μf or mfd) = 10^{-6} farad and micromicrofarad ($\mu\mu$f or mmfd) = 10^{-12} farad are more suitable. By integrating expression (3–21), the voltage drop across C is found to be

$$e_C = \frac{1}{C}\int_0^t i\,dt = \frac{q}{C} \qquad (3\text{–}22)$$

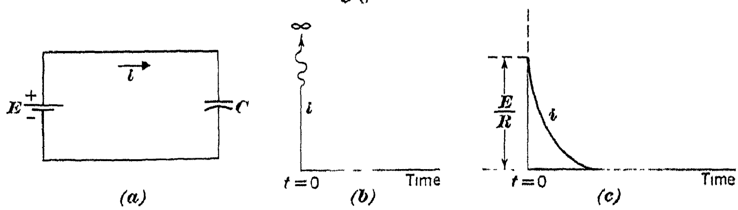


Fig 3–7

If an unvarying emf E is suddenly applied to the plates of an uncharged condenser,* as in Fig. 3–7(a), the negatively charged plate acquires a definite excess of electrons and the positively charged plate becomes deficient in electrons by a corresponding amount. A movement of electrons is necessary for this result to be accomplished. If it is assumed that the circuit possesses neither resistance nor self-inductance, the magnitude of the current is unlimited at the instant of applying the constant emf, as in Fig. 3–7(b). The condenser acquires the charge necessary for equilibrium in an infinitely short period of time. When equilibrium is reached, after $t = 0$, the current is zero since $\frac{dq}{dt} = 0$. Fig. 3–7(c) is a reproduction of an oscillogram showing the variation of current in a circuit which approaches the hypothetical C circuit just described. The presence of resistance R in the circuit is evident

* The symbol ⊣∈ has been used for capacitance, instead of ⊣⊢, to avoid confusion in circuits where the parallel bars are used to represent contacts. In this text, however, either symbol could have been used without confusion.

from the fact that the current starts at a definite value $\frac{E}{R}$ and decreases to practically zero in a very short time.

From the standpoint of dynamics, the driving emf must equal the counter voltage at every instant. The expression for dynamic equilibrium in a pure C circuit may be determined by studying Fig. 3–8. When the current is flowing in the direction of the current arrow, the top plate of the condenser connected to point 3 will have a plus polarity and the lower plate connected to point 4 will have a minus polarity. The voltage drop across C from 3 to 4 is

$$e_C = \frac{1}{C}\int_0^t i\,dt \tag{3–22}$$

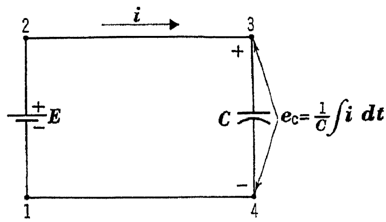


FIG 3–8

If the circuit is traced in the direction of the current arrow in Fig. 3–8, Kirchhoff's Emf Law gives for dynamic equilibrium

$$E = e_C \tag{3–23}$$

Again, in a pure C circuit, the voltage rise E is equal to the voltage drop e_C because not only their magnitudes are equal but also their polarities are the same. That is, the potential from the top wire to the bottom wire is from plus to minus for both E and e_C.

From equations (3–7*a*) and (3–7*b*), Kirchhoff's emf equation for the circuit shown in Fig. 3–8 may be written in either of the following forms:

$$E - e_C = 0 \tag{3–23a}$$

$$-E + e_C = 0 \tag{3–23b}$$

Example 3–2.—If the potential drop e_C across a condenser in a given circuit is varied according to the curves shown in Fig. 3–9, draw on the same graph the current flowing in the circuit during the corresponding time intervals.

Solution.—During the interval from t_1 to t_2, the voltage drop across the condenser is increasing at a constant rate. The slope $\frac{de_C}{dt}$ of the e_C-time curve is constant and positive. By applying equation (3–21), it is seen that the current flowing during this interval is positive and constant.

From t_3 to t_4, the voltage drop across the condenser is kept constant. The slope of the e_C-time curve is zero and the current in the circuit is zero.

When the voltage drop across the condenser decreases at a constant rate, the current in the circuit is made to flow in the opposite direction but has a constant value during the interval from t_5 to t_6.

Hence, the current flowing into or out of a condenser is directly proportional to the slope $\frac{de_C}{dt}$ of the e_C-time curve.

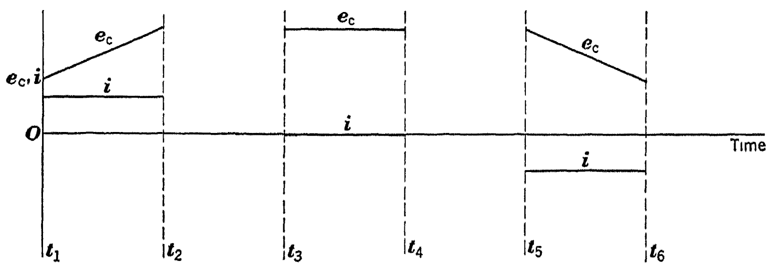


FIG. 3–9

(*b*) *Capacitance as an Intrinsic Property of a Condenser:* The capacitance calculated from the geometric dimensions of the condenser is sometimes called the geometric capacitance. In the rationalized MKS system, a condenser formed of two parallel plates which are separated by a dielectric material has the capacitance

$$C = \frac{A\epsilon}{d} \text{ farads} \tag{3–24}$$

where A = area of plates, in square meters;
ϵ = dielectric constant, in farads per meter;
d = separation of plates, in meters.

The capacitance, therefore, is directly proportional to the area of the plates and to the dielectric constant of the material separating them, and is inversely proportional to the distance between them. When the plates are separated by air, $\epsilon_0 = 8.85 \times 10^{-12}$ farad per meter and the capacitance is

$$C = 8.85 \times 10^{-12} \frac{A}{d} \text{ farad} \tag{3–25}$$

In order to increase the capacitance of a condenser a large number of plates are joined in multiple (and separated by very thin sheets of dielectric), as shown in Fig. 3–10.

(c) *Resistance of a Condenser:* The resistance to direct current of the dielectric material of a well-made condenser is practically infinite; and the resistance of the metal plates, connections, and leads is negligibly small. However, when alternating voltage is applied to a condenser and alternating current flows in and out of the condenser each half-cycle, the loss of power in the condenser is greater than that which can be accounted for on the basis of direct-current measurements. This loss is caused by imperfect behavior of the dielectric material, and will in general vary with frequency. To include this loss in circuit computations a hypothetical resistance is assumed to be in series with the condenser, the resistance

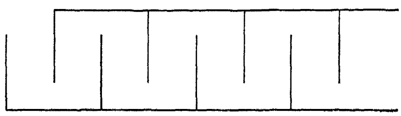


FIG 3–10

being just great enough to make the I^2R loss in it equal to the observed power loss in the actual condenser. It can be seen that this resistance may be considered as an *effective* resistance of the condenser to alternating current [see Art. 3–1(*d*)]. At power frequencies it is usually small enough to be neglected.

3–5. Summary.—(*a*) *The R Circuit:* In an R circuit, the resistance limits rate of movement of electric charge to a definite value by the equation

$$e_R = RI$$

This movement of electric charge persists at a particular rate as long as a given voltage drop e_R, in the direction of the current arrow, is maintained across R.

(*b*) *The L Circuit:* The self-inductance of the L circuit opposes any change in rate of movement of electric charge, but does not check the movement. The voltage drop across the inductance in the direction of the current arrow is

$$e_L = L\frac{di}{dt}$$

and L may be considered as a factor which merely fixes the rate at which the current can change.

(c) *The C Circuit:* The capacitance of the C circuit permits the flow of current as given by the expression

$$i = C\frac{de_C}{dt} = \frac{dq}{dt}$$

Also, the voltage drop across the condenser in the direction of the current arrow is

$$e_C = \frac{1}{C}\int_0^t i\,dt = \frac{q}{C}$$

PROBLEMS

3-1. The resistance of a copper rod 1 in. in diameter and 10 ft long is 0.000102 ohm. What will be its resistance when drawn into a wire 0.05 in. in diameter?

3-2. A 100-watt, 120-volt incandescent lamp is connected to a 120-volt d-c source. What is the current flowing in the circuit, and what is the resistance of the lamp?

3-3. A resistor of 10 ohms is connected to a 115-volt d-c supply. What current flows in the circuit, and what is the power, or the rate of change of energy dissipated in the resistor?

3-4. An incandescent lamp requires 0.5 amp to make it burn at full brilliancy. The hot resistance of the lamp is 240 ohms. What is the voltage applied across the terminals of the lamp?

3-5. The lamp in Problem 3-4 was left burning at the calculated voltage for 31 days. If the energy rate is 3 cents per kilowatt-hour, what is the cost for the energy used?

3-6. A current of 100 amp changes to 10 amp in $\frac{1}{100}$ second. What is the rate of change of the current during this time?

3-7. When a current of 10 amp flows through a coil, there is a flux of 10^6 maxwells; when the current is increased to 20 amp, the flux increases to 2×10^6 maxwells. What is the average rate of change of flux with current within these current limits?

3-8. A coil of 500 turns has a mean diameter of 5 cm and a length of 50 cm. Calculate its inductance.

3-9. When a current flowing in a coil is changing at the rate of 2×10^3 amp per second, it induces an emf of 1000 volts. What is the inductance of the coil?

3-10. At what rate must the potential across a 100-μf condenser be varied to have 5 amp flowing in the leads connected to the condenser?

3-11. A condenser with air as dielectric ($\epsilon_0 = 8.85 \times 10^{-12}$ farad per meter) has a capacitance of 0.01μf. If the dielectric is completely replaced by mica ($\epsilon = 5.8\ \epsilon_0$), what current will flow when the potential across the condenser is changing at a rate of 200 volts per second?

CHAPTER 4

HYPOTHETICAL CIRCUITS; SINUSOIDAL CURRENT AND VOLTAGE

In general, the wave forms of large commercial alternators do not differ materially from sinusoids. Analyses of practical problems based on the assumption of sinusoidal waves will generally be sufficiently accurate for engineering purposes. Sinusoidal waves and steady state conditions are assumed in the mathematical discus-

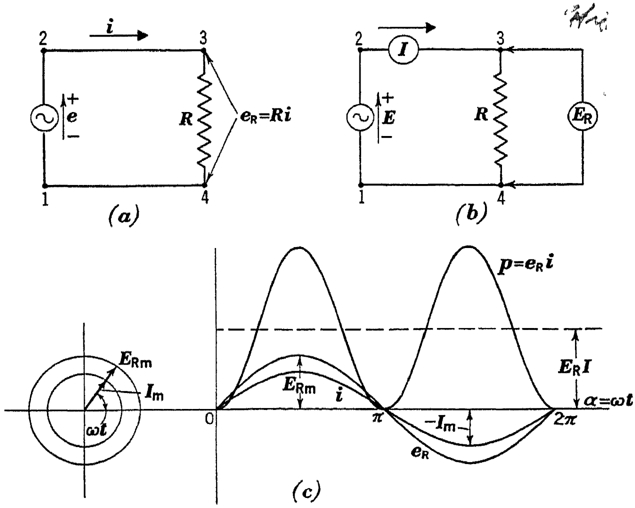


FIG. 4–1

sions of this chapter and those following unless it is otherwise stated. This should place no serious limitation on the mathematical analyses, since a non-sinusoidal wave may be separated into sinusoidal components (see Chapter 17).

The purpose of this chapter is to study sinusoidal waves in pure R, pure L, and pure C circuits. The relations given for the different elements may be considered as *experimental facts.*

4–1. The R Circuit.—Let a sinusoidal current be passed through an R circuit, Fig. 4–1(a). The current and voltage waves such as would be recorded by an oscillograph are given in Fig. 4–1(*c*). This may be considered as experimental fact. It can then be seen that the following conditions exist:

1. Both waves are sinusoidal and have the same frequency. The identity of their frequency may be concluded from the fact that the length on the abscissa scale, representing one cycle, is the same for both waves.
2. They are in phase for, when $t=0$, both waves cross the ωt-axis at $\omega t=0$ and are increasing positively.

Since the current is an alternating quantity, it may be well to consider first the significance of an arbitrary choice of the positive sense of current flow. It must be understood that the sense of the current arrow in Fig. 4–1(*a*) does not mean that the current is flowing in that direction at all times. It only means that, if at any instant of time the actual instantaneous current is flowing in the direction of the arrow (from 3 to 4 through R), then it is considered positive; and if at any instant of time the actual instantaneous current should be flowing in a direction opposite to that of the arrow (from 4 to 3 through R), then it must be considered negative. This is analogous to saying that if we agree in the arbitrary choice of passing the third floor of a building when going up as the positive direction, then passing the third floor when going down must be considered the negative direction.

It is desirable to consider next the significance of: (*a*) a voltage rise at the source and (*b*) a voltage drop in relation to the positive sense of current flow at the load. (See Appendix J.)

(*a*) *Voltage Rise at Source:* The arrow associated with e in Fig. 4–1(*a*) indicates only the direction of potential rise. That is, if at any instant point 2 has a plus polarity and point 1 has a minus polarity, then e is considered a positive potential rise in the direction of the arrow; and if at any instant point 2 has a minus polarity and point 1 has a plus polarity, then e must be considered a negative potential rise from 1 to 2 (a negative potential rise is equivalent to a potential drop).

It is not necessary to have both the arrow and the plus and minus signs to indicate the positive sense of voltage rise at a source. After the student has some experience in associating a plus instantaneous polarity with the arrowhead for a voltage rise at a source, the plus and minus signs for the polarities will be dropped.

(*b*) *Voltage Drop at Load:* The voltage drop across a load is considered to depend on the choice of the positive sense of current flow. Although the positive sense of current flow may be chosen quite arbitrarily, it is convenient to choose the current arrow in the same direction as that associated with e, Fig. 4–1(*a*). When the circuit is traced in the direction of the current arrow in Fig. 4–1(*a*), then the voltage drop across the resistance in the direction of the current arrow is from 3 to 4, and

$$e_R = Ri \tag{4–1}$$

The voltage $e_R = e_{34}$ is considered a voltage drop for, if at any instant i is flowing from 3 to 4 in the direction of the current arrow, then 3 is at a higher potential than 4; that is, 3 has a plus polarity and 4 has a minus polarity.

The relation shown in Fig. 4–1(*c*) may be given mathematically. Let it be assumed that the current flowing through the resistance is

$$i = I_m \sin \omega t \tag{4–2}$$

Then, since Ohm's Law must hold true for every instant, the voltage drop across the resistance from 3 to 4 is

$$e_R = Ri = RI_m \sin \omega t = E_{Rm} \sin \omega t \tag{4–3}$$

where $E_{Rm} = RI_m$.

Kirchhoff's Emf Law states that, at every instant, the voltage rise e must be equal to the voltage drop e_R when both are taken in the same direction around the circuit. Thus,

$$e = e_R \tag{4–4}$$

If $e = E_m \sin \omega t$, then

$$E_m \sin \omega t = E_{Rm} \sin \omega t \tag{4–5}$$

This relation indicates that, for a pure R circuit, *the wave representing e is the same as that representing e_R.* In other words, at every instant not only the voltage rise e and the voltage drop e_R are equal in magnitude but also their polarities are the same. That is, the potential from the top wire to the bottom wire in Fig. 4–1(*a*) is from plus to minus (or from minus to plus) for both e and e_R. Thus, e and e_R may be represented by the same vector in a vector diagram.

From equations (3–7a) and (3–7b), it may be seen that Kirchhoff's emf equation for the circuit shown in Fig. 4–1 may be written in either of the following forms:

$$e - e_R = 0 \tag{4–4a}$$

$$-e + e_R = 0 \tag{4–4b}$$

Thus, it can be seen that the current may be obtained from an expression involving the voltage rise e or from an equation involving the voltage drop e_R. For example,

$$i = \frac{e_R}{R} = \frac{e}{R} \tag{4–1a}$$

4–2. Analysis of the *R* Circuit.—Fig. 4–2 shows two vector diagrams for the R circuit, one using maximum values and the other using effective values. The length of each vector representing a maximum value is $\sqrt{2}$ times the length of the vector representing the effective value. There is no change in the phase angle between the voltage vector and the current vector. Since these revolving time vectors are to represent sine waves, it is theoretically preferable to use maximum values. However, in practice it is more convenient to use effective values in vector diagrams,

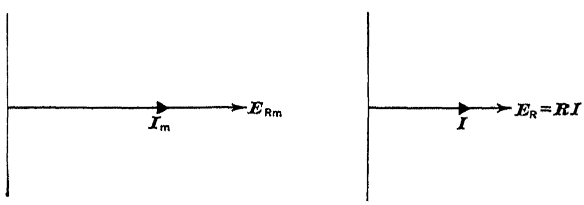


Fig. 4–2

since we are interested in checking meter readings (effective values) in a circuit with lengths of vectors obtained from a vector diagram. For a circuit containing resistance only, the voltage drop E_R is equal to the voltage rise E of the generator, and they are always in phase with the current. The two vectors E_R and I_m revolve together in the counter-clockwise direction.

Ohm's Law for an R circuit and power and energy in such a circuit will now be considered.

(*a*) *Ohm's Law for an R Circuit:* From the foregoing discussion, it can be seen that an ammeter reading of the circuit current and a voltmeter reading of the voltage drop across R, Fig. 4–1(b), will conform to the relation

$$E_R = RI \tag{4-6}$$

or

$$I = \frac{E_R}{R} \tag{4-7}$$

(b) *Power in an R Circuit:* For an R circuit, the voltage and current are in phase. The expressions for instantaneous power are:

$$p = ei \text{ (generated power)}$$
$$p = e_R i \text{ (absorbed power)}$$

Let it be assumed that the expressions for the voltage drop and current are:

$$e_R = E_{Rm} \sin \alpha$$
$$i = I_m \sin \alpha$$

Then the instantaneous power absorbed by the resistance, which is $e_R i = ei$, is

$$p = e_R i = E_{Rm} I_m \sin^2 \alpha = E_{Rm} I_m \frac{1 - \cos 2\alpha}{2}$$

or

$$p = E_R I - E_R I \cos 2\alpha \tag{4-8}$$

Also, the average power is

$$P = \frac{1}{2\pi} \int_0^{2\pi} p \, d\alpha = \frac{E_R I}{2\pi} \int_0^{2\pi} (1 - \cos 2\alpha) \, d\alpha = E_R I \tag{4-9}$$

That is, when the voltage and current are in phase, the average power absorbed by R is equal to the product of the effective values of voltage and current. Equation (4–9) gives the power in watts if E_R and I are in volts and amperes. The power curve for this case is shown in Fig. 4–1(c). During the entire first half-cycle both the voltage and the current are positive. During the entire second half-cycle, both the voltage and the current are negative. For every cycle of either voltage or current, the power curve touches the α-axis twice. Hence, the instantaneous power is zero twice in every cycle, but no part of the power curve is below the α-axis. This indicates that no energy is returned to the source. The energy delivered to the resistance is immediately changed into heat, and the process is not reversible.

It can be seen from equation (4–8) that the instantaneous power consists of a constant term $E_R I$ and a double frequency quantity $E_R I \cos 2\alpha$. The average power taken by an R circuit can also be written as follows:

$$P = E_R I \cos \theta$$

where $\cos \theta$ and θ are called the *power factor* and *power factor angle*, respectively. Since the voltage and current are sinusoidal and in phase, $\theta = 0$ and the power factor of an R circuit is unity. Then,

$$P = E_R I = I^2 R \tag{4–10}$$

This is represented by the average ordinate of the power curve. It is shown in Fig. 4–1(*c*) as the displacement of the power axis from the ωt-axis.

(*c*) *Energy in an R Circuit:* The energy dissipated in an R circuit is in heat. It is equal to the product of the average power I^2R and the time. The expression for energy may be obtained by integrating the power over a given interval of time. The instantaneous change of energy, which is $\frac{dW}{dt} = p$, may be written as follows:

$$dW = p\,dt = e_R\, i\, dt = R\, i^2\, dt \tag{4–11}$$

By substituting $I_m \sin \omega t$ for i and integrating over one complete cycle, the expression for energy is found to be

$$W = \int_0^T p\,dt = \int_0^T RI_m^2 \sin^2 \omega t\, dt$$

or $W = RI_m^2 \int_0^T \frac{1 - \cos 2\omega t}{2}\, dt = RI^2\, T$ joules or watt-seconds (4–12)

where R, I, and T are in ohms, amperes, and seconds, respectively.

Example 4–1.—A 120-volt mazda lamp having a constant resistance $R = 144$ ohms and negligible inductance is connected to a 120-volt, 60-cycle outlet in your home. Determine: (*a*) the average power taken by the lamp; (*b*) the energy supplied to the lamp in 1 hour.

Solution.—(*a*) The current taken by the lamp may be calculated from Ohm's Law. Thus,

$$I = \frac{E}{R} = \frac{120}{144} = 0.834 \text{ amp}$$

If a wattmeter were connected in the circuit, it would read the average power, which is

$$P = EI = 120 \times 0.834 = 100 \text{ watts}$$

(*b*) The energy supplied to the lamp in 1 hour may be obtained by taking the product of watts and time. It is

$$W = RI^2T = 144 \times 0.834^2 \times 3600 = 360{,}000 \text{ joules} = 100 \text{ watt-hours}$$

4–3. The L Circuit.—When the current is changing in an L circuit, Fig. 4–3(a), the applied emf will be equal and opposite to the emf of self-induction. The emf of self-induction is the only voltage in the circuit which opposes the change in current. (It corresponds to the back emf of a direct-current motor. In such a case, the line supplies a voltage equal and opposite to the back emf before any current can flow in the armature.) If the current is sinusoidal, there will be an alternate increase and decrease in the energy stored in the magnetic field. The establishment of energy

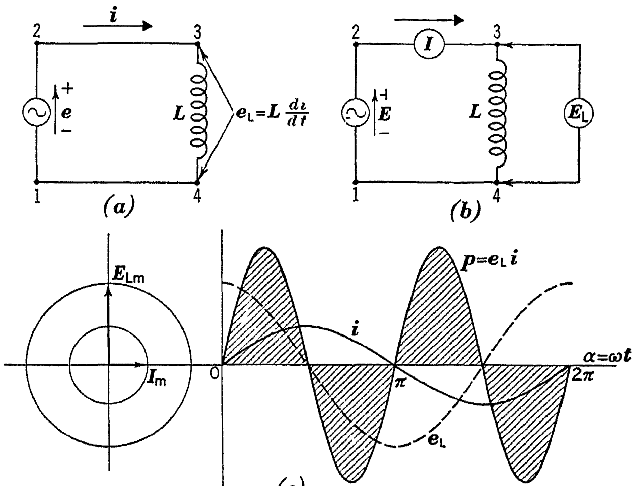


FIG. 4–3

in the field sets up an induced emf which is reactive in character and which retards the flow of current. As a result the current reaches a given value later than it would have reached that value if this induced emf were not present. Therefore, inductance in a circuit causes the current to lag behind the applied emf.

Let it be assumed that the current flowing in an L circuit is

$$i = I_m \sin \omega t$$

Then, by Lenz's Law, the induced emf which opposes the change of current is

$$e_i = -L\frac{di}{dt} = -\omega L I_m \cos \omega t$$

or $$e_i = X_L I_m \sin (\omega t - 90°) = E_{im} \sin (\omega t - 90°) \qquad \textbf{(4–13)}$$

where $X_L = \omega L$ and $E_{im} = X_L I_m$. Equation (4–13) states that the induced emf lags behind the current by 90 degrees.

When the current is changing in a pure L circuit, the impressed emf is equal and opposite to the induced emf. Thus,

$$e = -e_i \tag{4–14}$$

The voltage drop in Fig. 4–3(a) across the inductance from 3 to 4 is in the direction of the current arrow, and $e_L = e_{34}$. From Kirchhoff's Emf Law,

$$e = e_L \tag{4–15}$$

Hence,

$$e_L = -e_i = L\frac{di}{dt} \tag{4–16}$$

or

$$e_L = E_{Lm} \sin(\omega t + 90°) \tag{4–17}$$

where $E_{Lm} = X_L I_m$. From equations (4–15) and (4–17), the impressed emf has the form

$$e = E_m \sin(\omega t + 90°) = E_{Lm} \sin(\omega t + 90°) \tag{4–18}$$

where $E_m = X_L I_m$.

Equation (4–18) states that the impressed emf or the voltage drop leads the current by 90 degrees. This is equivalent to saying that, in a pure L circuit, the current i lags behind the impressed emf e or the voltage drop e_L by 90 degrees. The current and voltage waves in Fig. 4–3(c), which are like those that would be recorded by an oscillograph, show this relationship. This may be considered as an experimental fact.

It should be noted from equation (4–18) that for a pure L circuit the voltage rise e is equal to the voltage drop e_L because, at each instant, not only their magnitudes are equal but also the potential from the top wire to the bottom wire of Fig. 4–3(a) is from a plus polarity to a minus polarity (or from a minus polarity to a plus polarity) for both e and e_L. Thus, e and e_L may be represented by the same vector in a vector diagram.

From equations (3–7a) and (3–7b), Kirchhoff's emf equation for the circuit shown in Fig. 4–3 may be written in either of the following forms:

$$e - e_L = 0 \tag{4–15a}$$

$$-e + e_L = 0 \tag{4–15b}$$

4–4. Analysis of the L Circuit.—Ohm's Law for an L circuit and power and energy in such a circuit will now be considered.

(*a*) *Ohm's Law for an L Circuit:* If a voltmeter and an ammeter are connected to read the voltage drop across L and the current, Fig. 4–3(*b*), the readings will have the following relationship:

$$E_L = X_L I \tag{4–19}$$

or

$$I = \frac{E_L}{X_L} \tag{4–20}$$

where $X_L = \omega L = 2\pi f L$ is called the *inductive reactance* of the circuit. The inductive reactance is the choking effect or the opposition to the flow offered by the inductance. If E_L and I are in volts and amperes, respectively, then X_L is in ohms and L is in henrys. The current is directly proportional to the voltage drop E_L and is inversely proportional to both the frequency and the self-inductance.

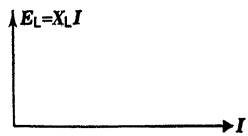


Fig. 4–4

Equation (4–19) is a relationship of magnitudes only, and it does not give the phase relation between E_L and I. The vector diagram of the L circuit in Fig. 4–4 shows that the inductive reactance drop E_L leads the current I by 90 degrees.

(*b*) *Power in an L Circuit:* When the voltage and current are 90 degrees out of phase, they are said to be in quadrature. The term quadrature applies whether the current is leading or lagging behind the voltage by 90 degrees. For an L circuit,

$$i = I_m \sin \alpha$$
$$e_L = E_{Lm} \sin (\alpha + 90°)$$

The expression for instantaneous power in the inductance L, which is $e_L i = e\, i$, is

$$p = e_L i = E_{Lm} I_m \sin \alpha \sin (\alpha + 90°)$$

$$= \frac{E_{Lm} I_m}{2} [\cos 90° - \cos (2\alpha + 90°)]$$

or

$$p = \frac{E_{Lm} I_m}{2} [0 + \sin 2\alpha] = E_L I \sin 2\alpha \tag{4–21}$$

Also, the average power is

$$P = \frac{1}{2\pi}\int_0^{2\pi} p\,d\alpha = \frac{E_L I}{2\pi}\int_0^{2\pi} \sin 2\alpha\,d\alpha = 0 \qquad (4\text{–}22)$$

The power curve is shown in Fig. 4–3(*c*). Energy is stored in the magnetic field while the current increases, and this energy is returned to the source when the current decreases. This process is automatically reversible. The cross-hatched area above the α-axis indicates that energy is received from the source of supply and stored in the magnetic field. The cross-hatched area below the α-axis indicates that energy is returned to the source of supply. The two areas are equal and opposite in sign; and the net energy delivered to the circuit over a complete cycle is zero. Thus, the average power taken by a circuit containing only inductance, averaged over a complete cycle, is zero. It should be noted, however, that although the average power is zero, the instantaneous power is zero only at four points during a cycle. There is an oscillation of power between the source and the inductance.

It can be seen from equation (4–21) that power in an L circuit is a double frequency quantity. The average power can also be written as follows:

$$P = E_L\, I \cos\theta = 0 \qquad (4\text{–}23)$$

This is so because the voltage and current are sinusoidal and in quadrature, in which case $\theta = 90°$ and the power factor is zero.

(*c*) *Energy in an L Circuit:* From the relation $\frac{dW}{dt} = p$ for the change in energy, the conditions in an L circuit may be represented as follows:

$$dW = p\,dt = e_L i\,dt = L\frac{di}{dt} i\,dt = Li\,di \qquad (4\text{–}24)$$

The total energy supplied to the circuit during any interval of time from t_1 to t_2 is

$$W = \int_{t_1}^{t_2} p\,dt = \int_{I_1}^{I_2} Li\,di = \tfrac{1}{2}L\,(I_2^2 - I_1^2) \qquad (4\text{–}25)$$

where L = constant self-inductance;
I_1 = current when $t = t_1$;
I_2 = current when $t = t_2$.

If I_2 is greater than I_1, the energy is positive and added to the circuit or stored in the magnetic field. If I_2 is less than I_1, the expression in equation (4–25) is negative and energy is received from the circuit or given up by the magnetic field. The total change in energy is independent of the path of change, or of the intermediate values of i during the interval of time chosen. It depends only on the terminal values of the current, or I_1 and I_2.

If $I_1 = 0$ when $t = t_1$ and I_2 is the final value of current, then the expression for energy becomes

$$W = \tfrac{1}{2}LI_2^2 \tag{4–26}$$

For sine waves of current and voltage some interesting conclusions may be drawn. Let

$$i = I_m \sin \omega t$$

Then from equation (4–24)

$$dW = \omega L I_m^2 \sin \omega t \cos \omega t \, dt$$

or

$$dW = \pi f L I_m^2 \sin 2\omega t \, dt \tag{4–27}$$

The energy change during any interval of time from t_1 to t_2 is

$$W = \pi f L I_m^2 \int_{t_1}^{t_2} \sin 2\omega t \, dt \tag{4–28}$$

For one complete cycle from 0 to T, the energy change is

$$W = \pi f L I_m^2 \int_0^T \sin 2\omega t \, dt = 0 \tag{4–29}$$

There is no total energy change. The same result will be obtained for a half-cycle.

For a quarter of a cycle from 0 to $\frac{T}{4}$,

$$W = \pi f L I_m^2 \int_0^{\frac{T}{4}} \sin 2\omega t \, dt$$

or

$$W = \frac{-\pi f L I_m^2}{2\omega} \Big[\cos 2\omega t \Big]_0^{\frac{T}{4}} = \tfrac{1}{2} L I_m^2 \text{ joules} \tag{4–30}$$

This amount of energy is stored in the magnetic field of the circuit. During the next quarter of the cycle from $\frac{T}{4}$ to $\frac{T}{2}$,

$$W = \pi f L I_m^2 \int_{\frac{T}{4}}^{\frac{T}{2}} \sin 2\omega t \, dt$$

or

$$W = \frac{-\pi f L I_m^2}{2\omega} \Big[\cos 2\omega t\Big]_{\frac{T}{4}}^{\frac{T}{2}} = -\tfrac{1}{2} L I_m^2 \text{ joules} \tag{4-31}$$

This indicates that the same amount of energy which was stored in the preceding quarter-cycle is returned to the supply. During the third quarter of the cycle, an amount of energy equal to $\frac{1}{2}LI_m^2$ is again stored. This amount is again returned to the supply during the fourth quarter of the cycle.

Although the net energy stored in an inductance over a complete cycle is zero, there is considerable surge of energy into and out of the circuit. The energy is stored in and released from the magnetic field surrounding and linking the circuit.

Example 4–2.—An air-core coil having a self-inductance of $L = 0.1$ henry and negligible resistance is connected to a 110-volt, 60-cycle source. What current will flow? If the current is written in the form $i = I_m \sin \omega t$, what is the energy stored in the magnetic field from the instant when $\omega t_1 = \frac{\pi}{6}$ to the instant when $\omega t_2 = \frac{3\pi}{2}$?

Solution.—The inductive reactance of the coil is

$$X_L = \omega L = 377 \times 0.1 = 37.7 \text{ ohms}$$

The current in quadrature with the voltage is

$$I = \frac{E}{X_L} = \frac{110}{37.7} = 2.92 \text{ amp}$$

The equation for the current may be written as

$$i = 2.92\sqrt{2} \sin \omega t$$

When $t = t_1$ . . . . . $I_1 = 2.92\sqrt{2} \sin \frac{\pi}{6} = 2.065$ amp

When $t = t_2$ . . . . . $I_2 = 2.92\sqrt{2} \sin \frac{3\pi}{2} = -4.13$ amp

Substituting values in equation (4–25) gives

$$W = \frac{0.1}{2} [(-4.13)^2 - (2.065)^2] = 0.642 \text{ joule}$$

4–5. The *C* Circuit.—The effect of capacitance in a circuit, as far as the phase relation between the current and the voltage is concerned, is just the opposite to that of inductance. The opposing voltage of a condenser is given by the value

$$e_C = \frac{q}{C} \qquad (4\text{–}32)$$

If the charge is zero, as in the case of an uncharged condenser, the opposing voltage is zero. If the charge $q = \int i\,dt$ is a maximum, the opposing voltage is a maximum. There can be no charge in the condenser until there has been a current flow, since $q = \int i\,dt$; the current therefore leads the charge. As the voltage

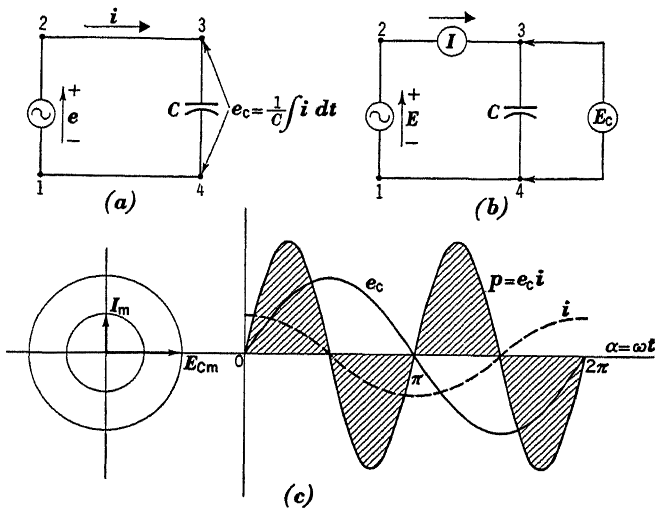


FIG. 4–5

drop across the condenser is $e_C = \frac{q}{C}$, the current will also lead this voltage. The effect of the opposing voltage on current flow is similar to the opposition which a coil spring offers when it is being stretched more and more.

To find the *phase relationship* between the voltage across the condenser and the current, it is convenient to start with an assumed wave form for the voltage. If $e_C = E_{Cm} \sin \omega t$ is the voltage drop across C from 3 to 4 in the direction of the current arrow, as in Fig. 4–5(a), the movement of electricity is expressed by the equation

$$i = C\frac{de_C}{dt} = \omega C E_{Cm} \cos \omega t = \omega C E_{Cm} \sin (\omega t + 90°) \qquad (4\text{–}33)$$

With an increasing voltage, the greater the rate of change of voltage the more quickly the condenser must be charged, and hence the greater will be the current flowing into it. With a decreasing voltage, the greater the rate of change of voltage the more quickly the condenser must discharge, and hence the greater will be the current. Also, the more rapidly the voltage alternates ($\omega = 2\pi f$) the greater will be the quantity of electricity charged and discharged per second, and the greater will be the flow of current [note that ω is a coefficient in equation (4–33)]. It is evident from equation (4–33) that capacitance causes the current to lead the voltage drop across a perfect condenser by 90°. This relationship is shown in Fig. 4–5(c), where the voltage and current waves are those which would be recorded by an oscillograph. The relationship also may be considered as an experimental fact.

From Kirchhoff's Emf Law, the potential rise e of the generator is equal to the voltage drop $e_C = e_{34}$, or

$$e = e_C \qquad (4\text{–}34)$$

This can be seen from Fig. 4–5(a). The equality in magnitudes is obvious. At any instant, when point 2 has a plus polarity the top plate connected to point 3 also has a plus polarity. When point 2 has a minus polarity, point 3 will have a minus polarity at the same instant. Thus, for a pure C circuit, e and e_C may be represented by the same sine wave and by the same vector in a vector diagram.

From equations (3–7a) and (3–7b), it is seen that Kirchhoff's emf equation for the circuit shown in Fig. 4–5 may be written in either of the following forms:

$$e - e_C = 0 \qquad (4\text{–}34a)$$

$$-e + e_C = 0 \qquad (4\text{–}34b)$$

4–6. Analysis of the *C* Circuit.—An alternating current does not actually flow conductively through the insulation of a condenser. With this point of view, a perfect condenser offers an infinite resistance to alternating current as well as to direct current. In an alternating-current circuit, however, the condenser is alternately charged and discharged; and a quantity of electricity flows alternately into the positive plate and then out again. This quantity of electricity, which flows to charge and to discharge the condenser, constitutes the alternating current. An ammeter placed in the line to such a condenser indicates a current.

Ohm's Law for a C circuit and power and energy in such a circuit will now be considered.

(*a*) *Ohm's Law for a C Circuit:* The readings of the voltmeter and ammeter connected as shown in Fig. 4–5(*b*) will be such that

$$E_C = X_C I \tag{4–35}$$

or

$$I = \frac{E_C}{X_C} \tag{4–36}$$

where $X_C = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C}$ is called the *capacitive reactance.* If E_C and I are in volts and amperes, respectively, then X_C is in ohms and C is in farads. The current is directly proportional to the applied emf or the voltage drop across the condenser; that is, $I = E\omega C = E_C \omega C$. The current is also proportional to both f and C.

Equation (4–35) is a relationship of magnitude only and it does not give the phase relation between E_C and I. The vector diagram for the C circuit in Fig. 4–6 shows that the capacitive reactance drop $E_C = -X_C I$ (since $E_L = +X_L I$) lags behind I by 90 degrees.

(*b*) *Power in a C Circuit:* The current in a C circuit leads the voltage drop by 90 degrees. Then

$$i = I_m \sin \alpha$$
$$e_C = E_{Cm} \sin (\alpha - 90°)$$

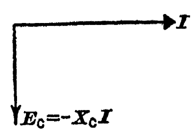


FIG. 4–6

The expression for instantaneous power taken by the condenser, or $e_C i = e\, i$, is

$$p = e_C i = E_{Cm} \sin (\alpha - 90°)\, I_m \sin \alpha$$

or

$$p = \frac{-E_{Cm} I_m}{2} \sin 2\alpha \tag{4–37}$$

The average power taken over one cycle is

$$P = \frac{1}{2\pi}\int_0^{2\pi} p\, d\alpha = 0 \qquad (4\text{–}38)$$

It is also zero when written

$$P = E_C I \cos\theta \qquad (4\text{–}39)$$

since the voltage and current are sinusoidal and in quadrature, $\theta = 90°$ and the power factor $\cos\theta$ of the C circuit is zero.

From equation (4–37) the power in a C circuit is a double frequency quantity. Energy is stored in the dielectric field when the voltage is increasing, and this energy is returned to the source when the voltage decreases. This process is also reversible. Although the average power is zero, the instantaneous power is zero only at four instants during each cycle.

(*c*) *Energy in a C Circuit:* The total energy taken by a perfect condenser during any interval of time from t_1 to t_2 is

$$W = \int_{t_1}^{t_2} p\, dt = \int_{t_1}^{t_2} e_C i\, dt = \int_{t_1}^{t_2} Ce_C \frac{de_C}{dt}\, dt$$

or

$$W = \int_{E_1}^{E_2} Ce_C\, de_C = \frac{C}{2}\left[E_2^2 - E_1^2\right] \text{ joules} \qquad (4\text{–}40)$$

where C = capacitance of condenser, in farads;
E_1 = condenser voltage when $t = t_1$;
E_2 = condenser voltage when $t = t_2$.

The total change in energy is independent of the path of the change of the voltage across the condenser. It depends only on the initial and final values of the voltage, or E_1 and E_2.

If $E_1 = 0$ when $t = t_1 = 0$, the expression in equation (4–40) becomes

$$W = \tfrac{1}{2} C E_2^2 \text{ joules} \qquad (4\text{–}41)$$

If the voltage and current waves are sinusoidal, some interesting conclusions may be drawn. Let

$$e_C = E_{Cm} \sin \omega t$$
$$i = I_m \sin(\omega t + 90°) = I_m \cos \omega t$$

Then the equation for the energy stored in the condenser during any interval of time from t_1 and t_2 is

$$W = \int_{t_1}^{t_2} p\, dt = \int_{t_1}^{t_2} e_C i\, dt \tag{4-42}$$

For a complete cycle from 0 to T,

$$W = \int_0^T E_{Cm} \sin \omega t\, I_m \cos \omega t\, dt \tag{4-43}$$

Since $I_m = \omega C E_{Cm}$, equation (4–43) becomes

$$W = \omega C E_{Cm}^2 \int_0^T \sin \omega t \cos \omega t\, dt$$

or

$$W = C E_{Cm}^2 \left[\frac{\sin^2 \omega t}{2}\right]_0^T = 0 \text{ joules} \tag{4-44}$$

For one-quarter of a cycle from 0 to $\frac{T}{4}$, the energy stored is

$$W = \omega C E_{Cm}^2 \int_0^{\frac{T}{4}} \sin \omega t \cos \omega t\, dt = \tfrac{1}{2} C E_{Cm}^2 \text{ joules} \tag{4-45}$$

When the area under the power curve in Fig. 4–5(c) is positive, energy is stored in the dielectric field. This energy is returned to the generator during an interval when the area is negative.

4–7. Summary.—(a) *Element R:* The current I flowing through an element R is always in phase with the voltage drop E_R in the direction of the current arrow. The magnitude of the current is given by the relation

$$I = \frac{E_R}{R}$$

The average power taken by an element R is

$$P = E_R I = I^2 R$$

The energy consumed by an element R is

$$W = I^2 R T$$

This energy is dissipated in heat.

(*b*) *Element L:* The voltage drop E_L in the direction of the current arrow through an element L is always 90 degrees ahead of the current flowing through the element, or I lags behind E_L by 90 degrees. The magnitude of the current is obtained from the relation

$$I = \frac{E_L}{X_L}$$

The average power taken by the element L over a complete cycle is zero. Energy is stored in the element when the current is increasing, while energy is given up by the element when the current is decreasing. In a circuit where L is a constant, the total change in energy depends only on the initial and final values of the current in the expression

$$W = \tfrac{1}{2}L\,(I_2^2 - I_1^2)$$

This change is independent of the intermediate values of i during the interval of time chosen. When I_1 is zero, then (with $I = I_2$)

$$W = \tfrac{1}{2}LI^2$$

(*c*) *Element C:* The voltage drop E_C in the direction of the current arrow through an element C is always 90 degrees behind the current causing the voltage drop, or I leads E_C by 90 degrees. The current has a magnitude of

$$I = \frac{E_C}{X_C}$$

The power taken by the element C over a complete cycle is zero. Energy is stored in the element when the voltage across the element is increasing, while energy is given up by the element when the voltage is decreasing. The expression for the total change in energy is

$$W = \tfrac{1}{2}C\,(E_2^2 - E_1^2)$$

where E_1 and E_2 are the initial and final values of the voltage, respectively. This change is independent of the path of change of the voltage. When E_1 is zero, then (with $E = E_2$)

$$W = \tfrac{1}{2}CE^2$$

PROBLEMS

4–1. If i is a function of time and e is a function of time, give the relation between e and i in: (a) an R circuit; (b) an L circuit; and (c) a C circuit.

4–2. An incandescent lamp (usually considered to be a pure resistive load) takes 2 amp from 220-volt, 60-cycle mains. If the lamp resistance is assumed to remain constant, what current will it take from 220-volt, 25-cycle mains? What will be the current from 110-volt, 60-cycle mains?

4–3. An incandescent lamp (considered a pure resistive load) is connected to a 110-volt, 60-cycle source. An ammeter in the circuit reads $\frac{1}{2}$ amp. If the lamp resistance is assumed to remain constant, what is the resistance of the lamp? What will be the reading of a wattmeter connected in the circuit? Draw a vector diagram, using effective values Also, draw two sine waves representing the voltage and the current, and indicate the phase difference.

4–4. Consider the R circuit shown in Fig. 4–1(a) If $R = 2$ ohms and $e = 10 \sin 377t$, find the numerical values of e, i, e_R, and e_{34} at the instants at which: (a) $t = \frac{1}{480}$ sec and (b) $t = \frac{1}{96}$ sec Check equations (4–1) and (4–4) for both magnitude and polarity at the top wire.

4–5. If the lamp in Problem 4–3 is left burning on the circuit for 10 hours, what energy is dissipated in the lamp?

4–6. A coil consists of 100 turns It is wound uniformly in a single layer on a non-magnetic core. The core is 20 cm long and 2 cm in diameter. When the current flowing in the coil is changing at a rate of 100 amp per sec, what is the emf induced in the coil?

4–7. A coil has an inductance of 0 5 henry (or 500 millihenrys). If the coil is connected to a 60-cycle source, what is its reactance?

4–8. The current flowing in a resistanceless air-core coil is semicircular, as shown in Fig. 4–7. Sketch on the same graph the wave representing the voltage drop e_L in the direction of the current arrow. Do e_L and i have the same wave form? Does e_L lead or lag behind the current by 90 degrees? Is it correct to consider angles of lead or lag in the case when the waves are dissimilar or non-sinusoidal?

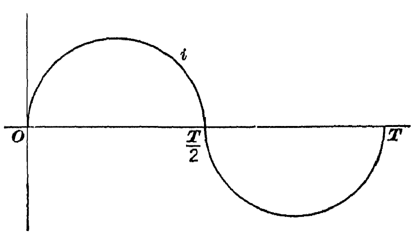


FIG. 4–7

4–9. If the emf applied to an L circuit is $e = E_m \sin \omega t$, Fig. 4–3(a), what is the expression for the current? Also, draw the vector diagrams showing the phase relation between the vectors representing e and i at $t = 0, \frac{T}{8}, \frac{T}{4}, \frac{T}{2}$, and T. What can you say about the phase angles at the different instants?

4–10. Repeat Problem 4–9 if the applied emf is

$$e = E_m \sin(\omega t + \theta)$$

where θ is an arbitrary angle.

4–11. Consider the pure L circuit shown in Fig. 4–3(*a*). If $L = 0.001$ henry and $e = 10 \sin 377t$, find the numerical values of e, i, e_L, and e_{34} at the instants at which: (*a*) $t = \frac{1}{480}$ sec and (*b*) $t = \frac{1}{96}$ sec. Check equations (4–15) and (4–16) for both magnitude and polarity at the top wire.

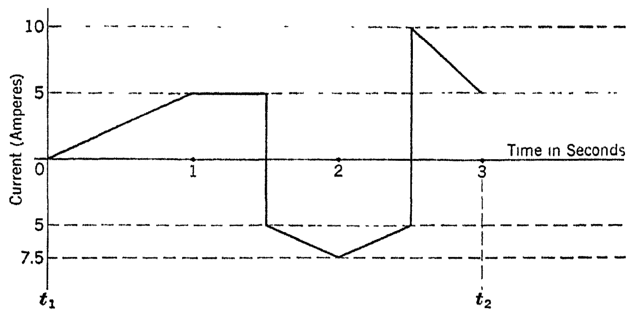


FIG. 4–8

4–12. The current in an air-core coil having an inductance of 0.3 henry is varied according to the graph shown in Fig. 4–8. What is the energy stored in the magnetic field from the instant t_1 to the instant t_2?

4–13. Consider the pure C circuit shown in Fig. 4–5(*a*). If $C = 100$ μf (or 100×10^{-6} farad) and $e = 10 \sin 377t$, find the numerical values of e, i, e_C, and e_{34} at the instants at which: (*a*) $t = \frac{1}{480}$ sec and (*b*) $t = \frac{1}{96}$ sec. Check equation (4–34) and the first part of equation (4–33) for both magnitude and polarity at the top wire.

4–14. A condenser for which $C = 5$ μf (or 5×10^{-6} farad) is connected to a 100-volt, 500-cycle source. What is the current flowing in the circuit?

4–15. A condenser has a capacitance of 20 μf. What must be the frequency of an emf applied to the condenser so that the capacitive reactance is 8 ohms?

4–16. What is the energy stored in a condenser of 0.2 farad when it is connected to a battery of 100 volts? The condenser is assumed to have no initial charge.

4–17. If the curve in Fig. 4–8 represents the variation of voltage across a condenser of 0.01 farad, what energy is stored in the condenser from the end of 1 second to the end of 3 seconds? Assume that each vertical division is equal to 10 volts.

4–18. The current in a pure element circuit is $i = I_m \sin \omega t$. Determine the expression for the voltage when the element is: (*a*) R; (*b*) L; (*c*) C.

CHAPTER 5

SERIES CIRCUITS

We have already seen how circuits containing resistance only, inductance only, or capacitance only behave when a sinusoidal emf is impressed upon them. Such circuits are hypothetical in that we cannot have, for example, a pure inductance; the resistance would actually be inseparably associated with the inductance. A *series circuit* is defined as a circuit in which two or more elements are connected end to end and the current is the same in all parts of the circuit. Thus, an inductive coil may be considered as a

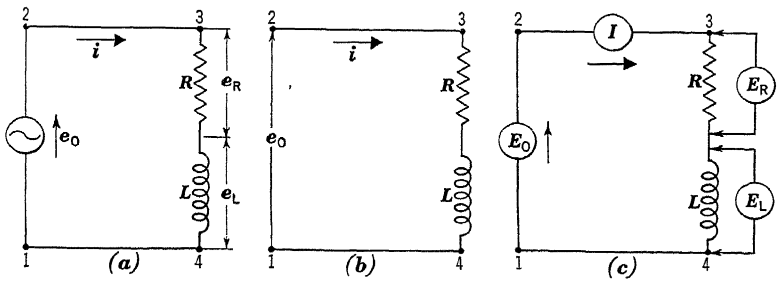


FIG. 5–1

circuit containing resistance and inductance in series since the same current flows in every turn of the coil.* Actually, the resistance R and the inductance L are distributed along the length of the wire forming the coil; but, for the present, it is sufficient to consider that all the resistances are lumped together and all the inductances are lumped together. To consider the action of the coil we shall assume that a pure resistance is placed in series with a pure inductance. It is the purpose of this chapter to consider the action of circuits containing series combinations of resistance R, inductance L and capacitance C.

5–1. Inductive Circuits.—An inductive circuit or an RL circuit is one which contains resistance and inductance in series. When a sinusoidal voltage is impressed upon a series circuit, such as is

* See Art. 7–6.

shown in Fig. 5–1, a sinusoidal current will flow. It is evident that at any instant the current must have the same value in every portion of the circuit. In order to know the value and the phase relation of the impressed emf with reference to the current and the other voltages in the circuit, resistance and inductance must be considered at the same time.

In the analysis of the circuits considered in this chapter, it is convenient to choose the positive sense of current flow in the same direction as that of the positive sense of voltage rise e_0 of the generator in Fig. 5–1. If the circuit is traced in the direction of the current arrow, the resistive drop $e_R = Ri$ is in phase with the current i, and the inductive drop $e_L = L\frac{di}{dt}$ is 90 degrees ahead

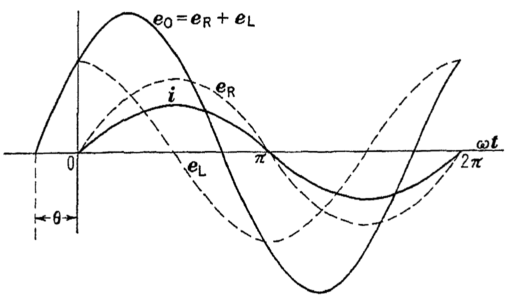


FIG. 5–2

of current i. It is reasonable to suppose that the applied emf e_0 leads the current i by some angle between 0° and 90°. If a two-element oscillograph is connected in the circuit, one element being in series with the circuit for i and the other element being across the terminals 1 and 2 for e_0, the phase relation between e_0 and i is recorded; the result is indicated in Fig. 5–2.

It can be seen from Fig. 5–1(a) that the total voltage drop* from 3 to 4 is the sum of the voltage drops e_R and e_L, or

$$e_{34} = e_R + e_L \tag{5–1}$$

By Kirchhoff's Emf Law, the algebraic sum of the instantaneous values of the voltage rises is equal to the algebraic sum of the

* Some authors prefer the use of the symbol v or V to represent voltage drops.

instantaneous values of the voltage drops when both are taken in the same direction around the circuit. Then,

$$e_0 = e_{34} \tag{5-2}$$

which indicates that they may be represented by the same sine wave and by the same vector in a vector diagram. The fundamental differential equation* for an RL series circuit is, therefore,

$$e_0 = e_R + e_L$$

or

$$e_0 = Ri + L\frac{di}{dt} \tag{5-3}$$

Since the current is the common quantity, it is convenient to assume a form for the current and then solve for the applied emf e_0. Let the current $i = I_m \sin \omega t$ be passed through the circuit shown in Fig. 5-1. Then, from equation (5-3),

$$e_0 = RI_m \sin \omega t + L\frac{d}{dt}(I_m \sin \omega t) = I_m (R \sin \omega t + \omega L \cos \omega t)$$

$$= I_m (R \sin \omega t + X_L \cos \omega t) = I_m Z\left(\sin \omega t \cdot \frac{R}{Z} + \cos \omega t \cdot \frac{X_L}{Z}\right)$$

$$= I_m Z (\sin \omega t \cos \theta + \cos \omega t \sin \theta) = I_m Z \sin (\omega t + \theta)$$

or

$$e_0 = E_{0m} \sin (\omega t + \theta) \tag{5-4}$$

where (see Fig. 5-4)

$$Z = \sqrt{R^2 + X_L^2}$$

$$\theta = \tan^{-1}\frac{X_L}{R}$$

$$E_{0m} = I_m Z$$

The applied emf has a maximum value of $E_{0m} = I_m Z$ and it leads the current by the phase difference angle θ. The quantity Z is called the *impedance* of the circuit. For the present, it may be considered as a factor by which the maximum value of the current is multiplied to get the maximum value of the voltage. It is similar to resistance and reactance, and its unit is the ohm. The quantity Z will be explained further in Art. 5-3.

In constructing Fig. 5-2 the reference i is drawn first. The resistive drop $e_R = Ri$ is proportional to the current at every

* The subscript 0 will be used to designate the applied emf or applied voltage considered as a voltage rise.

instant, and it is in phase with the current. The inductive drop $e_L = L\frac{di}{dt}$ is proportional to the time rate of change of the current, and it leads the current by 90 degrees. The applied emf e_0 is equal to the algebraic sum of the ordinates of the curves for e_R and e_L at every instant. The applied emf has the same frequency as the current, and it leads the current by the phase difference angle θ. It will be explained in Art. 5–5 that, since the voltage and current are sinusoidal, this is the power factor angle of the circuit.

For simplicity in appearance, Fig. 5–1(*b*) may be used to represent the circuit in Fig. 5–1(*a*).

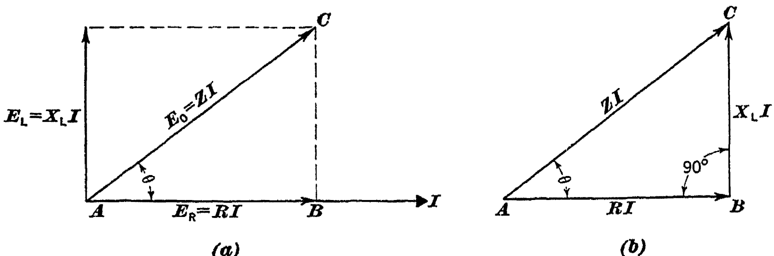


FIG. 5–3

5–2. Vector Diagram for an *RL* Circuit.—We are interested in vector diagrams with effective values, since they give us in a very simple manner the meter values of voltages and currents and the phase angles they make with each other. The current being the same in all parts of a series circuit, it is most convenient to use it as the reference vector (directed horizontally to the right). The vector diagram for the *RL* circuit shown in Fig. 5–1 is given in Fig. 5–3(*a*).

The effective voltages represented by RI, X_LI, and ZI are the values a voltmeter would read if it were connected across the resistance only, the inductance only, and the entire circuit, respectively. The resistance drop $E_R = RI$ is in phase with the current; and it is, therefore, laid off along the current vector. The inductance drop $E_L = X_LI$ leads the current by 90 degrees, and it is drawn 90 degrees ahead of the current vector. The generator voltage E_0 must be equal to the vector sum of these two voltages and leads the current by an angle θ. If the actual circuit is an inductive coil, the resistance and inductance cannot be separated as indicated in Fig. 5–1. Hence, the drops RI and X_LI cannot be measured by a voltmeter as shown in Fig. 5–1(*c*). The triangle ABC in Fig. 5–3(*b*) is called a *voltage triangle.*

5–3. Impedance Diagram for an RL Circuit.—Since the voltage triangle in Fig. 5–3(*b*) is a right triangle, the hypotenuse ZI is equal to the square root of the sum of the squares of the other two sides RI and X_LI; thus,

$$ZI=\sqrt{(RI)^2+(X_LI)^2} \tag{5–5}$$

or

$$Z=\sqrt{R^2+X_L^2} \tag{5–6}$$

Equation (5–6) states that, to obtain the impedance, it is necessary to combine resistance and reactance at right angles to each other, as shown in Fig. 5–4. The triangle $A'B'C'$ is called the *impedance triangle.* It can be obtained from the voltage triangle ABC by dividing each side by I.

An inspection of Fig. 5–4 shows that the angle θ, by which the current lags behind the applied emf, may be found as follows:

$$\tan\theta=\frac{X_L}{R}$$

$$\cos\theta=\frac{R}{Z}$$

$$\sin\theta=\frac{X_L}{Z}$$

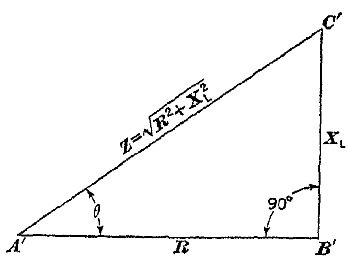


FIG. 5–4

5–4. Ohm's Law for an Inductive Circuit.—Ohm's Law for a direct-current circuit is $E=RI$. In alternating-current problems, the effect of inductance in a circuit must be considered. Since the voltage rise E_0 in Fig. 5–1 is equal to the voltage drop E_{34}, Ohm's Law when applied to an alternating-current circuit may be written as follows:

$$E_0=ZI=\left(\sqrt{R^2+X_L^2}\right)I \tag{5–7}$$

or

$$I=\frac{E_0}{Z}=\frac{E_0}{\sqrt{R^2+X_L^2}} \tag{5–7a}$$

The current is directly proportional to the voltage applied across the circuit and inversely proportional to the impedance of the

Example 5–1.—An air-core inductive coil, having a resistance of 3 ohms and an inductance of 10.6 millihenrys, is connected across a 60-cycle, 110-volt supply. (*a*) What are the impedance and power factor of the coil? (*b*) What is the reading of an ammeter placed in the circuit? (*c*) Find the average power which is read by a wattmeter in the circuit. (*d*) Draw the waves representing i, e_0, e_R, e_L, p_R, p_L, and p; and also draw the vector diagram.

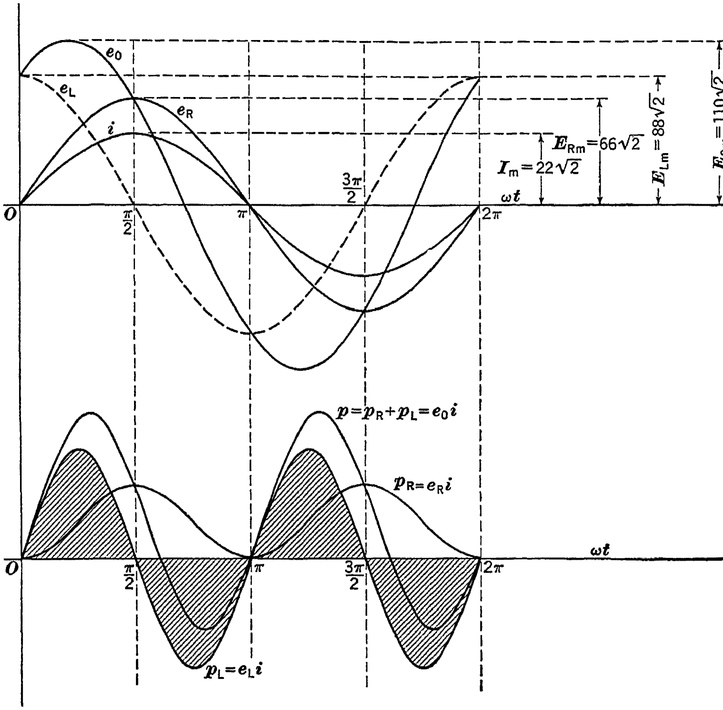


FIG. 5–5

Solution.—The inductive reactance is

$$X_L = 2\pi f L = 2\pi \times 60 \times 0.0106 = 4 \text{ ohms}$$

The impedance is

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{3^2 + 4^2} = 5 \text{ ohms}$$

The power factor is

$$\text{P.F.} = \cos\theta = \frac{R}{Z} = \frac{3}{5} = 0.6, \text{ lagging (inductive circuit)}$$

The ammeter reading is

$$I = \frac{E}{Z} = \frac{110}{5} = 22 \text{ amp}$$

The potential drops are:

$$RI = 3 \times 22 = 66 \text{ volts}$$
$$X_L I = 4 \times 22 = 88 \text{ volts}$$

The wattmeter reading is

$$P = E_0 I \cos\theta = 110 \times 22 \times 0.6 = 1452 \text{ watts}$$

or

$$P = I^2 R = 22^2 \times 3 = 1452 \text{ watts}$$

The waves representing i, e_0, e_R, e_L, p_R, p_L, and p are shown in Fig. 5–5.

Fig 5–5 shows that during the interval from $\omega t = 0$ to $\omega t = \frac{\pi}{2}$, energy is stored in the inductance L, this energy being represented by the positive area between the p_L-curve and the ωt-axis; while, during the interval from $\omega t = \frac{\pi}{2}$ to $\omega t = \pi$, the area between the p_L-curve and the ωt-axis is negative, the position of this part of the curve indicating that the same energy is released by the inductance. The p_R-curve is never below the ωt-axis, this condition indicating that energy is delivered to the circuit as a whole. This fact is also shown by the positive net area of the curve representing p

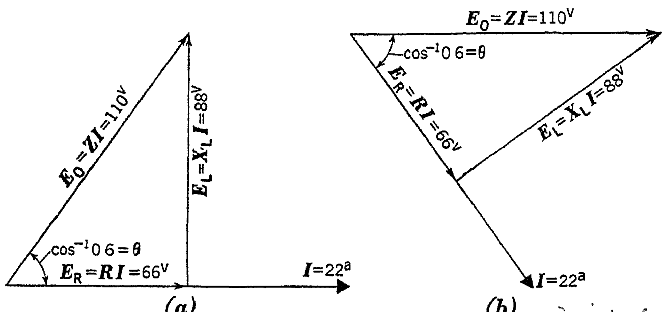


FIG. 5–6

The vector diagram* with I as reference is shown in Fig. 5–6(a); and that with E_0 as reference is shown in Fig. 5–6(b). These diagrams indicate that the current lags behind the applied emf by the power factor angle, which is $\theta = \cos^{-1} 0.6 = 53.2°$.

5–6. Current Locus of RL Circuit.—To better understand the RL circuit, it might be helpful to study the current variation as R or X_L is varied while the impressed emf and frequency remain constant. Any change in the elements of the circuit causes a corresponding change in the magnitude and phase angle of the current. The current variation may be graphically represented in polar coordinates (or in cartesian coordinates). The curve described by

* All voltage vectors must be drawn to the same scale, and all current vectors must be drawn to their own scale. The voltage scale and the current scale may be different.

the moving end of the current vector is called the *locus* of the current vector.

Whenever the elements of a circuit are changed, there appear transient terms which become zero in a very short time. Only the steady state conditions will be discussed in this chapter. We will now consider two cases: (*a*) constant R and variable X_L and (*b*) constant X_L and variable R.

(*a*) *Constant R and Variable X_L:* Fig. 5–7(*a*) shows a circuit with a fixed resistance R and a variable reactance X_L in series. The impressed emf and the frequency are held constant. A change in X_L means a change in the self-inductance L of the circuit. It is assumed here that where the inductance is varied the change in the resistance of the circuit is negligible.

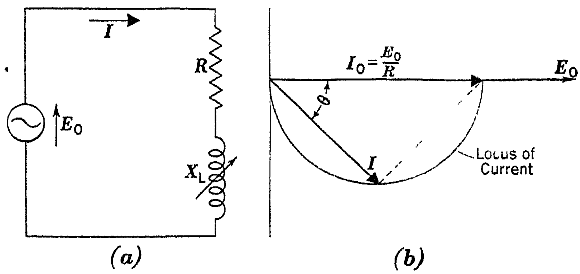


FIG. 5–7

If $X_L = 0$, the current is $I_0 = \frac{E_0}{R}$ and it is in phase with E_0 as indicated in Fig. 5–7(*b*). As X_L is increased from its zero value, the current I becomes less than $\frac{E_0}{R}$ in magnitude and lags more and more behind E_0. When X_L becomes great without limit, the power factor angle θ approaches 90° and the current approaches zero. It will now be shown that the locus of the current vector is the semicircle shown in Fig. 5–7(*b*).

The general expression for the current is

$$I = \frac{E_0}{Z}$$

Since $\cos\theta = \frac{R}{Z}$ or $Z = \frac{R}{\cos\theta}$, then

$$I = \frac{E_0}{R}\cos\theta \tag{5–12}$$

This is the polar equation of a circle with diameter $\frac{E_0}{R}$. As this is an inductive circuit, the current must always lag behind the applied emf; thus, the locus of the current is the semicircle given in Fig. 5–7(*b*). The advantage of the locus diagram (or circle diagram) is that it shows in a simple manner how the current varies with reactance in the circuit.

(*b*) *Constant X_L and Variable R:* A circuit containing a fixed X_L and a variable R is shown in Fig. 5–8(*a*). It is assumed that the impressed emf and the frequency are held constant. If $R=0$,

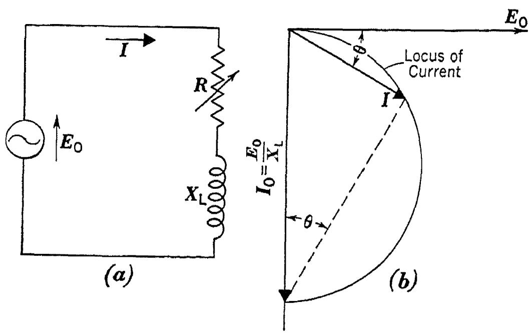


Fig. 5–8

the current is $I_0=\frac{E_0}{X_L}$ and it lags E_0 by 90 degrees as indicated in Fig. 5–8(*b*). As R is increased from its zero value, the magnitude of the current becomes less than $\frac{E_0}{X_L}$ and I becomes more nearly in phase with E_0. When R becomes great without limit, both the phase angle θ and the magnitude of the current approach zero. The determination of the locus for the current vector follows a procedure similar to that given in part (*a*). The expression for the current is

$$I=\frac{E_0}{Z}$$

Since $\sin\theta=\frac{X_L}{Z}$ or $Z=\frac{X_L}{\sin\theta}$, then

$$I=\frac{E_0}{X_L}\sin\theta \tag{5–13}$$

This is the polar equation of a circle with diameter $\frac{E_0}{X_L}$. As this is an inductive circuit, the locus diagram (or circle diagram) is that shown in Fig. 5–8(b).

5–7. Inductive Circuits in Series.—Let n air-core inductive coils be connected in series, as shown in Fig. 5–9(a). This can be represented by a circuit with several lumped resistances and

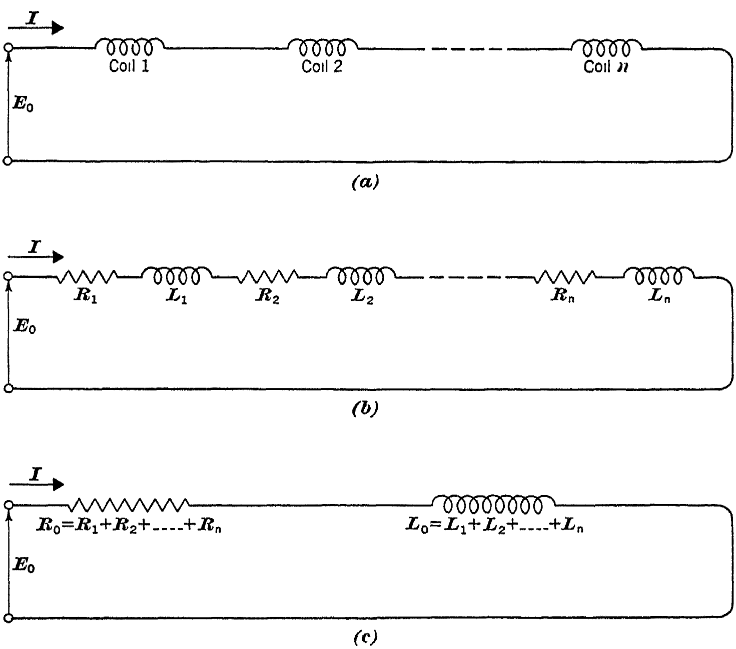


FIG. 5–9

inductances in series, as given in Fig. 5–9(b). If a current $i = I_m \sin \omega t$ flows in this circuit, the RI drops will be in phase with the current and the $X_L I$ drops will be 90 degrees ahead of the current. It will be shown that Fig. 5–9(b) can be reduced to a simple RL circuit, as represented in Fig. 5–9(c).

According to Kirchhoff's Emf Law,

$$e_0 = R_1 i + L_1 \frac{di}{dt} + R_2 i + L_2 \frac{di}{dt} + \ldots + R_n i + L_n \frac{di}{dt} \qquad (5\text{–}14)$$

Since the same current flows through all parts of the circuit, we may collect terms, as follows:

$$e_0 = (R_1 + R_2 + \ldots + R_n)i + (L_1 + L_2 + \ldots L_n)\frac{di}{dt}$$
$$= (R_1 + R_2 + \ldots + R_n)I_m \sin \omega t + \omega (L_1 + L_2 + \ldots + L_n)I_m \cos \omega t$$

or $$e_0 = (R_1 + R_2 + \ldots + R_n)\, I_m \sin \omega t + (X_{L1} + X_{L2} + \ldots + X_{Ln})\, I_m \cos \omega t \quad (5\text{–}15)$$

where $X_{Ln} = \omega L_n$.

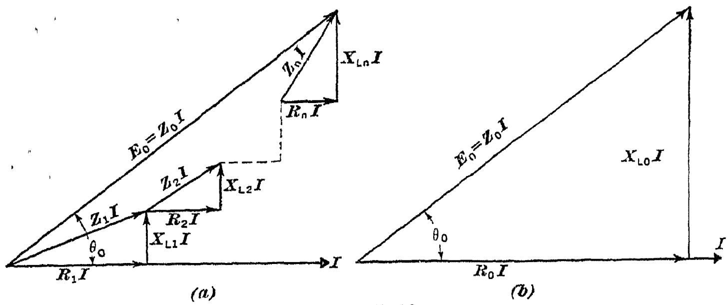


FIG. 5–10

If we let $R_0 = R_1 + R_2 + \ldots + R_n$ and $X_{L0} = X_{L1} + X_{L2} + \ldots + X_{Ln}$, then

$$e_0 = I_m[R_0 \sin \omega t + X_{L0} \cos \omega t] = Z_0 I_m \sin (\omega t + \theta_0) \quad (5\text{–}16)$$

where

$$Z_0 = \sqrt{R_0^2 + X_{L0}^2}$$

$$\theta_0 = \tan^{-1} \frac{X_{L0}}{R_0}$$

For sinusoidal voltage and current, the power factor of the entire circuit is

$$\text{P.F.} = \cos \theta_0 = \frac{R_0}{Z_0} \quad (5\text{–}17)$$

The total power absorbed is

$$P = I^2 R_0 = I^2 R_1 + I^2 R_2 + \ldots + I^2 R_n \text{ watts} \quad (5\text{–}18)$$

or the total power is the sum of the I^2R losses in the different resistive parts of the circuit.

It can be concluded from equations (5–15), (5–16), and (5–18) that the circuits in Figs. 5–9(b) and 5–9(c) are equivalent to the extent that each requires the same voltage for the given current and, each having the same phase angle, each absorbs the same power. This is a definition quite generally used for equivalence.

When the applied emf and voltage drops are represented by vectors, as in Fig. 5–10(a), all the RI vectors are in the same direction, and they may be added together Thus,

$$R_1I + R_2I + \ldots + R_nI = R_0I \tag{5-19}$$

Similarly, the X_LI vectors are added together to give

$$X_{L1}I + X_{L2}I + \ldots X_{Ln}I = X_{L0}I \tag{5-20}$$

Fig. 5–10(a), which is a vector diagram of Fig. 5–9(b), can then be reduced to Fig. 5–10(b). Hence,

$$E_0 = IZ_0 = I\sqrt{(R_1 + R_2 + \ldots + R_n)^2 + (X_{L1} + X_{L2} + \ldots + X_{Ln})^2}$$

or

$$Z_0 = \sqrt{(R_1 + R_2 + \ldots + R_n)^2 + (X_{L1} + X_{L2} + \ldots + X_{Ln})^2} \tag{5-21}$$

Example 5–2.—Two coils having the constants $R_1 = 3$ and $X_{L1} = 4$, and $R_2 = 8$ and $X_{L2} = 6$, are connected in series across a 60-cycle, 220-volt supply. Find: (a) the equivalent impedance Z_0; (b) the power factor; (c) the current; and (d) the power in the circuit. Also, draw the vector diagram

Solution.—The equivalent resistance R_0 and the equivalent inductive reactance X_{L0} are:

$$R_0 = R_1 + R_2 = 11 \text{ ohms}$$
$$X_{L0} = X_{L1} + X_{L2} = 10 \text{ ohms}$$

The equivalent impedance is

$$Z_0 = \sqrt{R_0^2 + X_{L0}^2} = \sqrt{11^2 + 10^2} = 14.85 \text{ ohms}$$

The power factor of the entire circuit is

$$\text{P.F.} = \cos\theta_0 = \frac{R_0}{Z_0} = \frac{11}{14.85} = 0.74$$

$$\theta_0 = 42.2^\circ$$

From Ohm's Law, the current is

$$I = \frac{E_0}{Z_0} = \frac{220}{14.85} = 14.82 \text{ amp}$$

The power is

$$P = E_0I\cos\theta_0 = 220 \times 14.82 \times 0.74 = 2420 \text{ watts}$$

or

$$P = I^2R_0 = 14.82^2 \times 11 = 2420 \text{ watts}$$

Two vector diagrams for this problem are drawn in Fig. 5–11. In one the coils are considered separately; and in the other the resistances of both coils are lumped together and the reactances of the coils are also lumped together. In Fig. 5–11(a), Z_1I is the impedance drop across coil 1 and Z_2I is the impedance drop across coil 2. Their vector sum gives the applied voltage.

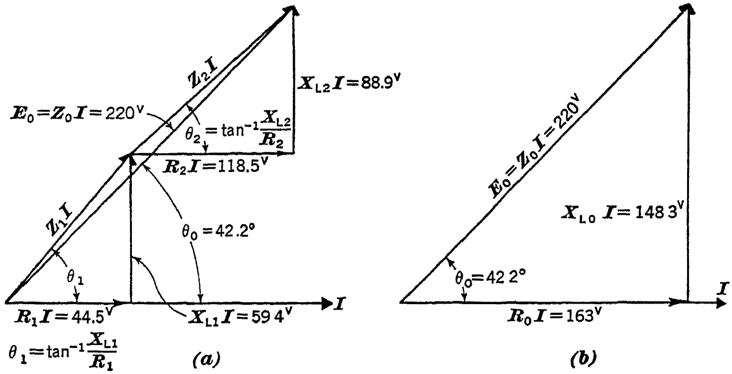


FIG. 5–11

5–8. Capacitive Circuits.—A capacitive circuit or an RC circuit is a circuit with resistance and capacitance in series. If a current $i = I_m \sin \omega t$ is flowing through the circuit in Fig. 5–12, the resistive drop $e_R = Ri$ is in phase with the current; and the potential drop $e_C = \frac{1}{C}\int i\, dt$ across the condenser lags behind the current by 90 degrees. The voltage rise e_0 of the generator is obtained by adding the two potential drops, and it will lag behind the current by some angle between 0° and 90°, as indicated in Fig. 5–13. Thus, with the current leading the applied emf in an RC circuit, a capacitive circuit is for this reason sometimes said to have a leading power factor.

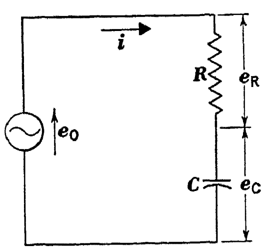


FIG. 5–12

By application of Kirchhoff's Emf Law, the fundamental differential equation for a circuit containing R and C in series, as in Fig. 5–12, is found to be

$$e_0 = e_R + e_C$$

or

$$e_0 = Ri + \frac{q}{C} = Ri + \frac{1}{C}\int i\, dt \tag{5–22}$$

For $i = I_m \sin \omega t$,

$$e_0 = RI_m \sin \omega t + \frac{I_m}{C}\int \sin \omega t\, dt$$

Integrating the second term on the right, we obtain the equation

$$e_0 = RI_m \sin \omega t - \frac{I_m}{\omega C} \cos \omega t + K \tag{5-23}$$

where K is a constant of integration.

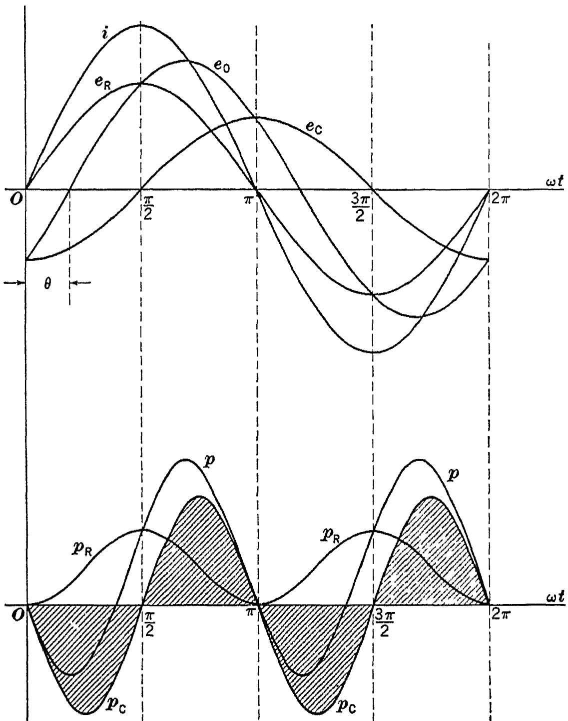


FIG. 5–13

Since it is assumed that the emf at the terminals of the circuit is sinusoidal, K is zero and equation (5–23) becomes

$$e_0 = I_m \left(R \sin \omega t - \frac{1}{\omega C} \cos \omega t \right)$$

or
$$e_0 = I_m (R \sin \omega t - X_C \cos \omega t) = ZI_m \sin (\omega t + \theta) \tag{5-24}$$

where
$$Z = \sqrt{R^2 + (-X_C)^2}$$

$$\theta = \tan^{-1} \frac{-X_C}{R}$$

The quantity Z is the impedance of an RC circuit. It may be considered as the factor by which the maximum value of the current is multiplied to get the maximum value of the applied emf. Its unit is the ohm. The angle θ is the power factor angle, and its value is negative when used in equation (5–24).

The curves representing i, e_R, e_C, and e_0 are shown in Fig. 5–13 The energy delivered to the circuit is consumed in the resistance only. The energy stored in the condenser during part of the cycle is released by the condenser during another part of the cycle, as shown by the cross-hatched areas. The net energy in the electrostatic field during a whole cycle is zero. The power dissipated in the resistance is represented by p_R; the power stored in the condenser is represented by p_C; and the total power is represented by p.

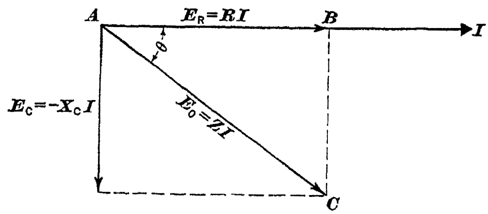


FIG. 5–14

5–9. Vector Diagram and Impedance Triangle of an *RC* Circuit.—If the current, the potential drops across the resistance and the condenser, and the applied emf are represented by vectors, the vector diagram shown in Fig. 5–14 gives the magnitudes of the different quantities and their phase relationship to each other. It can be seen that the line current leads the applied emf by the power factor angle θ.

Just as in the case of an RL circuit, the impedance triangle of an RC circuit may be obtained by dividing the sides of the voltage triangle ABC in Fig. 5–14 by I. The impedance diagram is given in Fig. 5–15.

5–10. Ohm's Law for an *RC* Circuit.—Ohm's Law for an RC circuit may be written as follows:

$$E_0 = ZI = (\sqrt{R^2 + (-X_C)^2})I \tag{5–25}$$

$$I = \frac{E_0}{Z} = \frac{E_0}{\sqrt{R^2 + (-X_C)^2}} \tag{5–26}$$

If the voltage and current are in volts and amperes, the impedance is in ohms; hence, R and X_C are in ohms and C is in farads. For sinusoidal voltage and current, the power factor of the circuit is $\cos\theta = \dfrac{R}{Z}$ and the power factor angle θ is $\tan^{-1}\left(\dfrac{-X_C}{R}\right)$.

The average power supplied to the circuit is dissipated in the resistance, and it has the value

$$P = E_0 I \cos\theta = I^2 R \text{ watts} \tag{5-27}$$

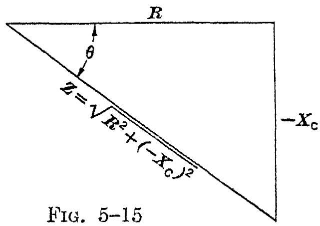


FIG. 5-15

Example 5-3.—A rheostat having a resistance of 6 ohms is connected in series with a condenser having a capacitance of 331.5 microfarads across a 60-cycle, 100-volt supply. Find: (*a*) the capacitive reactance; (*b*) the impedance; (*c*) the power factor; (*d*) the current; and (*e*) the reading of a wattmeter placed in the circuit. Draw the vector diagram.

Solution—The capacitive reactance is

$$X_C = \frac{1}{\omega C} = \frac{10^6}{377 \times 331.5} = 8 \text{ ohms}$$

The impedance is

$$Z = \sqrt{R^2 + \left(\frac{-1}{\omega C}\right)^2} = \sqrt{6^2 + (-8)^2} = 10 \text{ ohms}$$

The power factor is

$$\text{P.F.} = \cos\theta = \frac{R}{Z} = \frac{6}{10} = 0.6$$

$$\theta = \tan^{-1}\frac{-8}{6} = -53.2°$$

The current is

$$I = \frac{E_0}{Z} = \frac{100}{10} = 10 \text{ amp}$$

The power is

$$P = E_0 I \cos\theta = 100 \times 10 \times 0.6 = 600 \text{ watts}$$

The check on power is

$$P = I^2 R = 10^2 \times 6 = 600 \text{ watts}$$

The vector diagram with I as reference is drawn in Fig. 5-16(*a*), and that with E_0 as reference is given in Fig. 5-16(*b*).

5–11. Current Locus of *RC* Circuit.—Only a brief discussion of the determination of the current locus of an *RC* circuit will be given, as the procedure is similar to that given in Art. 5–6 for the *RL* circuit. We will consider two cases: (*a*) constant *R* and variable X_C; (*b*) constant X_C and variable *R*. In the analysis of the locus diagram (or circle diagram), the applied emf and the frequency are assumed to remain constant.

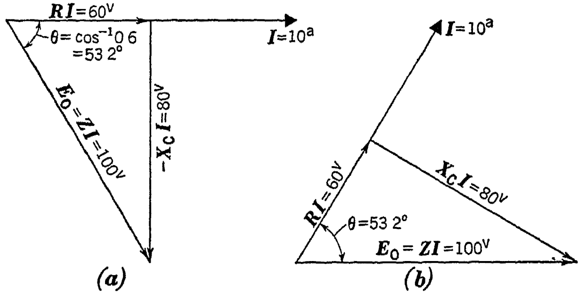


Fig. 5–16

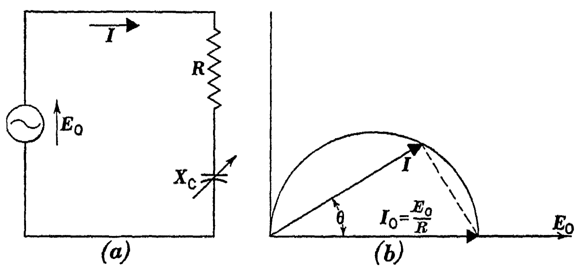


Fig. 5–17

(*a*) *Constant R and Variable* X_C: Fig. 5–17(*a*) shows an *RC* circuit with a variable X_C. If $X_C = 0$, the current is $I_0 = \frac{E_0}{R}$ and it is in phase with E_0, as shown in Fig. 5–17(*b*). As X_C is increased rom its zero value, *I* becomes less than $\frac{E_0}{R}$ in magnitude and leads E_0 more and more (the phase angle θ approaches 90° as a limit). The current locus is the semicircle with diameter $\frac{E_0}{R}$ shown in Fig. 5–17(*b*), as the current always must lead the applied emf.

The polar expression of the locus is found in the following way:

$$I = \frac{E_0}{Z}$$

Since $\cos\theta = \frac{R}{Z}$ or $Z = \frac{R}{\cos\theta}$, then

$$I = \frac{E_0}{R}\cos\theta \qquad (5\text{–}28)$$

(*b*) *Constant* X_C *and Variable* R*:* An RC circuit with a variable R is represented in Fig. 5–18(*a*). If $R=0$, the current is $I_0 = \frac{E_0}{X_C}$ and it leads E_0 by $\theta = 90°$. As R is increased from its zero value,

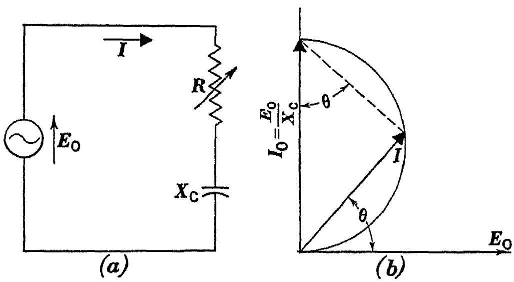


FIG. 5–18

the current I decreases in magnitude and becomes more nearly in phase with E_0 (that is, θ approaches zero as a limit). The current locus is the semicircle with diameter $\frac{E_0}{X_C}$ shown in Fig. 5–18(*b*), and its polar equation is determined as follows:

$$I = \frac{E_0}{Z}$$

Since $\sin\theta = \frac{X_C}{Z}$ or $Z = \frac{X_C}{\sin\theta}$, then

$$I = \frac{E_0}{X_C}\sin\theta \qquad (5\text{–}29)$$

5–12. Capacitive Circuits in Series.—From what has been said regarding a circuit containing several resistances and inductances in series, it is evident that a circuit containing two resistances and two capacitances in series, as in Fig. 5–19(a), can be reduced to a simple RC circuit shown in Fig. 5–19(b). The impedance of the entire circuit is

$$Z = \sqrt{(R_1 + R_2)^2 + (-X_{C1} - X_{C2})^2} \tag{5–30}$$

and the power factor angle θ is

$$\theta = \tan^{-1} \frac{-(X_{C1} + X_{C2})}{R_1 + R_2} \tag{5–31}$$

The combined capacitive reactance of two condensers in series is

$$(X_{C1} + X_{C2}) = \frac{1}{\omega}\left[\frac{1}{C_1} + \frac{1}{C_2}\right] = \frac{1}{\omega}\left[\frac{C_1 + C_2}{C_1 C_2}\right] = \frac{1}{\omega C} \tag{5–32}$$

where $C = \dfrac{C_1 C_2}{C_1 + C_2}$.

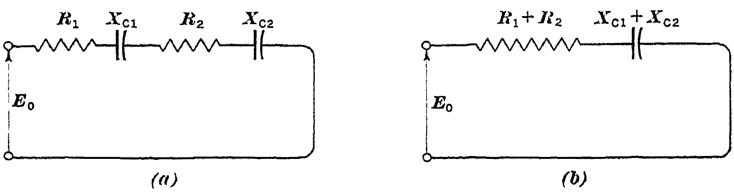


FIG. 5–19

Example 5–4.—Two rheostats, with resistances $R_1 = 2$ ohms and $R_2 = 1$ ohm, and two condensers, with capacitances $C_1 = 0.00265$ farad and $C_2 = 0.000883$ farad, are connected in series across a 60-cycle, 100-volt supply. Find the reading of an ammeter placed in the circuit and also the power delivered to the circuit. What is the power factor of the circuit?

Solution.—The capacitive reactances are.

$$X_{C1} = \frac{1}{377 \times 0.00265} = 1 \text{ ohm}$$

$$X_{C2} = \frac{1}{377 \times 0.000883} = 3 \text{ ohms}$$

The impedance of the entire circuit is

$$Z_0 = \sqrt{(R_1 + R_2)^2 + (-X_{C1} - X_{C2})^2}$$

$$= \sqrt{(2+1)^2 + (-1-3)^2} = 5 \text{ ohms}$$

The current is

$$I = \frac{E_0}{Z_0} = \frac{100}{5} = 20 \text{ amp}$$

The power factor is

$$\text{P.F.} = \cos\theta = \frac{R_1 + R_2}{Z_0} = \frac{3}{5} = 0.6$$

The power is

$$P = E_0 I \cos\theta = 100 \times 20 \times 0.6 = 1200 \text{ watts}$$

or

$$P = I^2(R_1 + R_2) = 20^2 \times 3 = 1200 \text{ watts}$$

5–13. Circuits Containing Resistance, Inductance, and Capacitance in Series.—Fig. 5–20 shows an *RLC* circuit or a circuit containing three elements *R*, *L*, and *C* in series. The current is the same in all parts of the circuit. Here, e_0 is a voltage rise and e_R, e_L, and e_C are voltage drops. The voltage drop $E_R = RI$ across the resistance is in phase with the current. The voltage drop $E_L = X_L I$ across the inductance leads the current by 90 degrees, and the voltage drop $E_C = X_C I$ across the capacitance lags behind the current by 90 degrees. The drops across the inductance and capacitance are 180 degrees apart; and their resultant is their arithmetical difference.

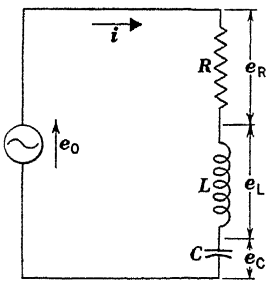


FIG. 5–20

The fundamental differential equation of an *RLC* circuit is

$$e_0 = e_R + e_L + e_C$$

or

$$e_0 = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \qquad (5\text{–}33)$$

If $i = I_m \sin \omega t$, then

$$e_0 = RI_m \sin \omega t + \omega L I_m \cos \omega t - \frac{1}{\omega C} I_m \cos \omega t$$

$$= RI_m \sin \omega t + (X_L - X_C)\, I_m \cos \omega t$$

or

$$e_0 = Z I_m \sin(\omega t + \theta) \qquad (5\text{–}34)$$

where

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\theta = \tan^{-1} \frac{X_L - X_C}{R}$$

The applied emf will lead or lag behind the current, the relation depending on whether X_L is greater or less than X_C. When X_L is greater than X_C, the circuit is inductive and the applied emf leads the current. When X_L is less than X_C, the circuit is capacitive and the applied emf lags behind the current. The power factor angle θ is positive when $X_L > X_C$ and it is negative when $X_L < X_C$.

Two impedance diagrams and two voltage diagrams are shown in Fig. 5–21, the proper ones to be used depending on whether X_L is greater or less than X_C. Curves representing i, e_R, e_L, e_C, and e_0 are shown in Fig. 5–22. In constructing this set of curves, ωL has been assumed to have a value greater than $\frac{1}{\omega C}$.

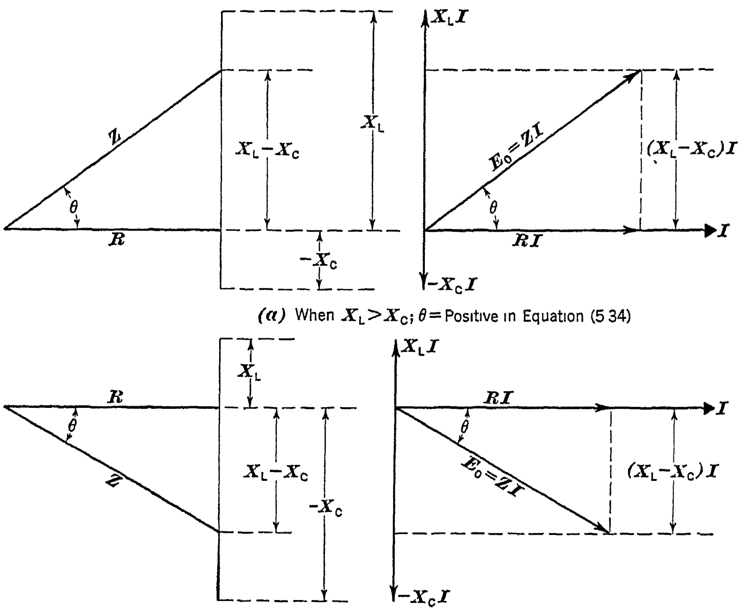


FIG. 5–21

5–14. Ohm's Law for an *RLC* Circuit.—Ohm's Law for an RLC circuit may be written as follows:

$$E_0 = ZI = \left(\sqrt{R^2 + (X_L - X_C)^2}\right) I \qquad (5\text{–}35)$$

or

$$I = \frac{E_0}{\sqrt{R^2 + (X_L - X_C)^2}} \qquad (5\text{–}36)$$

If X_L is greater than X_C, the circuit may be considered as equivalent to a simple RL circuit having a resistance R and an inductive reactance equal to $(X_L - X_C)$. On the other hand, if X_L is less than X_C, the circuit may be viewed as a simple RC circuit.

For sinusoidal voltage and current, the power factor of an RLC circuit is

$$\text{P.F.} = \cos\theta = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{5-37}$$

The power delivered to this circuit is

$$P = E_0 I \cos\theta \text{ watts} \tag{5-38}$$

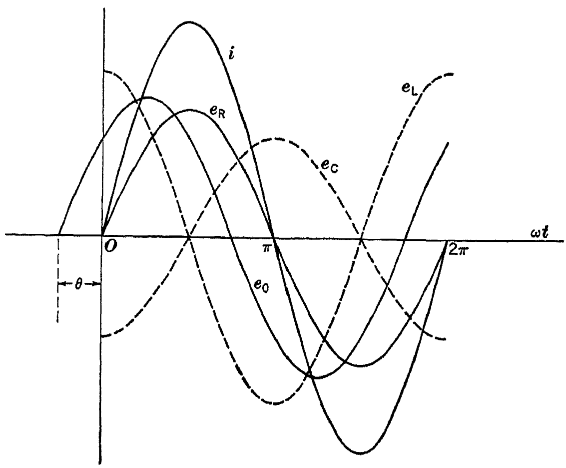


Fig. 5-22

Example 5-5.—A condenser having a capacitive reactance of 16 ohms is connected in series with an air-core coil having a resistance of 6 ohms and an inductive reactance of 8 ohms. This series circuit is connected to a 60-cycle, 100-volt source. Find: (*a*) the impedance of the coil; (*b*) the power factor of the coil; (*c*) the impedance of the entire circuit; (*d*) the power factor of the entire circuit; (*e*) the current; and (*f*) the power delivered to the circuit. Draw the vector diagram.

Solution.—The impedance of the coil is

$$Z_L = \sqrt{6^2 + 8^2} = 10 \text{ ohms}$$

The power factor of the coil is

$$\text{P.F.} = \cos\theta_L = \frac{R_L}{Z_L} = \frac{6}{10} = 0.6$$

The impedance of the entire circuit is

$$Z_0 = \sqrt{6^2 + (8 - 16)^2} = 10 \text{ ohms}$$

The power factor of the entire circuit is

$$\text{P.F.}_0 = \cos\theta_0 = \frac{R_0}{Z_0} = \frac{6}{10} = 0.6$$

The current is

$$I = \frac{E_0}{Z_0} = \frac{100}{10} = 10 \text{ amp}$$

It leads E_0 by 53.2°, since

$$\theta_0 = \tan^{-1}\frac{8-16}{6} = \tan^{-1}\frac{-8}{6} = -53.2°$$

The power is

$$P = E_0 I \cos\theta_0 = 100 \times 10 \times 0.6 = 600 \text{ watts}$$

or

$$P = I^2 R = 10^2 \times 6 = 600 \text{ watts}$$

The vector diagram is shown in Fig. 5–23.

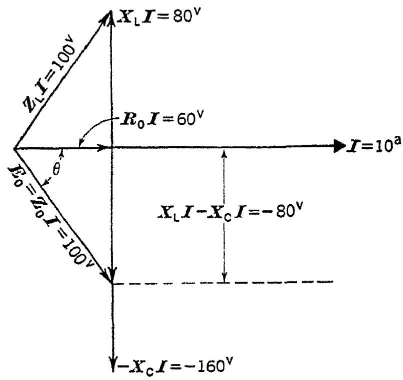


FIG. 5–23

5–15. Resonance in Series Circuits.—In a circuit containing R, L, and C in series, the effective current is given by the relation

$$I = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{5–39}$$

The impedance, which controls the magnitude and phase of the current for a given impressed voltage, is made up of three independent terms; of these the resistance (when skin effect can be neglected) is independent of the frequency, whereas the values of the two reactive terms depend on the frequency. When $\omega L = \frac{1}{\omega C}$, the impedance is a minimum and the current is a maximum. The voltage across the inductance is then equal in magnitude to the voltage across the capacitance, or

$$2\pi f L I = \frac{I}{2\pi f C} \tag{5–40}$$

As these two voltages are in exact opposition, they balance each other; and the RI drop is equal to the line voltage. If the circuit is considered from an energy point of view, the electrostatic field is giving up energy at a rate at which the magnetic field requires it; correspondingly, when the electrostatic field requires energy, the magnetic field is giving it up. The stored energy in the circuit thus stays constant, sometimes being resident in the magnetic field, at other times being resident in the electrostatic field, and at still other times being partly in each. The source of power is called upon only to furnish energy dissipated in the resistance.

When the foregoing conditions exist, the circuit is said to be in *resonance*. The current is then in phase with the impressed emf and has a value of $\frac{E_0}{R}$. With the power factor equal to unity, the power is $P = E_0 I$. That is, the circuit is equivalent to a pure R circuit.

The resonant frequency f_r may be obtained by setting the reactances equal to each other to form the equation $2\pi fL = \frac{1}{2\pi fC}$, and solving for the frequency. Thus,

$$f_r = \frac{1}{2\pi\sqrt{LC}} \text{ cycles per second} \tag{5-41}$$

where L and C are in henrys and farads, respectively.

Equation (5-41) indicates that resonance may be produced in any series circuit containing R, L, and C by: (*a*) varying the frequency for a given L and C; or (*b*) varying either L or C, or both for a given frequency.

Example 5-6.—A series circuit, consisting of a coil having a resistance of 3 ohms and an inductive reactance of 6 ohms and a condenser having a capacitive reactance of 2 ohms, is connected to a 100-volt supply. The values given for the reactances were measured with a 60-cycle source. Find the resonant frequency, the current flowing at this frequency, the drop across the coil, and the drop across the condenser.

Solution.—The inductance is

$$L = \frac{X_L}{\omega} = \frac{6}{377} = 0.0159 \text{ henry}$$

The capacitance is

$$C = \frac{1}{\omega X_C} = \frac{1}{377(2)} = 0.001324 \text{ farad}$$

The resonant frequency is

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.0159\times 0.001324}} = 34.6 \text{ cps}$$

At this frequency,

$$X_L = 2\pi\times 34.6\times 0.0159 = 3\ 47 \text{ ohms}$$

$$X_C = \frac{1}{2\pi\times 34.6\times 0\ 001324} = 3\ 47 \text{ ohms}$$

$$Z_0 = \sqrt{R^2 + (X_L - X_C)^2} = R = 3 \text{ ohms}$$

The current at resonant frequency is

$$I = \frac{E_0}{R} = \frac{100}{3} = 33.33 \text{ amp}$$

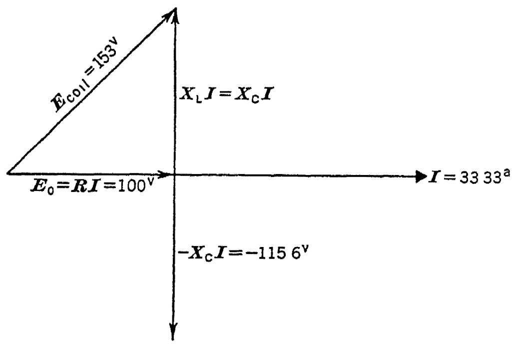


FIG. 5-24

The power factor at resonant frequency is

$$\text{P.F.} = \frac{R}{Z_0} = \frac{R}{R} = \frac{3}{3} = 1$$

The drop across the coil at resonant frequency is found as follows:

$$Z_{coil} = \sqrt{3^2 + 3.47^2} = 4.59 \text{ ohms}$$

$$E_{coil} = IZ_{coil} = 33.33\times 4.59 = 153.0 \text{ volts}$$

The drop across the condenser at resonant frequency is

$$E_{condenser} = -IX_C = -33.33\times 3.47 = -115.6 \text{ volts}$$

The vector diagram for resonant conditions is drawn in Fig. 5-24.

It is seen that in Example 5-6 the drop across the coil and the drop across the condenser are greater than the impressed voltage. Since the current is the same in each element of an RLC circuit, the voltage $X_C I$ or $X_L I$ will be greater than $E_0 = RI$ at resonance when X_C or X_L is greater than R. Hence, in the case of a series circuit containing inductance and capacitance, *consideration should be given to the possibility of high voltages appearing across the coil and across the condenser.*

5–16. Resonance With Variable Frequency.—Fig. 5–25 shows an RLC circuit and vector diagrams for the cases when the frequency is less than, equal to, and greater than the resonant frequency f_r. The curves for the three variables I, E_L, and E_C are shown in Fig. 5–26.

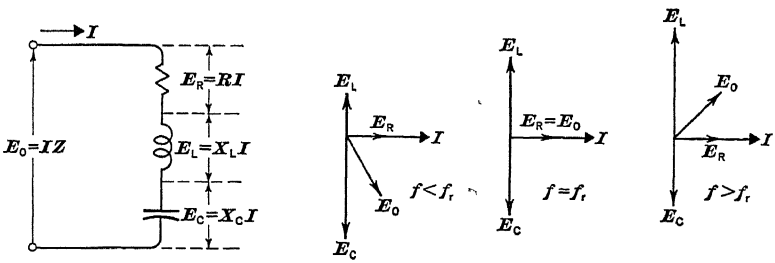


FIG. 5–25

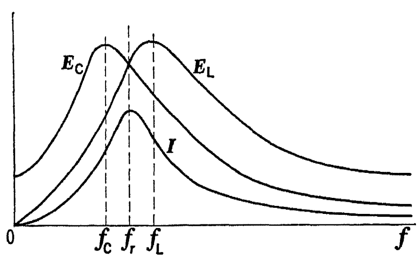


FIG. 5–26

At resonance, $I=\dfrac{E_0}{R}$ and the resistance of the circuit determines the peak or largest ammeter value of the current. The smaller the resistance the higher will be the largest ammeter value of the current.

It can be seen from equation (5–39) that, when $f<f_r$, the capacitive reactance $X_C=\dfrac{1}{\omega C}$ (being greater than the inductive reactance $X_L=\omega L$) will cause the current to be less than the

value of $\frac{E_0}{R}$. When $f > f_r$, the inductive reactance $X_L = \omega L$ $\left(\text{being greater than } X_C = \frac{1}{\omega C}\right)$ will cause the current to be less than the value of $\frac{E_0}{R}$. That is, any departure from the condition of resonance will cause the ammeter value of the current to diminish.

It should be noted from Fig. 5–26, where the resistance R of the circuit is not zero, that neither the voltage E_L across the inductance nor the voltage E_C across the capacitance has the maximum value at resonance. These two voltages are equal in magnitude and opposite in phase at the resonant frequency. The fact that the maximum value of E_C occurs at $f_C < f_r$ while that of E_L occurs at $f_L > f_r$ can be explained as follows. In the case of $E_C = \frac{1}{\omega C} I$, the current I increases from zero frequency up to the resonant frequency while the capacitive reactance $\frac{1}{\omega C}$ decreases in value with an increase in frequency. At the smaller frequencies below f_C, the increase of I is greater than the decrease of $\frac{1}{\omega C}$, and the net result is that E_C increases. The current I is maximum at resonance and decreases with frequencies above f_r. It is obvious that the maximum value of E_C cannot come at a frequency above f_r, as both I and $\frac{1}{\omega C}$ will be decreasing in value for a given change in frequency. In the frequency range between f_C and f_r, the decrease in $\frac{1}{\omega C}$ is greater than the increase in I, and the result is a net decrease in E_C. The maximum value of E_C will come at a frequency* below that for resonance, and that frequency may be found from the relation

$$f_C = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}} \tag{5–42}$$

When $R = 0$, equation (5–42) reduces to equation (5–41).

In the case of $E_L = \omega L I$, both ωL and I are increasing when the frequency is varied from zero to the resonant value. Consequently, their product is increasing over this range of frequencies. At resonance, the current reaches its highest value but ωL is still

* See Appendix B.

increasing; hence, the maximum value of E_L is expected to occur at resonance or at a frequency* above f_r. This frequency is

$$f_L = \frac{1}{2\pi\sqrt{LC - \dfrac{R^2C^2}{2}}} \tag{5-43}$$

which also reduces to equation (5–41) when $R = 0$.

It may be concluded from equations (5–42) and (5–43) that, for $R = 0$, the peak values of the curves for E_C and E_L in Fig. 5–26 will coincide at f_r. The separation between these peak values will be greater for larger values of R.

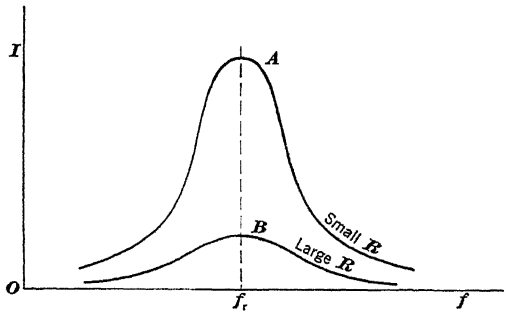


FIG. 5–27

A curve of current plotted against frequency, such as either of those shown in Fig. 5–27, is called a *resonance curve.* Curve A is for a circuit with a relatively small resistance. If the resistance is increased, curve B will be obtained. The precision of determination of frequency depends on what may be called the *sharpness* of resonance; this will be considered in Art. 5–18. The circuit for curve A in Fig. 5–27 will discriminate more in favor of frequencies in the region of the resonant frequency than will the circuit for curve B. Therefore, circuit A is said to be more "selective."

5–17. Reactance Curves.—In determining how the reactance $\left(\omega L - \dfrac{1}{\omega C}\right)$ of a series circuit varies with frequency, use can be made of sketches showing the way each individual element behaves. The inductive reactance $X_L = 2\pi f L$ varies directly with the fre-

* See Appendix B.

quency, as indicated in Fig. 5-28(*a*). The curve for the capacitive reactance $X_C = \frac{1}{\omega C}$ is a hyperbola, Fig. 5-28(*b*). The sum of these two is represented by the curve in Fig. 5-28(*c*) marked total reactance. At the point f_r, where the total reactance curve crosses the axis—that is, where $\omega L = \frac{1}{\omega C}$—the current is maximum and we thus have the value of the resonant frequency.

It should be noted that in Fig. 5-28 the slope of each reactance curve is positive at all points. It also should be noted in Fig. 5-28(*c*) that, for $f < f_r$, the total reactance curve is below the

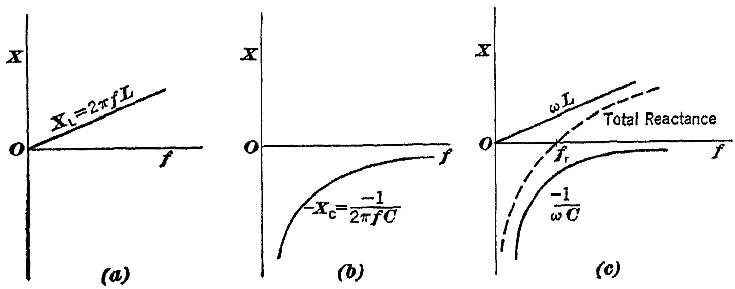


FIG. 5-28

horizontal axis and, hence, the *RLC* circuit is apparently capacitive; whereas, for $f > f_r$, the *RLC* circuit is apparently inductive. This checks the vector diagrams in Fig. 5-25.

5-18. Sharpness of Resonance.—Resonant circuits are often employed as frequency-selective devices. In such cases, it is desirable that the circuit will respond to one frequency or one narrow band of frequencies and preferably will not respond to any other frequencies. The narrower the band of frequencies, the more selective is the circuit. Thus, for the purpose of comparing the selectivity of different resonant circuits, it is convenient to have a measure of the effectiveness with which a resonant circuit performs as a frequency selector.

In specifying the selectivity or the degree of sharpness of resonance of an *RLC* circuit, it is customary to determine the frequency band width $(f_2 - f_1)$ within which the magnitude of the current exceeds a certain percentage of the maximum. This percentage usually is arbitrarily assumed in the following manner. At the resonant frequency, the reactance $X = X_L - X_C = 0$ and

$$I = \frac{E_0}{R} \tag{5-44}$$

$$P = I^2R = \frac{E_0^2}{R} \tag{5-45}$$

When the frequency is increased or decreased from f_r in Fig. 5-29(a), there will be two frequencies, f_1 and f_2, at which the net

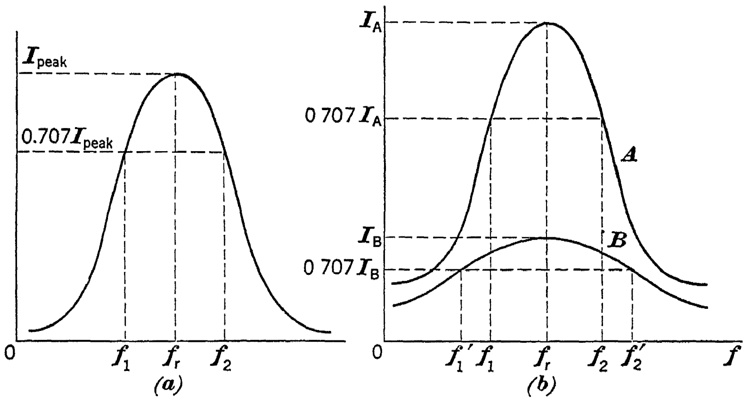


FIG. 5-29

reactance is equal in value to the resistance. At f_1, $X = X_C - X_L = R$; and, at f_2, $X = X_L - X_C = R$. At these points, the impedance is $Z = \sqrt{R^2 + R^2} = \sqrt{2}\,R$. Then,

$$I = \frac{E_0}{\sqrt{2}\,R} = 0.707\,\frac{E_0}{R} \tag{5-46}$$

$$P = I^2R = \frac{1}{2}\frac{E_0^2}{R} \tag{5-47}$$

That is, the magnitude of the current at f_1 or f_2, Fig. 5-29(a), is 70.7 per cent of the peak value $\frac{E_0}{R}$, and the power at each of these points is equal to one-half of the value $\frac{E_0^2}{R}$ at resonance. Sometimes the points at f_1 and f_2 are referred to as the *half-power points*.

The sharpness of resonance in an *RLC* circuit depends primarily on the ratio of the total circuit inductive reactance ωL to the total circuit resistance R. This ratio is usually designated by the symbol Q and is called the Q of the circuit, or sometimes the "Quality" of the circuit. Thus,

$$Q = \frac{\omega L}{R} = \frac{\text{Total circuit inductive reactance}}{\text{Total circuit resistance}} \tag{5-48}$$

In coil design, the ratio of the inductive reactance ωL of the coil to its resistance R is called the Q of the coil. Over the high frequency range, the Q of a coil is generally more nearly constant than is the resistance of the coil, as shown in Fig. 5-30. Because of skin effect, the resistance increases with frequency; and this offsets the increase in $2\pi fL$. It is quite possible that the radio-

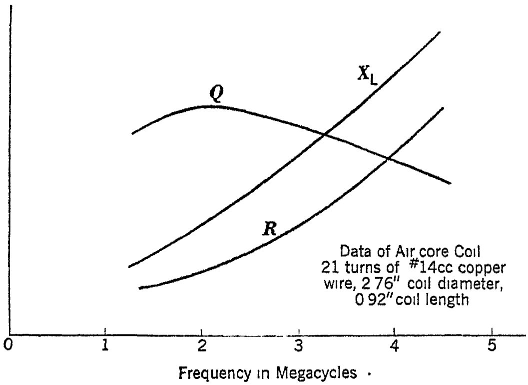


FIG. 5-30

frequency resistance of a coil may be as much as 100 or 200 times the direct-current resistance. For the usual air-core coils used at frequencies from 10^4 to 10^7 cycles per second, the Q of a coil may have values from 100 to 300.

The Q of a coil has little significance in determining the characteristics of a series resonant circuit, unless the details of the circuit to which it is connected are also known. The real figure of merit is the Q of the entire circuit, which is the ratio of the total reactance of the inductances to the total resistance of the circuit, including that of the generator. This factor Q plays an important role in radio circuit analysis. For example, the current at resonance is $I = \frac{E_0}{R}$ and the voltage across the inductance is

$$E_L = X_L I = \frac{E_0 \omega L}{R} = E_0 Q \tag{5-49}$$

The voltage across the condenser has the same value, since $\omega L = \frac{1}{\omega C}$. Thus, if the applied emf in an RLC circuit is 100 volts and the Q of the circuit is 100, the voltage across the condenser at resonance is $E_C = E_0 Q = 100 \times 100 = 10{,}000$ volts.

In cases where the Q of a coil is high, then ωL is very nearly equal to the impedance Z of the coil and, as an approximation,

$$Q = \frac{\omega L}{R} \approx \frac{Z}{R} \tag{5–50}$$

Hence, the power factor of the coil is approximately equal to the reciprocal of its Q

5–19. Sharpness With Variable Frequency.—The selectivity or sharpness of resonance, with variable frequency, in an RLC circuit will now be considered. It is desirable to study first the variation of reactance in the region of resonance and then to discuss the relationship between the frequency band $(f_2 - f_1)$ in Fig. 5–29 and the Q of the circuit for the determination of sharpness of resonance.

The variation of reactances with frequency may be seen from the curves in Fig. 5–28(c). The slope of the reactance curve for the capacitance is

$$\frac{\partial(-X_C)}{\partial f} = \frac{1}{2\pi f^2 C} \tag{5–51}$$

and the slope of the reactance curve for the inductance is

$$\frac{\partial X_L}{\partial f} = 2\pi L \tag{5–52}$$

At the resonant frequency

$$2\pi f_r L = \frac{1}{2\pi f_r C}$$

or

$$2\pi L = \frac{1}{2\pi f_r^2 C} \tag{5–53}$$

This relation states that the slope given by equation (5–51) and the slope given by equation (5–52) are equal at the resonant frequency, or

$$\frac{\partial(-X_C)}{\partial f} = \frac{\partial X_L}{\partial f} \tag{5–54}$$

Since the slope of any reactance curve is always *positive*, it can be seen also from Fig. 5–28(c) that, if the frequency is increased slightly above the resonant value f_r, the inductive reactance $2\pi fL$ will be increased and the capacitive reactance $\frac{1}{2\pi fC}$ will be decreased. This means that, in the region of resonance where the rates of change with frequency of the inductive and capacitive reactances are nearly equal, the net change in reactance $\left(2\pi fL - \frac{1}{2\pi fC}\right)$ due to a variation in frequency is approximately twice the change in the inductive reactance alone. That is, from the standpoint of the sharpness of resonance in a series circuit, a change in frequency is more effective than a change in C alone or a change in L alone.

The equation for the determination of sharpness of resonance, *as far as the variation of the current flowing in the circuit is concerned*, is

$$\frac{f_2 - f_1}{f_r} = \frac{R}{2\pi f_r L} = \frac{1}{Q} \tag{5–55}$$

The derivation of this equation is given in Appendix C. For a given value of f_r, it is obvious that when Q is large the difference $(f_2 - f_1)$ is small. A narrow frequency band $(f_2 - f_1)$ means a more selective circuit, as shown in Fig. 5–29(b). In a circuit with air-core coils, a satisfactory value for the Q of the circuit is 200 or higher. Equation (5–55) may be considered as a definition* for Q in terms of the frequency band for an RLC circuit.

In certain communication problems, it is desirable to have some information on the variation of E_C across the condenser in an RLC circuit. For such cases, a knowledge of the sharpness of resonance on the basis of the variation of E_C with frequency would be useful. When R is negligible, as in the case of a circuit with high Q, the peaks of the curves in Fig. 5–26 for I, E_C, and E_L occur approximately at f_r and the curves have practically the same sharpness. The three curves will check equation (5–55) quite well for sharpness of resonance; that is, the ratio of the difference $(f_2 - f_1)$ of the frequencies at the half-power points to the frequency at the maximum value of each curve will be approximately the same.

* See Appendix H for definition of Q (figure of merit).

5-20. Sharpness With Variable Capacitance.—In the measurements of electrical quantities in radio-frequency circuits, the capacitance is often made the variable from the standpoint of convenience. If C is varied, the resonance curves will be similar to those of Fig. 5–26 where f is made the variable. The curves for the variables I, E_L, and E_C are shown in Fig. 5–31(a). The resonance curve for the current is shown again in Fig. 5–31(b).

When the capacitance is varied to obtain the resonance curve for an RLC circuit, it is convenient to specify the sharpness of resonance in terms of the variation between the two values of

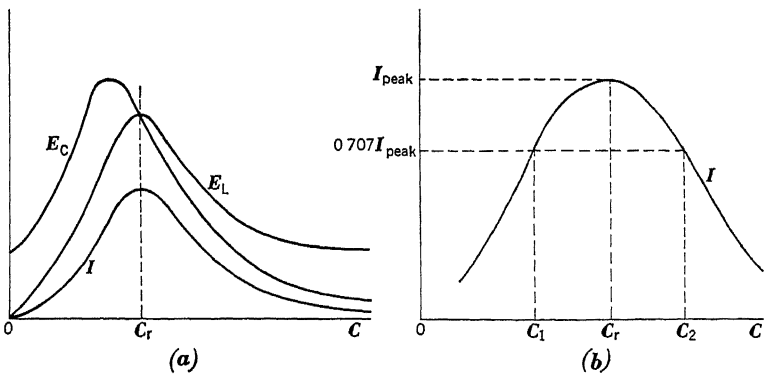


FIG. 5–31

capacitance, C_1 and C_2, where the net reactance of the circuit is equal to the total resistance of the circuit. On this basis we have at the points where the current is 0.707 I_{peak},

$$\frac{1}{\omega C_1} - \omega L = R \tag{5–56}$$

$$\omega L - \frac{1}{\omega C_2} = R \tag{5–57}$$

Adding equations (5–56) and (5–57) gives

$$\frac{1}{\omega C_1} - \frac{1}{\omega C_2} = 2R$$

or

$$\frac{C_2 - C_1}{\omega C_1 C_2} = 2R \tag{5–58}$$

In general, C_r^2 is approximately equal to C_1C_2. If the assumption is made that $C_r^2 = C_1C_2$, then equation (5–58) becomes

$$\frac{C_2 - C_1}{\omega C_r^2} = 2R \tag{5–59}$$

Since at resonant condition $\omega L = \frac{1}{\omega C_r}$, equation (5–59) can be written as follows:

$$\frac{C_2 - C_1}{C_r} = 2R\omega C_r = \frac{2R}{\omega L} = \frac{2}{Q} \tag{5–60}$$

where C_1 = capacitance at the lower $0.707 I_{\text{peak}}$ point;
C_2 = capacitance at the upper $0.707 I_{\text{peak}}$ point;
C_r = capacitance at resonance

5–21. Summary.—(*a*) *RL Circuits:* In an RL circuit, such as a coil, the current lags behind the applied voltage.

(*1*) *Instantaneous values:* If the current is $i = I_m \sin \omega t$, then

$$e_0 = ZI_m \sin (\omega t + \theta)$$

where
$$Z = \sqrt{R^2 + (\omega L)^2} = \sqrt{R^2 + X_L^2}$$

$$\theta = \tan^{-1} \frac{X_L}{R} \text{ (positive power factor angle)}$$

(*2*) *Effective values:* Ohm's Law for an RL circuit may be written:

$$E_0 = ZI = (\sqrt{R^2 + X_L^2})I$$

$$I = \frac{E_0}{\sqrt{R^2 + X_L^2}}$$

The power factor is $\cos \theta = \frac{R}{Z}$ and it is considered a lagging power factor as the current lags behind the applied voltage. The average power in the circuit is

$$P = E_0 I \cos \theta = I^2 R$$

(*3*) If n coils are connected in series, the impedance of the entire circuit is

$$Z = \sqrt{(R_1 + R_2 + \dots + R_n)^2 + X(_{L1} + X_{L2} + \dots + X_{Ln})^2}$$

and the current is

$$I = \frac{E_0}{Z}$$

The lagging power factor is

$$\text{P.F.} = \cos\theta = \frac{R_1 + R_2 + \ldots + R_n}{Z}$$

The power in the circuit is

$$P = E_0 I \cos\theta = I^2 (R_1 + R_2 + \ldots + R_n)$$

(*b*) *RC Circuits:* In an RC circuit, the current leads the applied voltage.

(*1*) *Instantaneous values:* If $i = I_m \sin \omega t$, then

$$e_0 = Z I_m \sin(\omega t + \theta)$$

$$Z = \sqrt{R^2 + \left(\frac{-1}{\omega C}\right)^2} = \sqrt{R^2 + (-X_C)^2}$$

$$\theta = \tan^{-1} \frac{-X_C}{R} \text{ (negative power factor angle)}$$

(*2*) *Effective values:* Ohm's Law for an RC circuit may be written:

$$E_0 = ZI = \left(\sqrt{R^2 + (-X_C)^2}\right) I$$

$$I = \frac{E_0}{\sqrt{R^2 + (-X_C)^2}}$$

The power factor is equal to $\cos\theta = \frac{R}{Z}$ and it is a leading power factor as the current leads the applied voltage. The expression for the power is

$$P = E_0 I \cos\theta = I^2 R$$

(*3*) If n circuits with resistance and capacitance are connected in series, the impedance of the entire circuit is

$$Z = \sqrt{(R_1 + R_2 + \ldots + R_n)^2 + (-X_{C1} - X_{C2} - \ldots - X_{Cn})^2}$$

and the current is

$$I = \frac{E_0}{Z}$$

The leading power factor is

$$\text{P.F.} = \cos\theta = \frac{R_1 + R_2 + \ldots + R_n}{Z}$$

The power in the circuit is

$$P = E_0 I \cos\theta = I^2 (R_1 + R_2 + \ldots + R_n)$$

(*c*) *RLC Circuits:* The impedance of an *RLC* circuit is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

When X_L is greater than X_C, the circuit is inductive and the current lags behind the applied emf. When X_L is less than X_C, the circuit is capacitive and the current leads the applied emf.

(*d*) *Resonance in Series Circuits:* A series circuit containing *R*, *L*, and *C* is said to be in resonance when

$$2\pi f L = \frac{1}{2\pi f C}$$

Under this condition, the current is in phase with the applied emf and has its maximum value of $\frac{E_0}{R}$. The circuit is then equivalent to an *R* circuit and its power factor is unity. The resonant frequency is

$$f_r = \frac{1}{2\pi\sqrt{LC}} \text{ cycles per second}$$

where *L* and *C* are in henrys and farads, respectively.

(*e*) *The Q of a Circuit:* The *Q* (or "Quality") of a coil is the ratio of its reactance ωL to its resistance *R*, or

$$Q_{\text{coil}} = \frac{\omega L}{R}$$

The *Q* of a coil has little significance in determining the characteristics of a series resonant circuit unless the details of the circuit to which it is connected are also known. The real figure of merit is the *Q* of the entire circuit, which is

$$Q = \frac{\omega L}{R} = \frac{\text{Total circuit inductive reactance}}{\text{Total circuit resistance}}$$

(*f*) *Sharpness With Variable Frequency:* When the frequency is the only variable, sharpness of resonance of a circuit is defined by the equation

$$\frac{f_2 - f_1}{f_r} = \frac{R}{2\pi f_r L} = \frac{1}{Q}$$

where $(f_2 - f_1)$ is the frequency band width and *Q* is the *Q* of the entire circuit.

(*g*) *Sharpness With Variable Capacitance:* When the capacitance is the only variable, the sharpness of resonance is

$$\frac{C_2 - C_1}{C_r} = 2R\omega C_r = \frac{2R}{\omega L} = \frac{2}{Q}$$

where Q is the Q of the entire circuit.

PROBLEMS

5–1. A coil having a resistance of 3 ohms and an inductance of 0.01 henry is connected across a 60-cycle source. If the sinusoidal current flowing in the coil has an effective value of 20 amp, derive the expression for the applied emf. Plot the current wave and the curves for e_R, e_L, and e_0. Determine the power factor angle by which the current lags behind the applied emf.

5–2. Plot the power curves for p_R, p_L, and p of Problem 5–1. What is the average power taken by the coil? Indicate this value on the diagram.

5–3. A coil having a resistance of 4 ohms and a reactance of 3 ohms is connected across a 115-volt, 60-cycle circuit What are the readings of an ammeter and a wattmeter connected in the circuit? Sketch the waves of i, e_R, e_L, and e_0.

5–4. A coil having a resistance of 10 ohms is connected to a 110-volt, 60-cycle source. An ammeter placed in the circuit reads 5 amp. (*a*) What is the inductance of the coil? (*b*) What is the power factor of the circuit? (*c*) What is the average power taken by the circuit? (*d*) Draw the impedance and voltage triangles.

5–5. What current will the coil in Problem 5–4 take: (*a*) from a 220-volt, 60-cycle source and (*b*) from a 110-volt, 25-cycle source?

5–6. A series circuit consists of a non-inductive resistance of 5 ohms and an inductive coil having $R = 5$ ohms and $L = 0.1$ henry. The alternating emf that is impressed on the circuit produces a current whose equation is $i = 14.14 \sin 377t$. (*a*) What is the equation of the emf impressed on the circuit? (*b*) At an instant when $377t = \pi$, what are the instantaneous values of the impressed emf, the potential across the non-inductive resistance, and the potential across the coil? (*c*) What is the energy stored in the magnetic field from $t = 0$ to the instant when $377t = \pi$? (*d*) What is the maximum energy of the magnetic field?

5–7. If the applied emf across an RL circuit is $e_0 = E_{0m} \sin \omega t$, explain by the use of sine-wave pictures why the expression for the current can be written as follows:

$$i = \frac{E_{0m}}{Z} \sin(\omega t - \theta)$$

where

$$Z = \sqrt{R^2 + (\omega L)^2} = \sqrt{R^2 + X_L^2}$$

$$\theta = \tan^{-1} \frac{X_L}{R}$$

(Hint: If an emf leads a current by an angle θ, the current is said to lag behind the emf by the angle θ.)

5–8. An emf $e_0 = 141.4 \sin (377t + 30°)$ is impressed on an impedance coil having a resistance of 4 ohms and an inductive reactance of 1.25 ohms measured at 25 cycles. What is the equation of the current? Sketch the waves for i, e_R, e_L, and e_0. What are the power factor of the coil and the average power absorbed by the entire circuit?

5–9. Two coils having resistances $R_1 = 3$ ohms and $R_2 = 2$ ohms and inductances $L_1 = 0.01$ henry and $L_2 = 0.02$ henry are connected in series. A 60-cycle sinusoidal emf with an effective value of 100 volts is applied to this combination. Determine the current and power in this circuit. Draw two vector diagrams for this problem, similar to those shown in Fig. 5–10.

5–10. A resistor of 10 ohms is connected in series with a coil for which $R_L = 3$ ohms and $X_L = 4$ ohms. If the applied emf is 110 volts (effective), what is the current flowing in the circuit? Determine the power dissipated in the coil. Also, draw the vector diagram.

5–11. If the emf applied across an RC circuit is $e_0 = E_{0m} \sin \omega t$, explain why the expression for the current can be written in the form

$$i = \frac{E_{0m}}{Z} \sin (\omega t - \theta)$$

where

$$Z = \sqrt{R^2 + \left(\frac{-1}{\omega C}\right)^2} = \sqrt{R^2 + (-X_C)^2}$$

$$\theta = \tan^{-1}\left(\frac{-X_C}{R}\right)$$

5–12. A 50-watt, 110-volt lamp having a constant pure resistance is connected in series with a condenser having a capacitance of 10 μf. An emf $e_0 = 220\sqrt{2} \sin 157t$ is impressed on the circuit. What is the equation of the current? Sketch the waves for i, e_R, e_C, e_0, p_R, p_C, and p.

5–13. A current $i = 7.07 \sin 377t$ is flowing in a series circuit in which $R = 5$ ohms and $X_C = 8$ ohms measured at 60 cycles. What is the equation of the impressed emf? If the circuit is connected to a 110-volt, 25-cycle source, what are the readings of an ammeter and a wattmeter connected in the circuit? What is the power factor of the entire circuit in each case?

5–14. If an emf $e_0 = 141.4 \sin (377t - 80°)$ is impressed on a circuit and a current $i = 28.28 \sin (377t - 30°)$ flows, what is the average power supplied to the circuit? What is the resistance of the circuit? What is the reactance of the circuit?

5–15. In an RL circuit having a lagging current of 10 amp at 220 volts and having a power factor of 0.80 at 60 cycles, the frequency is to be changed to 25 cycles. If the magnitude of the applied emf is to remain the same, what change in resistance must be made to have 20 amp flowing at this new frequency? What is the new power factor?

5–16. A condenser of 100 μf is connected in series with a circuit in which $R_1 = 3$ ohms and $X_{C1} = 4$ ohms are in series. When this combination is connected across 120-volt, 60-cycle mains, calculate the current. Draw two vector diagrams for this problem similar to those in Fig. 5–11. Also, draw an impedance triangle.

5-17. The current in a coil is made to vary at a uniform rate of 500 amp per sec. At the instant when the current is equal to 200 amp, the voltage across the terminals of the coil is 350 volts; and, when the instantaneous current is 125 amp, the voltage is 200 volts. What are the values of the resistance and inductance in the coil?

5-18. An impedance coil takes 20 amp when connected to the terminals of a 110-volt storage battery. It takes only 7 amp when connected to a 220-volt, 60-cycle alternator. Calculate the resistance and inductance of the coil.

5-19. An impedance coil having a resistance of 10 ohms is connected in series with a second impedance coil. When 220 volts is applied to the series combination, the current is 8 amp and the voltages across the coils are 110 (first coil) and 125 volts. Determine the resistance of the second coil and the impedance of the two coils.

5-20. An impedance coil has a fixed resistance of 10 ohms and a variable reactance. What is the inductance of the coil when it takes 900 watts from a 220-volt, 60-cycle source?

5-21. A non-inductive resistance is connected in series with a condenser across a 110-volt, 60-cycle main. The resistance and the condenser are adjusted so that the power absorbed is 200 watts and the voltmeter reading across the condenser is 90 volts. Calculate the resistance and the capacitance.

5-22. A series circuit that contains a resistance of 10 ohms, an inductance of 0.1 henry, and a capacity of 100 μf is connected to a 200-volt, 60-cycle source. (*a*) What is the current flowing in the circuit? (*b*) What is the voltage consumed in each part of the circuit? (*c*) Draw the vector diagram. (*d*) What is the power factor? (*e*) With the impressed emf maintained constant and the frequency varied, determine the maximum value of the current. (*f*) Plot i, e_R, e_L, e_C, and e_0 for a range of frequencies from 0 to 200 cps. Calculate values for frequencies of 0, 10, 40, 50, 60, 100, and 200 cps.

5-23. An impedance coil having a resistance of 10 ohms and a reactance of 20 ohms at 60 cycles is connected in series with a condenser of variable capacitance. If the impressed emf is 200 volts at 60 cycles, for what capacitance will the potentials across the coil and the condenser be equal? At what frequency will resonance occur for this value of capacitance?

5-24. A series circuit containing a resistance and a condenser takes 7.2 amp from a 220-volt, 25-cycle source. A voltmeter connected across the condenser reads 180 volts. What are the values of the resistance and the capacitance?

5-25. A series circuit containing resistance, inductance, and capacitance is connected to a 110-volt supply. By changing the frequency of the impressed emf the current can be varied from 5 amp at 25 cycles to a maximum of 30 amp at 60 cycles. A further increase in the frequency decreases the current. Determine the values of the resistance, inductance, and capacitance of the circuit.

5-26. If the applied emf across an *RLC* series circuit is $e_0 = E_{0m} \sin \omega t$, explain why the current can be expressed in the form

$$i = \frac{E_{0m}}{Z} \sin (\omega t - \theta)$$

where

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\theta = \tan^{-1} \frac{X_L - X_C}{R}$$

5–27. A sinusoidal emf is applied to a circuit containing R, L, and C in series. If the effective value of the applied emf is held constant, find the capacitance or the inductance or the frequency at which I, E_R, E_L, and then E_C is maximum when each of the following factors is varied: (a) C, (b) L, and (c) f. If the frequency is the resonant value and R is varied, what is the value of R which will give the maximum voltage across the resistance?

5–28. An impedance coil has an inductance of 500 microhenrys and a resistance of 25 ohms at 1000 kilocycles. What is the Q of the coil at this frequency?

5–29. If the coil in Problem 5–28 is connected to a generator having a constant internal resistance of 10 ohms and negligible inductance, what is the Q of the circuit at 1000 kilocycles?

5–30. An inductance of 300 microhenrys having a constant Q of 100 is connected in series with a condenser of 100 $\mu\mu$f across a generator of 1 volt. The generator has a constant internal resistance of 50 ohms and negligible inductance. What is the current in the circuit at resonance when the frequency is varied? Plot the curves of I, E_L, and E_C against f.

5–31. Repeat Problem 5–30 for a generator with 10,000 ohms instead of 50 ohms. Compare the curves and explain which circuit is more selective. What effect has the resistance of a generator on sharpness of resonance?

CHAPTER 6

COMPLEX QUANTITIES

From the preceding chapter, it is apparent that alternating-current problems cannot be solved ordinarily by the use of simple algebra, since geometrical relationships must be taken into consideration. Alternating currents and voltages are vector rather than scalar quantities and they must be handled by methods which are applicable to vectors. The justification for the use of complex quantities is that it furnishes a simple method for solving alternating-current problems. In fact, without the use of complex algebra, it would be extremely difficult, if not impossible, to solve many types of problems.

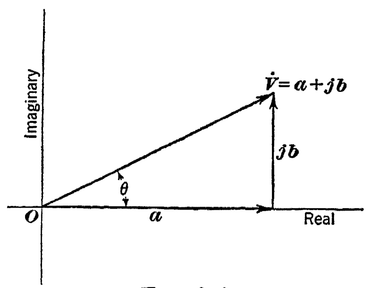


FIG. 6–1

6–1. Complex Quantities. A very advantageous method of handling vectors is known as the method of complex quantities or the *symbolic method*. In this method each vector is resolved into two components, respectively, along and at right angles to some conveniently chosen axis of reference. In Fig. 6–1 is shown a vector $\dot{V}$ making an angle θ with the *reference axis*. The dot over the letter $\dot{V}$ indicates that it is a vector. It will be noted that the vector is resolved into components along the two axes. If the two components were always labeled and kept separate, any convenient notation might be adopted to identify fully each vector of a coplanar system. It could be expressed in terms of the x and y components, as, for example,

$$\dot{V}=a_x+b_y \tag{6–1}$$

In electrical engineering, a shorter notation than that just given is universally used. The x component is left without any label and the y component is given the prefix j. Thus,

$$\dot{V}=a+jb \tag{6–2}$$

The horizontal components are measured along an axis called the axis of reals, Fig. 6–1, positive values being laid off to the right of O and negative values to the left of O. The vertical components, to which j is prefixed, are measured along an axis called the axis of imaginaries, with positive values, as $+jb$, drawn above the horizontal axis and negative values, as $-jb$, drawn below the horizontal axis. If, for example, $\dot{V}=5+j3$, then the horizontal component a has a value of $+5$ units, or 5 units to the right of O; and the component jb along the vertical axis has a value of $+j3$ or 3 units above the horizontal axis. The plane represented by the coordinate axes is called the *complex plane*. A quantity consisting of a real part a and an imaginary part jb is called a *complex quantity* or a *complex number*.

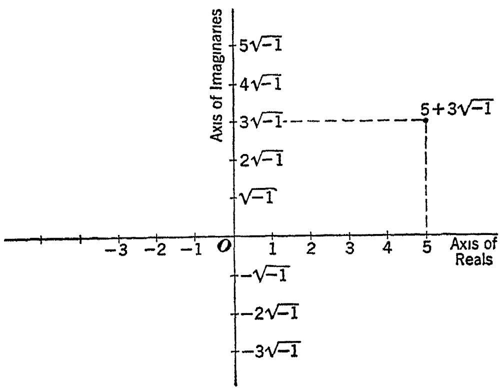


Fig. 6–2

The complex plane was devised by mathematicians for plotting complex quantities. Real numbers, such as 1 and 5 and -2, are plotted as points along the "axis of reals," and imaginary numbers are plotted on the "axis of imaginaries." See Fig. 6–2. The number $5\sqrt{-1}$, for example, would be plotted 5 units above the origin on the axis of imaginaries. Each point in the complex plane (the plane of the two axes) represents a complex number; a complex number has a real part and an imaginary part, and the coordinates of the point give the components of the complex number. For example, a point whose coordinates are 5 horizontally to the right and 3 vertically upward represents the quantity $5+3\sqrt{-1}$. If a vector is drawn from the origin of coordinates to any point in the complex plane, its components will be equal to the coordinates of the point to which it is drawn. A complex quantity may thus be used to represent a vector, and it is in this way that complex quantities are used in electrical engineering.

For convenience it is customary to write j instead of $\sqrt{-1}$. (The mathematician writes i for $\sqrt{-1}$, but electrical engineers use j to avoid confusion with current.) By definition, then, we say that $j=\sqrt{-1}$ and it follows that $j^2=-1$, $j^3=-j$, and $j^4=1$. Considering where the vectors representing these numbers would lie in the complex plane, and making use of the angle θ of Fig. 6–1, we can represent complex quantities by vectors. Thus,

for the quantity 1, the vector has length 1 at an angle 0°
for the quantity j, the vector has length 1 at an angle 90°
for the quantity j^2, the vector has length 1 at an angle 180°
for the quantity j^3, the vector has length 1 at an angle 270°
for the quantity j^4, the vector has length 1 at an angle 360°

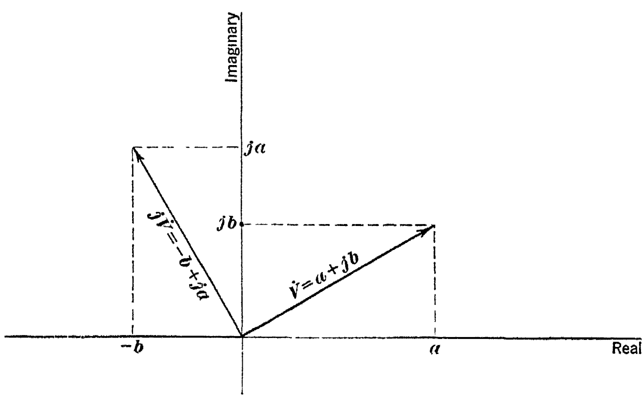


FIG. 6–3

The important thing to note is that when a complex quantity is multiplied by j its vector is rotated through an angle of 90° in a counter-clockwise direction. This result is true, not only for a vector along one of the axes, but for any vector; multiplication by j rotates any vector through a positive angle of 90°. To prove this, consider any vector $\dot{V}=a+jb$ with components a and b in the complex plane as shown in Fig. 6–3. Now multiply by j to obtain the vector

$$j\dot{V}=ja+j^2b=-b+ja \tag{6–3}$$

As indicated in Fig. 6–3, each component of $j\dot{V}$ is displaced 90° from the corresponding component of $\dot{V}$; thus, ja is 90° from a, and j^2b is 90° from jb. Therefore, the vector $j\dot{V}$ is displaced 90° (in a counter-clockwise direction) from $\dot{V}$; and, since $\dot{V}$ can be any vector whatever, it appears that j can be looked upon as *an operator that rotates a vector in the complex plane through a positive angle of* 90°.

It is unfortunate, from the standpoint of the electrical engineering student, that the words imaginary and complex are present to startle him as he commences the use of complex numbers. Although $j = \sqrt{-1}$ is an imaginary quantity, neither the component b of the vector to which j is prefixed nor the axis along which it lies is imaginary. The application of complex numbers simplifies the solutions of problems on alternating current circuits. The imaginary part jb in equation (6–2) might have been called the j part or the j number with less confusion.

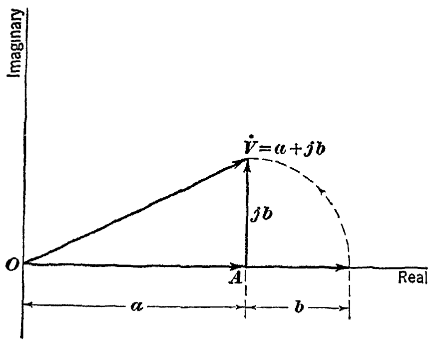


FIG. 6–4

6–2. The Operator j.—As shown in the preceding article, the operator $j = \sqrt{-1}$ associated with a complex quantity is an operator having the property of rotating the vector representing the quantity through an angle of 90° in a counter-clockwise direction. The component jb in equation (6–2) may be interpreted as a quantity measured b units to the right of A, Fig. 6–4, and rotated through an angle of 90° in the counter-clockwise direction.

The different powers of j carry the following meanings:

$$j = 90° \text{ counter-clockwise rotation} = \sqrt{-1}$$

$$j^2 = 180° \text{ counter-clockwise rotation} = -1$$

$$j^3 = 270° \text{ counter-clockwise rotation} = -\sqrt{-1}$$

$$= 90° \text{ clockwise rotation} = -j = -\sqrt{-1}$$

$$j^4 = 360° \text{ counter-clockwise rotation} = 1$$

Also,

$$(+j)(-j) = -j^2 = -(\sqrt{-1})^2 = 1$$

$$(-j)(-j) = +j^2 = (\sqrt{-1})^2 = -1$$

$$(-j)(+j^2) = -j^3 = -(\sqrt{-1})^3 = \sqrt{-1}$$

$$(+j)(+j^2) = +j^3 = (\sqrt{-1})^3 = -\sqrt{-1}$$

6–3. Representation of Vectors.—A vector may be represented in any of the following forms:

(*a*) Vectorial form
(*b*) Rectangular form
(*c*) Trigonometric or circular form
(*d*) Exponential form
(*e*) Polar form

(*a*) *Vectorial Form:* A vector is said to be in the vectorial form when it is expressed by a single letter, as $\dot{V}$, $\hat{V}$, $\bar{V}$, $\underline{V}$, a bold-face capital letter, etc. The form with the dot over the letter will be used here exclusively.

(*b*) *Rectangular Form:* The vector $\dot{A} = a + jb$, in Fig. 6–5, is said to be expressed in the rectangular form. It may be preferable to call this the "$a + jb$" form. The particular quadrant in which the vector lies is fixed by the signs of a and b, as in trigonometry. Thus, as shown in Fig. 6–5,

$\dot{A} = a + jb = 3 + j4$ is a vector in the first quadrant
$\dot{B} = -a + jb = -3 + j2$ is a vector in the second quadrant
$\dot{C} = -a - jb = -2 - j3$ is a vector in the third quadrant
$\dot{D} = a - jb = 4 - j3$ is a vector in the fourth quadrant

The vector $\dot{C}$ may also be written as $\dot{C} = a + jb = -2 - j3$ where $a = -2$ and $b = -3$.

(*c*) *Trigonometric or Circular Form:* By inspection of Fig. 6–5 the components of the vector $\dot{A} = a + jb$ may be expressed as

$$a = A \cos \theta \tag{6-4}$$

$$jb = jA \sin \theta \tag{6-5}$$

where $A = \sqrt{a^2 + b^2}$ = magnitude

$\theta = \tan^{-1} \frac{b}{a}$ = direction angle

Thus,

$$\dot{A} = a + jb = A(\cos \theta + j \sin \theta) \tag{6-6}$$

This is called the trigonometric or circular form for representing a vector. The position of $\dot{A}$ depends on the angle θ through which the vector is rotated in the positive or counter-clockwise direction from the reference axis. For negative direction of rotation, the expression for vector $\dot{A}$ becomes

$$\dot{A} = A[\cos(-\alpha) + j \sin(-\alpha)]$$

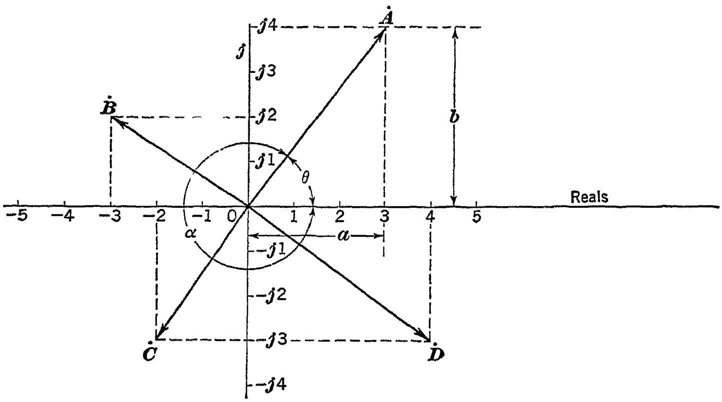


FIG. 6–5

(*d*) *Exponential Form:* From calculus, cos θ and sin θ may be expressed in the form of infinite series, as follows:

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \tag{6-7}$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \tag{6-8}$$

where θ is measured in radians. Thus,

$$\cos \theta + j \sin \theta = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} + \cdots \tag{6-9}$$

$$\cos \theta - j \sin \theta = 1 - j\theta - \frac{\theta^2}{2!} + j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - j\frac{\theta^5}{5!} - \cdots \tag{6-10}$$

By Maclaurin's Theorem, e^θ may be expanded into an infinite series, as follows:

$$e^\theta = 1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+ \cdots \tag{6–11}$$

Hence,

$$e^{j\theta} = 1+j\theta-\frac{\theta^2}{2!}-j\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+j\frac{\theta^5}{5!}+ \cdots \tag{6–12}$$

Comparison of equations (6–12) and (6–9) shows that

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{6–13}$$

Similarly,

$$e^{-j\theta} = \cos\theta - j\sin\theta \tag{6–14}$$

Thus, the vector $\dot{A}$ as given by equation (6–6) may be written in the exponential form

$$\dot{A} = Ae^{j\theta} \tag{6–15}$$

Strictly speaking θ should be given in circular measure (radians), but in most cases it may be given in degrees, as in the following case:

$$\dot{A} = 5e^{j0.927} = 5e^{j53.2^\circ}$$

(*e*) *Polar Form:* A very convenient way of writing a vector, commonly called the polar form, is

$$\dot{A} = A\underline{/\theta} \tag{6–16}$$

This comes directly from the exponential form. It must be interpreted as a vector with a magnitude A making an angle of θ degrees with the horizontal reference axis in the positive direction of rotation. In this notation, the magnitude A and the direction angle θ do not form a product and cannot be treated as such. That is, the vector $\dot{A} = 3+j4 = 5\underline{/53.2^\circ}$ in Fig. 6–5 is a vector having a magnitude of 5 and making an angle of 53.2° with the reference axis. It is wrong to say that $\dot{A}$ is equal to the product of the magnitude 5 and the direction angle 53.2°.

6–4. Equality of Complex Quantities.—If two vectors denoted by $\dot{V}_1 = a_1+jb_1$ and $\dot{V}_2 = a_2+jb_2$ are equal, then

$$a_1+jb_1 = a_2+jb_2 \tag{6–17}$$

and

$$\left.\begin{aligned} a_1 &= a_2 \\ b_1 &= b_2 \end{aligned}\right\} \tag{6–18}$$

When a vector is equal to zero,

$$\dot{V} = a + jb = 0 \qquad (6\text{–}19)$$

Then the following is true:

$$a = 0 \text{ and } b = 0$$

It should be noted that for every complex algebraic equation, as equation (6–17), there are two algebraic equations involving real quantities, as equation (6–18). Thus, two unknowns appearing in one complex equation can be determined if one is in the real part of the equation and the other is in the imaginary part.

6–5. Complex Expressions of Voltage, Current, and Impedance.—It has been mentioned that it is very useful to represent electrical quantities by vectors,* and it will now be seen that for purposes of computation it is particularly useful to represent them by vectors in the complex plane. The reason for using vectors in the complex plane to represent electrical quantities is that the laws of addition and multiplication of complex quantities happen to be identical with the laws of combination of currents and voltages, as will be shown in the succeeding paragraphs.

Electrical vectors are *rotating* vectors. A vector in the complex plane can be made a rotating vector by letting its angle increase with time. For instance, the vector $\dot{V} = V\underline{/\omega t}$ is obviously changing its angle with time in such a manner that it is rotating with a constant angular velocity of ω radians per second. Such a vector is the kind of vector that was found in Chapter 2 to be so useful for representing an alternating sinusoidal voltage or an alternating sinusoidal current. It will be remembered that instantaneous values of voltage and current are given by the projections of the rotating vectors on a fixed axis.

In drawing a vector diagram of voltage and current, some instant is selected and the positions of the vectors relative to the axes at that instant are shown. This is equivalent to taking a "snapshot" of the rotating vectors. Usually it is helpful to select an instant at which one of the vectors coincides with one of the axes, as in the following example.

* The term *phasor* has been suggested to replace vector and complex quantity in order to avoid confusion with a true space vector. Vector diagrams then become phasor diagrams.

The term *sinor* has also been suggested by Professor W. R. LePage, in "Symbolic Nomenclature for Sinusoids," *Electrical Engineering*, July, 1949.

Voltage and current may be expressed in any of the forms given in Art. 6–3.

$$\left.\begin{aligned} \dot{E} &= E_1 + jE_2 = E\underline{/\theta} = \text{etc.} \\ \dot{I} &= I_1 + jI_2 = I\underline{/\beta} = \text{etc.} \end{aligned}\right\} \tag{6–20}$$

where E_1 and I_1 are components along the real axis;
E_2 and I_2 are components along the j-axis;

$$\tan\theta = \frac{E_2}{E_1} \text{ and } \tan\beta = \frac{I_2}{I_1}.$$

Since the expressions in equation (6–20) are to represent a sinusoidal emf and a sinusoidal current of the same frequency, the time vectors $\dot{E}$ and $\dot{I}$ rotate at the same angular velocity and have an angle of $(\theta - \beta)$ between them.

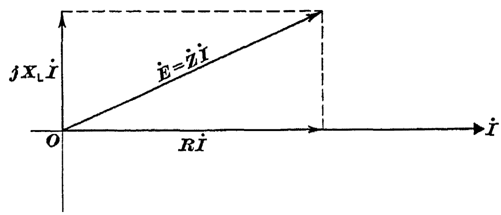


FIG. 6–6

In a simple series circuit consisting of resistance R and inductive reactance $X_L = 2\pi fL$, the resistance drop is in phase with the current and the inductive drop leads the current by 90°. If the direction of $\dot{I}$ is so chosen that it coincides with the reference axis, as shown in Fig. 6–6, then the complex expression for the current is

$$\dot{I} = I + j0 = I\underline{/0°}$$

Since the RI-vector is in phase with $\dot{I}$, its complex expression is $R\dot{I}$. The X_LI-vector is at right angles to and leads $\dot{I}$, and its complex expression is $jX_L\dot{I}$.

By Kirchhoff's Emf Law, or from Fig. 6–6, the voltage applied across the terminals of the series circuit is

$$\dot{E} = R\dot{I} + jX_L\dot{I} = \dot{Z}\dot{I} \tag{6–21}$$

where

$$\dot{Z} = R + jX_L = Z\underline{/\tan^{-1}\frac{X_L}{R}} \tag{6–22}$$

and

$$\dot{Z} = \sqrt{R^2 + X_L^2} = \text{magnitude} \tag{6–23}$$

For a capacitive circuit consisting of resistance R and capacitive reactance $X_C = \frac{1}{2\pi fC}$ in series, the impedance is

$$\dot{Z} = R - jX_C = Z \big/ \tan^{-1} \frac{-X_C}{R} \qquad (6\text{–}24)$$

where $$Z = \sqrt{R^2 + (-X_C)^2} = \text{magnitude}$$

Although the impedance of a circuit is expressed in the same form as that for voltage and current, its components R and X do not change in value with time. Thus, perhaps, it is better not to consider $\dot{Z}$ as a revolving vector like that of voltage or current, but to treat it as a complex operator or an impedance operator. An impedance operator has the property of modifying the magnitude and changing the direction angle of the vector when the impedance operator is associated with the vector. For example, when $\dot{Z} = R$, the operator simply changes the magnitude of the vector $\dot{I}$ in the expression $\dot{Z}\dot{I} = R\dot{I}$. When $\dot{Z} = jX_L$, the operator modifies the length of the vector $\dot{I}$ by the factor X_L and advances the resultant by an angle of 90° in the expression $\dot{Z}\dot{I} = jX_L\dot{I}$.

It is a common practice to represent an impedance $\dot{Z}$ and its components R and X as vectors in an impedance triangle with the understanding that they are not revolving time vectors. This practice will be followed hereafter.

Example 6–1.—An emf has the form $e = 100\sqrt{2} \sin \omega t$. Write its complex expression in both polar form and rectangular form when: (*a*) $\omega t = 0$, (*b*) $\omega t = 60°$, and (*c*) $\omega t = 90°$.

Solution.—(*a*) For $\omega t = 0$,

$$\dot{E} = 100 \underline{/0°} = 100(\cos 0° + j \sin 0°) = 100 + j0$$

(*b*) For $\omega t = 60°$,

$$\dot{E} = 100 \underline{/60°} = 100(\cos 60° + j \sin 60°) = 50 + j86.6$$

(*c*) For $\omega t = 90°$,

$$\dot{E} = 100 \underline{/90°} = 100(\cos 90° + j \sin 90°) = 0 + j100$$

6–6. Addition of Complex Quantities.—Although voltages, currents, and impedances are represented in the same form, they must not be added indiscriminately. Kirchhoff's Emf Law justifies the addition of the emf's in a series or closed circuit. Currents at a point may be added in accordance with Kirchhoff's Current Law. Also, it has been shown in Chapter 5 that impedances connected in series may be added vectorially; that is, the resistive components are added together, and the reactive components are added together.

The addition of two complex quantities $\dot{V}_1 = a_1 + jb_1$ and $\dot{V}_2 = a_2 + jb_2$ can be performed graphically as shown in Fig. 6–7 or analytically as follows:

$$\dot{V}_3 = \dot{V}_1 + \dot{V}_2$$

or

$$\dot{V}_3 = (a_1 + jb_1) + (a_2 + jb_2) \tag{6–25}$$

By inspection of Fig. 6–7, it can be seen that the real part of $\dot{V}_3$ is the sum of the real parts of $\dot{V}_1$ and $\dot{V}_2$ and the j part of $\dot{V}_3$ is found by adding the j parts of $\dot{V}_1$ and $\dot{V}_2$. Hence,

$$\dot{V}_3 = (a_1 + a_2) + j(b_1 + b_2) \tag{6–26}$$

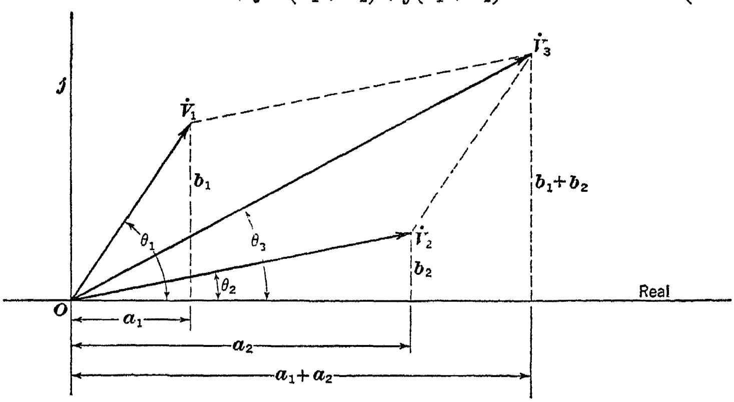


FIG. 6–7

The length of $\dot{V}_3$ (hypotenuse of a right triangle) is given by the following expression:

$$V_3 = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} \tag{6–27}$$

and the direction angle is

$$\theta_3 = \tan^{-1}\left(\frac{b_1 + b_2}{a_1 + a_2}\right) \tag{6–28}$$

Hence, the polar form is:

$$\dot{V}_3 = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} \,\Big/ \tan^{-1}\frac{b_1 + b_2}{a_1 + a_2} = V_3 \underline{/\theta_3} \tag{6–29}$$

If $\dot{V}_1$ and $\dot{V}_2$ are given in the polar or exponential form, they must be changed to the rectangular or the "$a + jb$" form before the addition can be performed. The addition of any number of complex quantities can be performed by extending this same operation, that is, by adding the real parts together and then adding the j parts together, taking into account the signs of the reals and the imaginaries. To add the real parts to the j parts would be meaningless. Thus, the law of ordinary addition applies with the one provision that real and j components must be taken separately. Accordingly,

$$\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dots + \dot{V}_n$$
$$= (a_1 + jb_1) + (a_2 + jb_2) + (a_3 + jb_3) + \dots + (a_n + jb_n)$$

or $$\dot{V} = (a_1 + a_2 + \dots + a_n) + j(b_1 + b_2 + \dots + b_n) \quad (6\text{–}30)$$

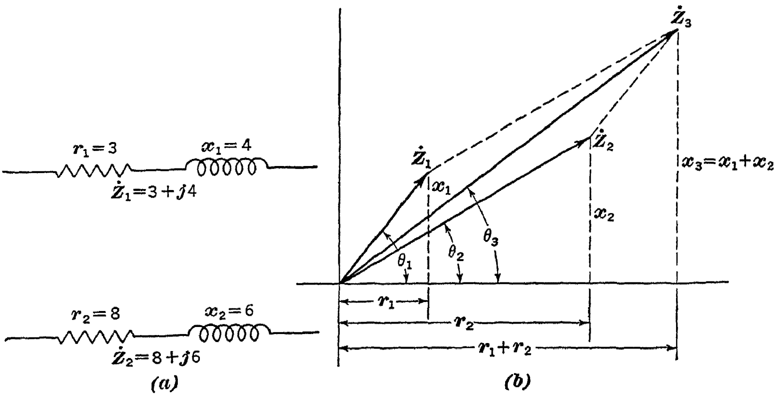


FIG. 6–8

Example 6–2.—Two impedances each consisting of resistance and inductive reactance are shown in Fig. 6–8(*a*). It is desired to find, by the method of complex notation, the resultant impedance when the two impedances are connected in series.

Solution.—The impedances are

$$\dot{Z}_1 = 3 + j4$$
$$\dot{Z}_2 = 8 + j6$$

By adding the resistances or real parts together and then the reactances or j parts together, the resultant impedance is found to be

$$\dot{Z}_3 = \dot{Z}_1 + \dot{Z}_2 = (3 + 8) + j(4 + 6) = 11 + j10$$
$$= \sqrt{(11)^2 + (10)^2} \Big/ \tan^{-1}\frac{10}{11} = 14.9\underline{/42\ 3^\circ}$$

Hence, this series combination is equivalent to a series circuit having a resistance of 11 ohms and an inductive reactance of 10 ohms. The resultant $\dot{Z}_3$ may also be obtained graphically as shown in Fig. 6–8(*b*). It should be noted that $11 + j10$ is a complete specification of the impedance $\dot{Z}_3$, and is as adequate as $14.9\underline{/42.3^\circ}$.

Example 6–3.—An *RL* circuit having an impedance of $5\angle 36.8°$ and an *RC* circuit having an impedance of $10\angle -53.2°$ are shown in Fig. 6–9. Find the impedance of a series circuit equivalent to the two impedances connected in series.

Solution.—The given impedances in the polar form or the exponential form are:

$$\dot{Z}_1 = 5\angle 36.8° = 5e^{j36.8°}$$

$$\dot{Z}_2 = 10\angle -53.2° = 10e^{-j53.2°}$$

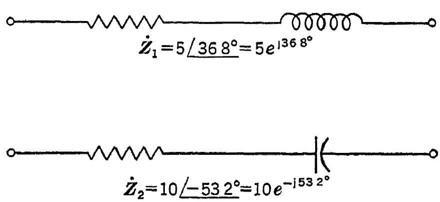


FIG. 6–9

Before the addition can be performed, they must be changed to the "$a+jb$" form, as follows:

$$\dot{Z}_1 = 5\angle 36.8° = 5e^{j36.8°}$$

$$= 5(\cos 36.8° + j\sin 36.8°) = 4 + j3 = r_1 + jx_1$$

$$\dot{Z}_2 = 10\angle -53.2° = 10e^{-j53.2°}$$

$$= 10[\cos(-53.2°) + j\sin(-53.2°)] = 6 - j8 = r_2 - jx_2$$

Hence,

$$\dot{Z} = \dot{Z}_1 + \dot{Z}_2 = (4+j3) + (6-j8) = 10 - j5$$

$$= \sqrt{(10)^2 + (-5)^2}\,\Big/\tan^{-1}\frac{-5}{10} = 11.2\angle -26.5° = 11.2e^{-j26.5°}$$

Since, by definition, $(+x)$ is inductive reactance and $(-x)$ is capacitive reactance, the series combination is equivalent to a circuit having a resistance of 10 ohms and a capacitive reactance of 5 ohms in series.

6–7. Subtraction of Complex Quantities.—As in the subtraction of algebraic quantities, one vector may be subtracted from another by changing its sign and adding. The subtraction of the vector $\dot{V}_2 = a_2 + jb_2$ from another vector $\dot{V}_1 = a_1 + jb_1$ can be performed graphically as shown in Fig. 6–10, where $\dot{V}_3$ is the resultant vector.

Analytically, the operation is indicated as follows:

$$\dot{V}_3 = \dot{V}_1 - \dot{V}_2 = (a_1 + jb_1) - (a_2 + jb_2) \qquad \textbf{(6–31)}$$

This is equivalent to

$$\dot{V}_3 = (a_1 + jb_1) + (-a_2 - jb_2) = (a_1 - a_2) + j(b_1 - b_2)$$

or
$$\dot{V}_3 = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2} \Big/ \tan^{-1} \frac{b_1 - b_2}{a_1 - a_2} = V_3 \underline{/\theta_3} \qquad (6\text{–}32)$$

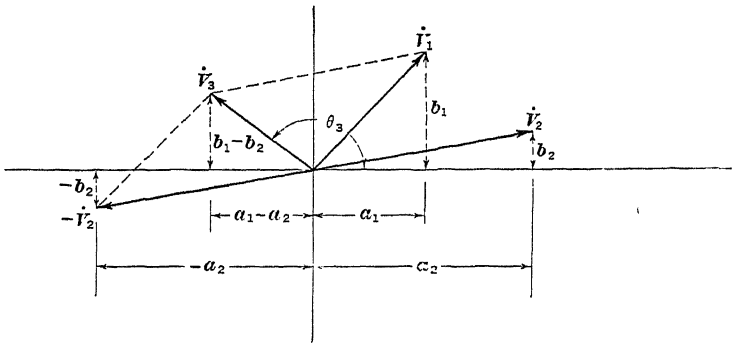


FIG. 6–10

The operation can be extended to cover the addition and subtraction of any number of vectors. The law of subtraction applies with the provision that real and j components must be taken separately, as follows:

Let
$$\dot{V} = \dot{V}_1 - \dot{V}_2 + \dot{V}_3 - \dot{V}_4$$

where $\dot{V}_1 = a_1 + jb_1$;
$\dot{V}_2 = a_2 - jb_2$;
$\dot{V}_3 = -a_3 - jb_3$;
$\dot{V}_4 = -a_4 + jb_4$.

Then
$$\dot{V} = (a_1 + jb_1) - (a_2 - jb_2) + (-a_3 - jb_3) - (-a_4 + jb_4)$$

or
$$\dot{V} = (a_1 - a_2 - a_3 + a_4) + j(b_1 + b_2 - b_3 - b_4) \qquad (6\text{–}33)$$

6–8. Multiplication of Complex Quantities.—For multiplication, the complex quantities may be expressed in the rectangular, polar, or exponential form. The polar form is preferable because it requires the least amount of labor.

(a) *The "$a + jb$" Form:* If $\dot{V}_1 = a_1 + jb_1$ and $\dot{V}_2 = a_2 + jb_2$, and $\dot{V} = \dot{V}_1 \times \dot{V}_2$, then

$$\dot{V} = (a_1 + jb_1)(a_2 + jb_2) = a_1a_2 + ja_1b_2 + jb_1a_2 + j^2b_1b_2 \qquad (6\text{–}34)$$

Since $j^2 = -1$, equation (6–34) becomes

$$\dot{V} = (a_1a_2 - b_1b_2) + j(a_1b_2 + b_1a_2)$$

or

$$\dot{V} = \sqrt{(a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2} \underline{/\tan^{-1} \frac{a_1b_2 + b_1a_2}{a_1a_2 - b_1b_2}} \quad (5\text{–}35)$$

Example 6–4.—Find the product of $(3+j4)$ and $(8-j6)$.

Solution.—The product is:

$$\dot{V} = (3+j4)(8-j6)$$
$$= (3\times 8) - (3\times j6) + (j4\times 8) - (j^2 4\times 6)$$
$$= 48 + j14 = 50\underline{/16.4°}$$

(b) *The Exponential or Polar Form:* Let the vectors be given in the forms

$$\dot{V}_1 = V_1 e^{j\theta_1} = V_1\underline{/\theta_1}$$

$$\dot{V}_2 = V_2 e^{j\theta_2} = V_2\underline{/\theta_2}$$

Then their product is a complex quantity whose magnitude is equal to the product of the two magnitudes of the original quantities, and whose angle is the algebraic sum of the original angles. Thus, if $\dot{V} = \dot{V}_1 \times \dot{V}_2$,

$$\dot{V} = V_1 e^{j\theta_1} \times V_2 e^{j\theta_2} = V_1 V_2 e^{j(\theta_1+\theta_2)} \quad (6\text{–}36)$$

or

$$\dot{V} = V_1\underline{/\theta_1} \times V_2\underline{/\theta_2} = V_1V_2\underline{/\theta_1 + \theta_2} \quad (6\text{–}37)$$

It has been demonstrated in Chapter 5 that Ohm's Law, or $\dot{E} = \dot{Z}\dot{I}$, may be applied as follows: The magnitude of $\dot{E}$ is equal to the product of the magnitudes of $\dot{Z}$ and $\dot{I}$; the angle of $\dot{E}$ is obtained by adding or subtracting the angle of $\dot{Z}$, or $\theta = \tan^{-1} \frac{X}{R}$, from that of the current. In an RL circuit the angle is added; while in an RC circuit the angle is subtracted.

Thus, Ohm's Law, which is a law of multiplication, lends itself very nicely to the treatment by complex quantities.

Example 6–5.—The current flowing in a series circuit containing a resistance $r = 3$ and an inductive reactance $x = 4$ is $\dot{I} = 80 + j60$. Find the expression for the voltage across the terminals of the circuit.

Solution.—The current may be expressed in polar form as follows:

$$\dot{I} = 80 + j60 = 100\underline{/36.8°}$$

The impedance of the circuit is

$$\dot{Z} = 3 + j4 = 5\underline{/53.2°}$$

By Ohm's Law for alternating current circuits, the voltage is

$$\dot{E} = \dot{Z}\dot{I}$$
$$= 5\underline{/53.2°} \times 100\underline{/36.8°} = 5 \times 100\underline{/53.2° + 36.8°} = 500\underline{/90°} = 0 + j500$$

Thus, when the vector $\dot{I}$ is operated upon by $\dot{Z}$, its length is increased from 100 to 500 and its angle is changed from 36.8° to 90°. The result is the voltage vector $\dot{E}$.

6–9. Division of Complex Quantities.—For division, as for multiplication, it is usually convenient to have the vectors expressed in the polar form.

(*a*) *Rectangular Form:* If the vectors are expressed in the "$a + jb$" form,

$$\dot{V} = \frac{\dot{V}_1}{\dot{V}_2} = \frac{a_1 + jb_1}{a_2 + jb_2} \tag{6–38}$$

In order to separate the real and imaginary parts of this expression, it is necessary to multiply both the numerator and denominator of the fraction by $(a_2 - jb_2)$. This operation, which is equivalent to multiplying by unity, is called *rationalizing the fraction.* The expression $(a - jb)$ is the conjugate of $(a + jb)$, and vice versa. They differ only in the sign of the j term. To rationalize any fraction involving complex numbers, the numerator and the denominator are multiplied by the conjugate of the denominator. Then,

$$\dot{V} = \frac{\dot{V}_1}{\dot{V}_2} = \frac{a_1 + jb_1}{a_2 + jb_2} \times \frac{a_2 - jb_2}{a_2 - jb_2}$$

or

$$\dot{V} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + j\,\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2} \tag{6–39}$$

It should be noted that the denominator $(a_2^2 + b_2^2)$ of each term in equation (6–39) is a real number; therefore, the first term is purely real and the second is purely imaginary.

(*b*) *Exponential or Polar Form:* The division of two vectors expressed in the exponential or polar form gives a complex quantity whose magnitude is the quotient derived by division of the magnitudes of the original quantities and whose angle is the difference between the angle of the dividend and the angle of the divisor. Thus,

$$\dot{V} = \frac{\dot{V}_1}{\dot{V}_2} = \frac{V_1 e^{j\theta_1}}{V_2 e^{j\theta_2}} = \frac{V_1}{V_2} e^{j(\theta_1 - \theta_2)} \tag{6-40}$$

or

$$\dot{V} = \frac{V_1 \underline{/\theta_1}}{V_2 \underline{/\theta_2}} = \frac{V_1}{V_2} \underline{/\theta_1 - \theta_2} \tag{6-41}$$

Example 6-6.—The expression $\dot{E} = 100\underline{/30°}$ represents the voltage across a series circuit whose impedance is $\dot{Z} = 5\underline{/53°} = 3 + j4$. Find the vector of the current flowing in the circuit. Draw the vector diagram.

Solution.—The current flowing in a series circuit is equal to the voltage applied across the terminals of the circuit divided by the impedance of the circuit. Thus,

$$\dot{I} = \frac{\dot{E}}{\dot{Z}}$$

Substituting values for $\dot{E}$ and $\dot{Z}$ gives

$$\dot{I} = \frac{100\underline{/30°}}{5\underline{/53°}} = \frac{100}{5}\underline{/30° - 53°} = 20\underline{/-23°} = 18.41 - j7.81$$

The vector diagram is shown in Fig. 6-11.

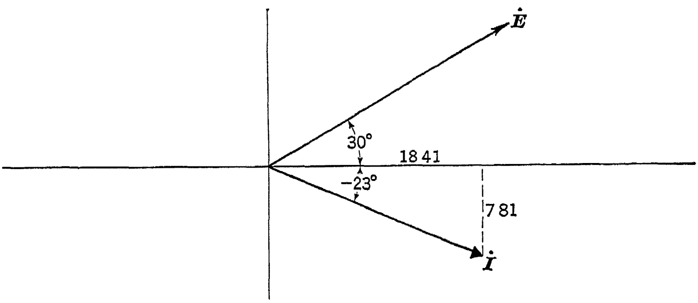


FIG. 6-11

6-10. Evolution (Powers).—When a complex number is to be raised to the nth power, the rule for multiplication applies. It will be shown that the result is a complex quantity whose magnitude is the nth power of the original magnitude and whose angle is n times the original angle.

(*a*) *The Exponential Form:* If a vector is expressed in the exponential form, as $\dot{V} = Ve^{j\theta}$, then

$$(\dot{V})^2 = (Ve^{j\theta})^2 = VVe^{j\theta}e^{j\theta} = V^2e^{j2\theta}$$
$$(\dot{V})^3 = (Ve^{j\theta})^3 = V^2Ve^{j2\theta}e^{j\theta} = V^3e^{j3\theta}$$

. . . .

and

$$(\dot{V})^n = (Ve^{j\theta})^n = V^n e^{jn\theta} \tag{6-42}$$

(*b*) *The Polar Form:* If $\dot{V} = V\angle\theta$, then

$$(\dot{V})^2 = (V\angle\theta)^2 = V^2\angle 2\theta$$

$$\cdots\cdots\cdots\cdots\cdots$$

and $$(\dot{V})^n = (V\angle\theta)^n = V^n\angle n\theta \qquad (6\text{–}43)$$

(*c*) *The Circular Form:* If a vector is expressed in the circular form, its *n*th power is obtained as follows:

$$(\dot{V})^2 = [V(\cos\theta + j\sin\theta)]^2 = V^2(\cos 2\theta + j\sin 2\theta)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and $$(\dot{V})^n = [V(\cos\theta + j\sin\theta]^n = V^n(\cos n\theta + j\sin n\theta) \qquad (6\text{–}44)$$

Example 6–7.—Let it be required to raise the complex number $\dot{Z} = 4 + j3$ to the second power.

Solution.—It is most convenient to use either the exponential form or the polar form. Thus,

$$\dot{Z} = 4 + j3 = 5e^{j36.8^\circ} = 5\angle 36.8^\circ$$

$$\begin{aligned}\dot{Z}^2 &= (4 + j3)^2\\ &= (5e^{j36.8^\circ})^2 = 25e^{j73.6^\circ}\\ &= (5\angle 36.8^\circ)^2 = 25\angle 73.6^\circ = 7 + j24\end{aligned}$$

6–11. Involution (Roots).—The relation (6–44) is known as DeMoivre's Theorem and it holds for both integral and fractional values of *n*. If a fractional value of *n* is used, this theorem furnishes a simple means for finding a root of a complex quantity. Thus, if $\dot{V} = Ve^{j\theta} = V\angle\theta = V(\cos\theta + j\sin\theta)$, then

$$\sqrt[m]{\dot{V}} = V^{\frac{1}{m}}e^{j\frac{\theta}{m}} = V^{\frac{1}{m}}\angle\frac{\theta}{m} = V^{\frac{1}{m}}\left(\cos\frac{\theta}{m} + j\sin\frac{\theta}{m}\right) \qquad (6\text{–}45)$$

where *m* is any integer, as 1, 2, 3, etc.

Hence, the *m*th root of a complex quantity is a complex quantity whose magnitude is the *m*th root of the original magnitude and whose angle is $\frac{1}{m}$ times the original angle.

6–12. Computation of Power in Circuit.—If the voltage and the current in a circuit are expressed in the polar form, or $\dot{E} = E\angle\theta_E$ and $\dot{I} = I\angle\theta_I$, then the average power is

$$P = EI\cos(\theta_E - \theta_I) \qquad (6\text{–}46)$$

If the voltage and the current are given in the rectangular form, the data are as follows:

$$\dot{E} = E_1 + jE_2$$
$$\dot{I} = I_1 + jI_2$$

The average power cannot be determined directly from the product of (E_1+jE_2) and (I_1+jI_2). The average power may, however, be obtained from the voltage components and the current components, without converting $\dot{E}$ and $\dot{I}$ to the polar form, in the following way. By Ohm's Law,

$$\dot{Z}=\frac{\dot{E}}{\dot{I}}=\frac{E_1+jE_2}{I_1+jI_2} \tag{6–47}$$

Rationalizing gives

$$\dot{Z}=\frac{E_1+jE_2}{I_1+jI_2}\times\frac{I_1-jI_2}{I_1-jI_2}$$

or

$$\dot{Z}=\frac{E_1I_1+E_2I_2}{I_1^2+I_2^2}+j\,\frac{E_2I_1-E_1I_2}{I_1^2+I_2^2}=R+jX \tag{6–48}$$

The average power is

$$P=I^2R$$

This expression for power is not a vector equation as only the *magnitude* of the current is used; therefore, $\dot{I}^2=(I_1^2+I_2^2)$. Then, substituting for R its value from equation (6–48), we have

$$P=(I_1^2+I_2^2)\times\frac{E_1I_1+E_2I_2}{(I_1^2+I_2^2)}$$

or

$$P=E_1I_1+E_2I_2 \tag{6–49}$$

Equation (6–49) states that, if the voltage and the current are given in the rectangular form, the power is the sum of the product of the real components and the product of the j components. The signs of the product terms are determined in the ordinary algebraic manner.

Example 6–8.—Find the power delivered to the circuit in Example 6–6.

Solution.—The expressions for the voltage and current are:

$$\dot{E}=100\underline{/30°}=86.6+j50 \text{ vector volts}$$

$$\dot{I}=20\underline{/-23°}=18.41-j7.81 \text{ vector amperes}$$

The average power is

$$P=100\times 20\cos[30°-(-23°)]=1200 \text{ watts}$$

or

$$P=86.6\times 18.41+[50\times(-7.81)]=1200 \text{ watts}$$

or

$$P=I^2R=20^2\times 3=1200 \text{ watts}$$

6–13. Summary.—(*a*) *Operator j:* The operator $j=\sqrt{-1}=1\underline{/90°}$ has the property of rotating a vector through an angle of 90° in the counter-clockwise or positive direction.

(*b*) *Representation of Vectors:* The common forms of representing a vector are the vectorial, the rectangular, and the polar forms. Thus,

$$\dot{V} = a + jb = V\angle\theta$$

where $\theta = \tan^{-1}\frac{b}{a}$.

(*c*) *Addition of Complex Quantities:* Write the quantities in the "$a+jb$" form. Add (algebraically) the real parts together and then add (algebraically) the j parts together.

(*d*) *Subtraction of Complex Quantities:* Write the quantities in the "$a+jb$" form. Subtract (algebraically) the real parts and then subtract (algebraically) the j parts.

(*e*) *Multiplication:* Write the quantities in the polar form, as $\dot{V}_1 = V_1\angle\theta_1$ and $\dot{V}_2 = V_2\angle\theta_2$. Multiply the magnitudes together and add the angles. Thus,

$$\dot{V}_1\dot{V}_2 = V_1V_2\angle\theta_1 + \theta_2$$

(*f*) *Division:* Write the quantities in the polar form. Take the quotient of the magnitudes and subtract the angles. Thus,

$$\frac{\dot{V}_1}{\dot{V}_2} = \frac{V_1}{V_2}\angle\theta_1 - \theta_2$$

(*g*) *Power:* When the voltage and the current are expressed as $\dot{E} = E\angle\theta_E$ and $\dot{I} = I\angle\theta_I$, the average power is

$$P = EI\cos(\theta_E - \theta_I)$$

If the voltage and current are expressed in the rectangular form, as $\dot{E} = E_1 + jE_2$ and $\dot{I} = I_1 + jI_2$, then the average power is the algebraic sum of the product of the real parts, or E_1I_1, and the product of the j parts, or E_2I_2. Thus,

$$P = E_1I_1 + E_2I_2$$

PROBLEMS

6–1. Show graphically in the complex plane the following vectors: (*a*) $\dot{V}_1 = 3+j4$; (*b*) $\dot{V}_2 = -10+j5$; (*c*) $\dot{V}_3 = +5-j5$; (*d*) $\dot{V}_4 = -6-j8$.

6–2. Show graphically the following vector expressions: (*a*) $\dot{V}_0 = 2+j2$; (*b*) $j\dot{V}_0$; (*c*) $-j\dot{V}_0$; (*d*) $j^2\dot{V}_0$; (*e*) $-j^2\dot{V}_0$; (*f*) $j^3\dot{V}_0$; (*g*) $-j^3\dot{V}_0$; (*h*) $j^4\dot{V}_0$; (*i*) $-j^4\dot{V}_0$.

6–3. Evaluate: j^6, j^{17}, j^{29}, j^{68}, j^{212}.

6–4. The current flowing in an RL circuit is $\dot{I} = 10 - j4$ when the expression for the applied emf is $\dot{E} = 80 + j60$. Draw the two vectors in the complex plane.

6–5. Convert the rectangular form of the vectors in Problem 6–1 into the circular form. Find the magnitude of each vector and the angle it makes with the reference axis.

6–6. Transform the expressions $\dot{V}_1 = -3 - j4$ and $\dot{V}_2 = 6 - j3$ into: (*a*) the circular form; (*b*) the exponential form; and (*c*) the polar form. Sketch the given vectors in the complex plane.

6–7. Give the values of a and b in the following expressions:

(*a*) $a + jb = 7 - j3$
(*b*) $a + jb = 10\underline{/-30°}$
(*c*) $a + jb = 100e^{j150°}$
(*d*) $a + jb = 50(-\cos 75° + j \sin 75°)$

6–8. A vector having a magnitude of 100 makes an angle of 60° with the axis of reals and is in the third quadrant. Show how it is expressed in: (*a*) rectangular form; (*b*) circular form; (*c*) exponential form; (*d*) polar form. Draw the vector in the complex plane.

6–9. The vector in Problem 6–8 is rotated through an angle of 90° in the positive direction. Show how it is expressed in: (*a*) rectangular form; (*b*) circular form; (*c*) exponential form; (*d*) polar form.

6–10. Add the following vectors and show graphically in the complex plane their individual positions and the position of their resultant:

$$\dot{V}_1 = 6 + j4 \text{ and } \dot{V}_2 = -3 + j5$$

What are the complex expression and magnitude of the resultant vector?

6–11. The expressions for the voltage drops across two impedance coils connected in series are $\dot{E}_1 = 30 + j40$ and $\dot{E}_2 = 50 + j30$, respectively. What is the vector expression for the emf impressed across the two coils?

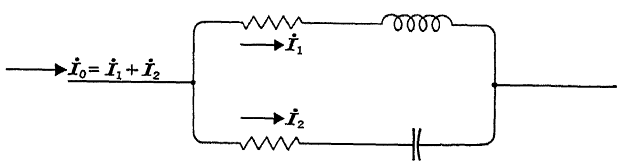


FIG. 6–12

6–12. The expressions for $\dot{I}_0$ and $\dot{I}_1$ in the circuit of Fig. 6–12 are $11 + j4$ and $3 - j1$, respectively. Find the vector current $\dot{I}_2$. Sketch the three vector currents in the complex plane.

6–13. Multiply $(-50 + j75)$ by $(60 - j80)$; $(40 + j60)$ by $(-30 - j30)$. Sketch graphically each quantity and their products.

6–14. Find the reciprocals of the vectors $\dot{V}_1 = 10 - j7$ and $\dot{V}_2 = -4 - j3$. Sketch graphically the position of each vector and of its reciprocal.

6–15. Rationalize the following expressions:

(*a*) $\dfrac{-2 + j3}{-4 + j8}$ (*b*) $\dfrac{-8 - j8}{10 - j7}$

6–16. Evaluate:

(a) $\dfrac{10+j5}{20-j16}+10\underline{/30°}$

(b) $50(\cos 30°+j \sin 30°)-10\epsilon^{j60°}$

(c) $(100\underline{/53.2°})\dfrac{50\underline{/20°}}{80-j60}$

6–17. Find:

(a) $(3-j7)^2$

(b) $(100\underline{/63°})^2$

(c) $[(3-j7)+100\underline{/63°}]^2$

6–18. Three vectors $\dot{V}_1$, $\dot{V}_2$, and $\dot{V}_3$ are related by the following expressions

$$\dot{V}_1-\dot{V}_2-\dot{V}_3=7+j20$$
$$\dot{V}_1+\dot{V}_2+\dot{V}_3=10-j17$$

The magnitude of $\dot{V}_1$ is twice that of $\dot{V}_2$ and $\dot{V}_1$ leads $\dot{V}_2$ by 90°. What are the complex expressions of the three vectors?

6–19. Two time vectors $\dot{A}$ and $\dot{B}$, having constant magnitudes of 10 and 5, respectively, revolve in the counter-clockwise direction at a constant angular velocity of 157 radians per second. If $\dot{A}$ lags behind $\dot{B}$ by 60 degrees, and at $t=0$ vector $\dot{A}$ coincides with the axis of reals, what are the rectangular and polar expressions for the vectors?

6–20. An inductive coil having $R=3$ ohms and $X_L=4$ ohms is connected to a 110-volt, 60-cycle source. At an instant when the emf vector makes a positive angle of 30° with the reference axis (first quadrant), what are the polar and rectangular expressions for the current? Draw the vector diagram.

6–21. An emf $\dot{E}=80-j60$ produces a current $\dot{I}=6+j8$ in a circuit. What are the polar and rectangular expressions for the impedance of the circuit? If the sinusoidal expression for the emf is $e=141.4 \sin 377t$, what are the polar and rectangular expressions for the current at the instant when $377t=\dfrac{\pi}{2}$?

6–22. A 500-watt, 220-volt lamp having a constant pure resistance is connected in series with a coil having $R=2$ ohms and $X_L=30$ ohms measured with a 25-cycle source. What are the polar and rectangular expressions for the current when the emf applied across the combination has the form $\dot{E}=110\underline{/75°}$? Find the power supplied to the combination by the complex method.

6–23. A coil having $R=9.3$ ohms and $L=0.01$ henry is connected to a 60-cycle source. If the expression for the current is $6.5-j12.4$, what is the corresponding polar expression for the impressed voltage at the same instant? Find the power taken by the complex method.

6–24. Two coils of an alternator have emfs given by the expressions.

$$e_1=100 \sin (377t-10°)$$
$$e_2=100 \sin (377t+20°)$$

Determine the complex expressions for the vectors which represent these emfs and their sum at the instant when: (a) $t=0$; (b) $t=0.01$ second

6–25. A circuit takes 850 watts from a 60-cycle source. The complex expression for the current is $\dot{I} = 10 + j30$ amp (effective). If the imaginary part of the emf is $-j75$ volts (effective), what is the reading of a voltmeter measuring the voltage of the source?

6–26. An impedance coil takes 30 amp from a 100-volt, 60-cycle source. When the coil is connected in series with a non-inductive resistance, the combination takes 10 amp from the same source. If the current is taken as $\dot{I} = 8 + j6$ and the impressed emf as $\dot{E}_0 = 60 + j80$, what are the complex expressions for the voltage drops across the coil and the non-inductive resistance?

CHAPTER 7

PARALLEL CIRCUITS

Two impedances are said to be connected in parallel when the *same* voltage is across each of them, as in Fig. 7–1. Thus, in a parallel circuit the voltage is the common factor. This is analogous to current as the common factor in a series circuit. In practice, parallel circuits are very common, as nearly all incandescent lamps and motors are connected for parallel operation.

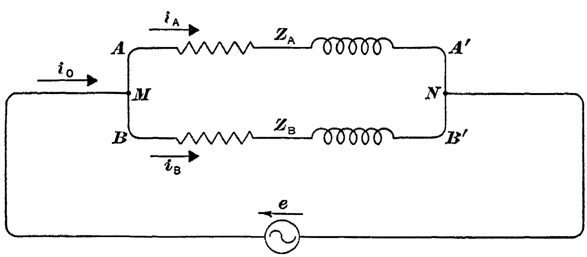


FIG. 7–1

One point to notice is that each branch of Fig. 7–1 may be treated as a simple series circuit, as though the other branch were not there. The current obtained for each branch will be the same in magnitude and phase no matter how many branches are connected in parallel with it, provided the impressed emf is not altered by connecting in the additional branches. The solution of a problem with two or more loads in parallel involves the finding of the current in each branch of the circuit and the combining of these currents by Kirchhoff's Current Law to give the resultant current in the line wires. Kirchhoff's Current Law may be expressed as follows:

(*a*) *For Instantaneous Values:* At any given instant the sum of the instantaneous values of all the currents flowing toward a point is equal to the sum of the instantaneous values of all the currents flowing away from the point.

(b) *For Vectors:* The vector sum of all the vector currents flowing toward a point must be equal to the vector sum of all the vector currents flowing away from the point.

7–1. Addition of Currents by Using Instantaneous Values.—In a series circuit the current is common and voltages across the different parts of the circuit are added to obtain the applied emf; while in a parallel circuit there is the same voltage across the different branches and the currents flowing in the different loads are added to obtain the line current. For simplicity, consider the case with two impedance coils connected in parallel as in Fig. 7–1.

Since the applied voltage is the common factor, it is convenient to use it as a reference quantity. Thus,

$$e = E_m \sin \omega t$$

The current flowing in branch AA' is

$$i_A = \frac{E_m}{Z_A} \sin (\omega t - \theta_A) = I_{Am} \sin (\omega t - \theta_A) \tag{7–1}$$

where $I_{Am} = \frac{E_m}{Z_A}$ and θ_A is the power factor angle of branch AA'. In an RL circuit the current lags behind the applied emf and $\theta_A = \tan^{-1} \frac{\omega L}{R}$.

The current in branch BB' is

$$i_B = \frac{E_m}{Z_B} \sin (\omega t - \theta_B) = I_{Bm} \sin (\omega t - \theta_B) \tag{7–2}$$

where $I_{Bm} = \frac{E_m}{Z_B}$ and θ_B is the power factor angle of branch BB'.

By Kirchhoff's Current Law, the line current is

$$i_0 = i_A + i_B$$

or

$$i_0 = I_{Am} \sin (\omega t - \theta_A) + I_{Bm} \sin (\omega t - \theta_B) \tag{7–3}$$

Expanding the sine terms gives:

$$i_0 = I_{Am} \sin \omega t \cos \theta_A - I_{Am} \cos \omega t \sin \theta_A + I_{Bm} \sin \omega t \cos \theta_B - I_{Bm} \cos \omega t \sin \theta_B \tag{7–4}$$

Collecting terms, we have:

$$i_0 = (I_{Am} \cos \theta_A + I_{Bm} \cos \theta_B) \sin \omega t - (I_{Am} \sin \theta_A + I_{Bm} \sin \theta_B) \cos \omega t \tag{7–5}$$

Equation (7–5) may be simplified with the aid of Fig. 7–2. Let

$$\left.\begin{aligned} I_{Am} \cos \theta_A + I_{Bm} \cos \theta_B &= I_{0m1} \\ I_{Am} \sin \theta_A + I_{Bm} \sin \theta_B &= I_{0m2} \end{aligned}\right\} \quad (7\text{–}6)$$

Then,

$$I_{0m} = \sqrt{I_{0m1}^2 + I_{0m2}^2} \quad (7\text{–}7)$$

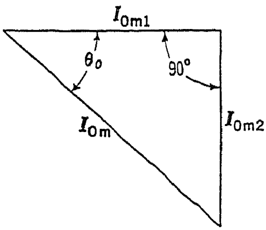


Fig. 7–2

$$\left.\begin{aligned} \cos \theta_0 &= \frac{I_{0m1}}{I_{0m}} \\ \sin \theta_0 &= \frac{I_{0m2}}{I_{0m}} \end{aligned}\right\} \quad (7\text{–}8)$$

Substituting values in equation (7–5) and multiplying by $\frac{I_{0m}}{I_{0m}}$, we have:

$$i_0 = I_{0m}\left[(\sin \omega t) \times \frac{I_{0m1}}{I_{0m}} - (\cos \omega t) \times \frac{I_{0m2}}{I_{0m}}\right]$$

$$= I_{0m}\left[\sin \omega t \cos \theta_0 - \cos \omega t \sin \theta_0\right]$$

or

$$i_0 = I_{0m} \sin (\omega t - \theta_0) \quad (7\text{–}9)$$

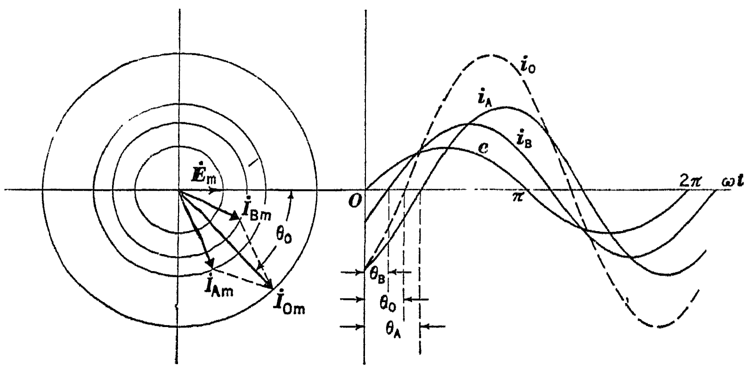


Fig. 7–3

Thus, the resulting current obtained by adding the ordinates of the sinusoidal branch currents i_A and i_B at each instant is a sine wave i_0, as shown in Fig. 7–3. Equations (7–6) to (7–9) will become clearer after studying Art. 7–2 and Fig. 7–4.

7–2. Addition of Currents by Using Vectors.—It has been shown in Chapter 5 that the magnitudes and phase angles of $\dot{I}_A$

and $\dot{I}_B$, Fig. 7-1, can be calculated. In Fig. 7–4, the vectors representing the effective values of these sinusoidal currents are drawn with the proper magnitudes and phase angles from the generator emf $\dot{E}$, and the line current $\dot{I}_0$ is found graphically by

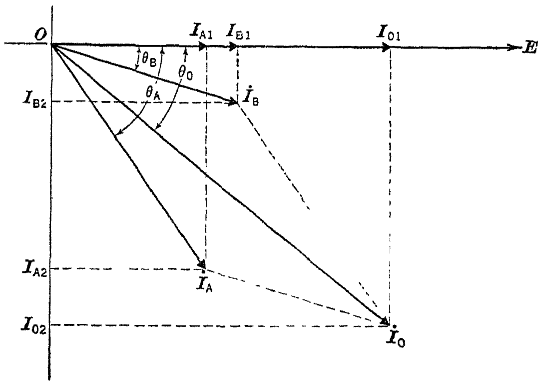


Fig. 7–4

completing the parallelogram. Its magnitude is equal to the length of the vector $\dot{I}_0$, and its phase angle θ_0 can be obtained by using a protractor. It can be seen from the vector diagram and equation (7–6) that: (*a*) the projection of the total current vector on the reference axis is equal to the algebraic sum of the projections of

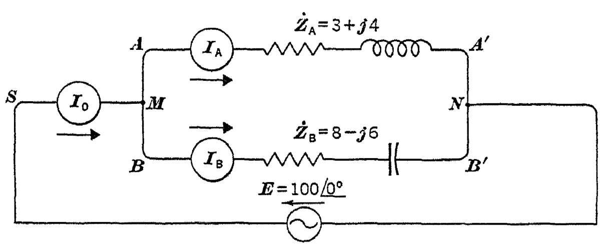


Fig. 7–5

the two branch current vectors on the same reference axis, or $I_{01} = I_{A1} + I_{B1}$; and (*b*) the projection of the total current vector on the axis at right angles to the reference axis is equal to the algebraic sum of the projection of the two branch current vectors on the same axis, or $I_{02} = I_{A2} + I_{B2}$.

Example 7–1.—Two loads $\dot{Z}_A = 3 + j4$ and $\dot{Z}_B = 8 - j6$ are connected in parallel across a 100-volt, 60-cycle source, as shown in Fig. 7–5. Find the total current flowing in the line: (*a*) by using instantaneous values and (*b*) by using vectors.

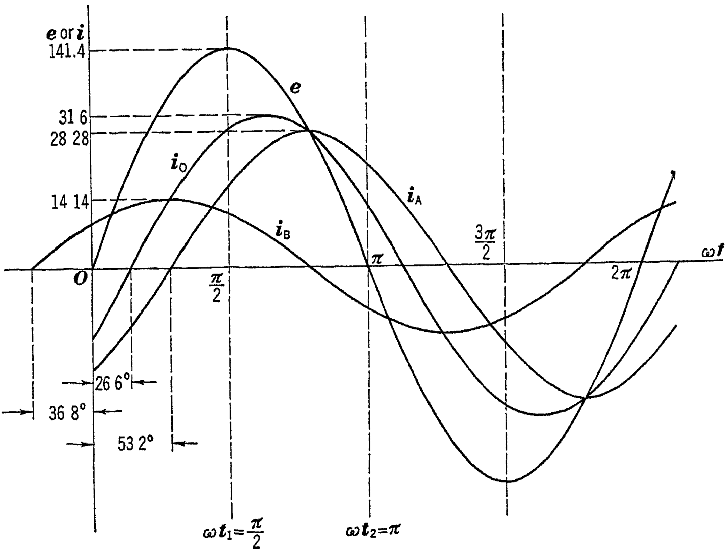


FIG. 7–6

Solution.—(*a*) *Instantaneous values:* Let the generator emf be

$$e = 141\ 4 \sin 377t$$

Then the current in load Z_A is

$$i_A = \frac{E_m}{Z_A} \sin(377t - \theta_A)$$

$$= \frac{141.4}{5} \sin (377t - 53.2°) = 28.28 \sin (377t - 53.2°)$$

where $\theta_A = \tan^{-1} \frac{4}{3} = 53.2°$.

The current in load Z_B is

$$i_B = \frac{E_m}{Z_B} \sin (377t - \theta_B)$$

$$= \frac{141.4}{10} \sin (377t + 36.8°) = 14.14 \sin (377t + 36.8°)$$

where $\theta_B = \tan^{-1} \frac{-6}{8} = -36.8°$.

By applying Kirchhoff's Current Law at point M, the line current i_0 is found to be the sum of the two load currents, or

$$\begin{aligned} i_0 &= i_A + i_B \\ &= 28.28 \sin (377t - 53.2°) + 14.14 \sin (377t + 36.8°) \\ &= 31.6 \sin (377t - 26.6°) = I_{0m} \sin (\omega t - \theta_0) \end{aligned}$$

The currents i_A, i_B, and i_0 are shown in Fig. 7–6.

(b) *Vectors:* The effective values of the two load currents are:

$$I_A = \frac{28.28}{\sqrt{2}} = 20 \text{ amp}$$

$$I_B = \frac{14.14}{\sqrt{2}} = 10 \text{ amp}$$

The impedance angles which give the phase angles are

$$\theta_A = \tan^{-1} \frac{4}{3} = 53.2°$$

$$\theta_B = \tan^{-1} \frac{-6}{8} = -36.8°$$

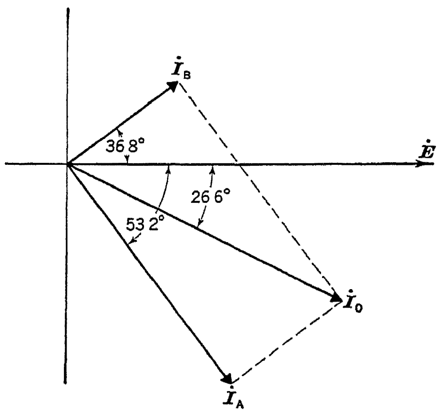


FIG. 7–7

When the applied voltage $\dot{E}$ is used as a reference vector, the vectors representing the two currents are shown in Fig. 7–7. By completing the parallelogram, the vector $\dot{I}_0$ representing the line current is obtained. Its length is equal to 22.35 amp (or $22.35\sqrt{2} = 31.6$) and its phase angle is 26.6°.

7–3. Addition of Currents by Using Symbolic Notation. Another method of solving the parallel circuit in Fig. 7–1 is to use the *Symbolic Notation.* Both $\dot{I}_A$ and $\dot{I}_B$ are resolved into their real and j components. The sum of the real components will give the real component of $\dot{I}_0$, and the sum of the j components will give the j component of $\dot{I}_0$. In the solution of a parallel circuit, the mathematics is much simplified when the applied emf is used as a reference vector. Let the generator voltage be written as

$$\dot{E} = E\underline{/0^\circ} = E + j0$$

The current in branch AA' is

$$\dot{I}_A = \frac{\dot{E}}{\dot{Z}_A} = I_A\underline{/-\theta_A} = I_{A1} - jI_{A2} \tag{7-10}$$

and the current in branch BB' is

$$\dot{I}_B = \frac{\dot{E}}{\dot{Z}_B} = I_B\underline{/-\theta_B} = I_{B1} - jI_{B2} \tag{7-11}$$

The real components are represented by the subscript 1, and the j components by the subscript 2.

The vector sum of the two branch currents gives the vector line current. Thus,

$$\dot{I}_0 = (I_{A1} + I_{B1}) - j(I_{A2} + I_{B2}) \tag{7-12}$$

or

$$\dot{I}_0 = I_{01} - jI_{02} \tag{7-13}$$

where $I_{01} = I_{A1} + I_{B1}$ (real components added together)
$I_{02} = I_{A2} + I_{B2}$ (j components added together)

7-4. Power and Power Factor.—The average power delivered to each branch in Fig. 7-1 is obtained by taking the product of the applied emf and the component of the branch current in phase with the applied emf. With the generator emf $\dot{E}$ as reference,

$$\left.\begin{aligned} P_A &= EI_{A1} = EI_A \cos\theta_A \\ P_B &= EI_{B1} = EI_B \cos\theta_B \end{aligned}\right\} \tag{7-14}$$

The total power delivered to the entire circuit is the numerical sum of the average powers delivered to the individual branches; thus,

$$P_0 = P_A + P_B = E(I_{A1} + I_{B1})$$

or

$$P_0 = EI_{01} = EI_0 \cos\theta_0 \tag{7-15}$$

The power factors of the branches and of the entire circuit are given in several forms, as follows:

$$\text{P.F.}_A = \cos\theta_A = \frac{R_A}{Z_A} = \frac{I_{A1}}{I_A} \tag{7-16}$$

$$\text{P.F.}_B = \cos\theta_B = \frac{R_B}{Z_B} = \frac{I_{B1}}{I_B} \tag{7-17}$$

$$\text{P.F.}_0 = \cos\theta_0 = \frac{I_{01}}{I_0} \tag{7-18}$$

It should be remembered that equations (7–14) to (7–18) are dependent on the fact that the voltage was selected as the reference, that is, the voltage is purely real with no j component.

Example 7–2.—Find the three currents in Example 7–1 by the Symbolic Method. Calculate the power factor and the power of each branch and the parallel combination.

Solution.—When $\dot{E}$ is used as a reference vector, the currents in the branches are:

$$\dot{I}_A = \frac{\dot{E}}{\dot{Z}_A} = \frac{100\underline{/0^\circ}}{5\underline{/53.2^\circ}} = 20\underline{/-53.2^\circ} = 12 - j16$$

$$\dot{I}_B = \frac{\dot{E}}{\dot{Z}_B} = \frac{100\underline{/0^\circ}}{10\underline{/-36.8^\circ}} = 10\underline{/36.8^\circ} = 8 + j6$$

The line current is the vector sum of the branch currents, or

$$\dot{I}_0 = \dot{I}_A + \dot{I}_B = 20 - j10 = 22.35\underline{/-26.6^\circ}$$

The power factor of the load Z_A is obtained by taking the cosine of the angle between the vectors $\dot{E} = 100\underline{/0^\circ}$ and $\dot{I}_A = 20\underline{/-53.2^\circ}$. Thus,

$$\text{P.F.}_A = \cos 53.2^\circ = 0.6$$

Similarly,

$$\text{P.F.}_B = \cos 36.8^\circ = 0.8$$
$$\text{P.F.}_0 = \cos 26.6^\circ = 0.894$$

The power delivered to each branch is

$$P_A = EI_A \cos\theta_A = 100 \times 20 \times \cos 53.2^\circ = 1200 \text{ watts}$$
$$P_B = EI_B \cos\theta_B = 100 \times 10 \times \cos 36.8^\circ = 800 \text{ watts}$$

The total power is

$$P_0 = P_A + P_B = 1200 + 800 = 2000 \text{ watts}$$

or

$$P_0 = EI_0 \cos\theta_0 = 100 \times 22.35 \times 0.894 = 2000 \text{ watts}$$

7–5. Consideration of Instantaneous Current Flow and Vector Diagram in Parallel Circuits.—Refer to Example 7–1. In order to determine the directions of instantaneous currents flowing in the loads and the line, it is necessary to decide arbitrarily on the positive sense of current flow. It is immaterial whether an instantaneous current flowing from A to A' is considered positive or a current flowing in the opposite direction, or from A' to A, is assumed to be in the positive sense. The convention set up should be such that the solution of a circuit problem is as simple as possible. It is perhaps more natural and more convenient in a two-terminal network, such as is shown in Fig. 7–5, to consider an instantaneous value of i_0 flowing toward the point M as positive and instantaneous values of i_A and i_B flowing away from the point M also as positive. These positive senses will be indicated by the arrows shown in the diagram. *The student must not consider the sense of the arrows as the sense of current flow at all times.*

Let the positive senses of current flow be those indicated by the arrows in Fig. 7–5 and let the ordinates above the ωt-axis in Fig. 7–6 be positive. At an instant when $\omega t_1 = \frac{\pi}{2}$, it can be seen from Fig. 7–6 that all three current ordinates are positive. Hence, in the circuit in Fig. 7–5,

i_A is flowing from A to A'
i_B is flowing from B to B'
i_0 is flowing from S to M

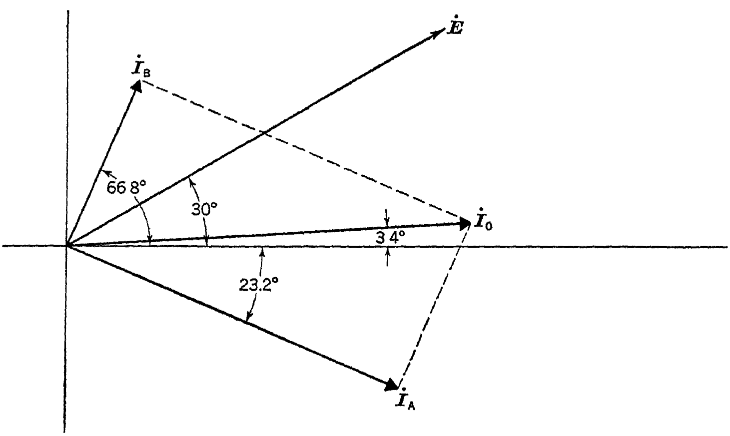


FIG. 7–8

At another instant when $\omega t_2 = \pi$, the ordinates of i_A and i_0 in Fig. 7–6 are positive, but the ordinate of i_B is negative. In the circuit in Fig. 7–5,

i_A is flowing from A to A'
i_B is flowing from B' to B
i_0 is flowing from S to M

A similar analysis can be made for any instant.

Since we are dealing with revolving time vectors, different vector diagrams can be drawn for different instants. The vector diagram in Fig. 7–7 is drawn for the instant when $t=0$ and $\dot{E}$ coincides with the reference axis. A vector diagram may be drawn for an instant when the emf vector $\dot{E}$ makes an angle of 30° with the reference line, as in Fig. 7–8. At the corresponding instant in Fig. 7–6, $\omega t = \frac{\pi}{6}$. The expression for the voltage is now $\dot{E} = 100\underline{/30^\circ}$ and all the currents advance in the same direction by 30°. Then,

$$\dot{I}_A = \frac{100\underline{/30^\circ}}{5\underline{/53.2^\circ}} = 20\underline{/-23.2^\circ}$$

$$\dot{I}_B = \frac{100\underline{/30^\circ}}{10\underline{/-36.8^\circ}} = 10\underline{/66.8^\circ}$$

$$\dot{I}_0 = \dot{I}_A + \dot{I}_B = 22.35\underline{/3.4^\circ}$$

The vertical projections of these vectors with maximum values, such as $\dot{I}_{Am}$, will give the instantaneous values at the instant when $\omega t = \frac{\pi}{6}$. Thus, there will be a different vector diagram for every instant of time along the ωt-axis in Fig. 7–6, and the angle of the voltage equation must be calculated for the time considered. However, the magnitudes of the current vectors, or the ammeter readings, will remain the same. The average power as given by a wattmeter is not affected by the choice of expression for the applied emf, since the phase angles between the voltage and the currents are not changed.

7–6. Admittance Method of Viewing a Circuit.—So far, in dealing with alternating-current circuits, we have analyzed them from the idea of the impedance of the circuit, that is, the quantity by which the voltage must be *divided* to obtain the current. In dealing with parallel circuits, however, it is generally easier to deal with the quantity by which the voltage is *multiplied* to obtain the current; this is evidently the reciprocal of the impedance.

To illustrate the *Admittance Method* of viewing a circuit, consider the branch AA' in Fig. 7–9. The emf E is the voltage rise of a generator. The current is

$$\dot{I} = \frac{\dot{E}}{\dot{Z}} = \dot{E}\dot{Y} \qquad (7\text{–}19)$$

where $\dot{Y} = \frac{1}{\dot{Z}}$ is called the admittance of the branch. When the emf and the current are given in volts and amperes, respectively, then the admittance is in mhos (mho is ohm spelled backward). Similar to impedance, which has a resistance component and a reactive component, admittance may be resolved into two components. Thus,

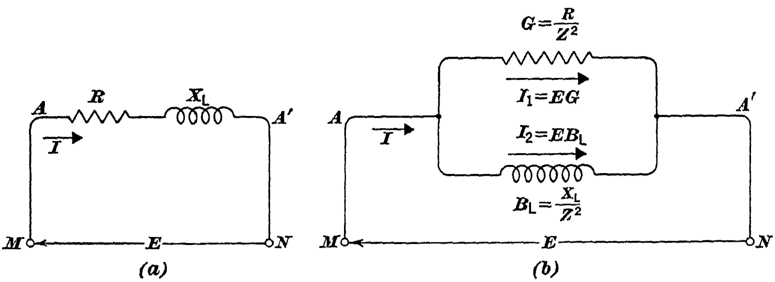


FIG. 7–9

$$\dot{Y} = \frac{1}{\dot{Z}} = \frac{1}{R+jX_L} \times \frac{R-jX_L}{R-jX_L}$$

or

$$\dot{Y} = \frac{R}{Z^2} - j\frac{X_L}{Z^2} = G - jB_L \qquad (7\text{–}20)$$

where $G = \frac{R}{Z^2}$ and is called the *conductance*, in mhos;

and $B_L = \frac{X_L}{Z^2}$ and is called the *susceptance*, in mhos.

Also, the magnitude of the admittance is

$$Y = \sqrt{G^2 + (-B_L)^2} \qquad (7\text{–}21)$$

It should be noted that, as a rule,

$G \neq \frac{1}{R}$ except when the branch has resistance only

$B_L \neq \frac{1}{X_L}$ except when the branch has reactance only

Substitution for $\dot{Y}$ in equation (7–19) gives

$$\dot{I} = \dot{E}\dot{Y} = \dot{E}(G - jB_L)$$

or

$$\dot{I} = \dot{E}G - j\dot{E}B_L = I_1 - jI_2 \tag{7–22}$$

As shown by the last equation, the conductance G is the factor by which the generator voltage E must be multiplied to give the component $EG = I_1$ of the current in phase with the voltage; this component is indicated in Fig. 7–10(*c*). The susceptance B_L is the factor by which the generator voltage must be multiplied to

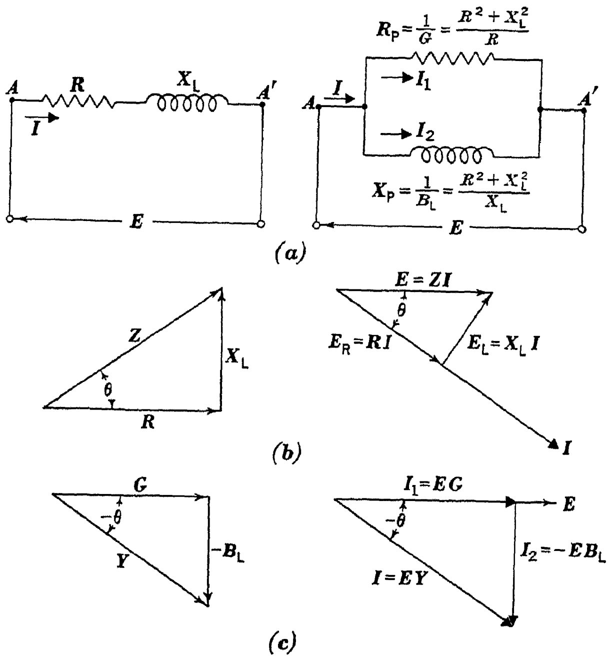


FIG. 7–10

give the component $EB_L = I_2$ of the current in quadrature with the voltage, as indicated in Fig. 7–10(*c*). Just as the reactance of a circuit has an algebraic sign, being positive or negative according as the circuit is inductive or capacitive, so susceptance must be used with its proper sign. When X is positive, as in an inductive circuit, the sign before B is minus; when X is negative, as in a capacitive circuit, the sign before B is plus.

It may be concluded from equation (7–22) that, for a single frequency, the impedance branch AA' of Fig. 7–9(a) may be replaced by an *equivalent parallel circuit*, as in Fig. 7–9(b), as far as the current $\dot{I}$ and the total power delivered by the voltage source E are concerned. Each gives the same in-phase and quadrature components of $\dot{I}$. If $\dot{E}$ is the reference vector, then the power is

$$P = EI_1 = E^2G \tag{7–23}$$

The actual equivalent resistance in the G branch, Fig. 7–10(a), is

$$R_p = \frac{1}{G} = \frac{R^2 + X_L^2}{R} = R\left[1 + \left(\frac{X_L}{R}\right)^2\right] \tag{7–24}$$

and the actual equivalent inductive reactance in the B_L branch is

$$X_p = \frac{1}{B_L} = \frac{R^2 + X_L^2}{X_L} = X_L\left[1 + \left(\frac{R}{X_L}\right)^2\right] \tag{7–25}$$

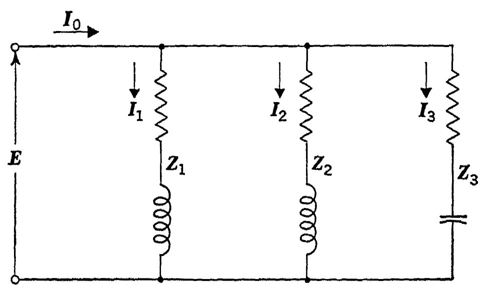


FIG. 7–11

It is obvious from equations (7–24) and (7–25) that the quantities R_p and X_p of the parallel combination will always be greater than the corresponding components R and X_L of the series combination. Since the G branch is a pure resistance R_p and the B_L branch is a pure reactance X_p, the component I_1 flowing through R_p is in phase with the voltage E and the component I_2 flowing through X_p is in quadrature with the voltage E.

When an impedance coil is considered as a circuit containing resistance and inductive reactance in series, there exist two triangles, one for the impedance and another for the voltage, as indicated in Fig. 7–10(b). If the same impedance coil is considered from the point of view of admittance, there are two triangles, one for the admittance and another for the current, as in Fig. 7–10(c).

When the power factor angle of branch AA' in Fig. 7–9 is desired, it may be obtained from Fig. 7–10(c). Thus,

$$\tan(-\theta) = \frac{-B_L}{G} \tag{7–26}$$

7–7. Admittance Method of Considering Parallel Circuits. The line current I_0 of a parallel circuit consisting of several impedances in parallel, as in Fig. 7–11, may be found by first determining the current in each branch and then applying Kirchhoff's Current Law at a junction point. Thus,

$$\dot{I}_1 = \frac{\dot{E}}{\dot{Z}_1};\quad \dot{I}_2 = \frac{\dot{E}}{\dot{Z}_2};\quad \dot{I}_3 = \frac{\dot{E}}{\dot{Z}_3}$$

$$\dot{I}_0 = \dot{I}_1 + \dot{I}_2 + \dot{I}_3 = \frac{\dot{E}}{\dot{Z}_1} + \frac{\dot{E}}{\dot{Z}_2} + \frac{\dot{E}}{\dot{Z}_3} \tag{7–27}$$

or

$$\dot{I}_0 = \dot{E}\dot{Y}_1 + \dot{E}\dot{Y}_2 + \dot{E}\dot{Y}_3 = \dot{E}\dot{Y}_0 \tag{7–27a}$$

The last equation indicates that the total or equivalent admittance $\dot{Y}_0$ is the vector sum of the admittances of the different branches. Hence,

$$\dot{Y}_0 = \frac{\dot{I}_0}{\dot{E}} = \dot{Y}_1 + \dot{Y}_2 + \dot{Y}_3 \tag{7–28}$$

or

$$\dot{Y}_0 = \frac{1}{\dot{Z}_0} = \frac{1}{\dot{Z}_1} + \frac{1}{\dot{Z}_2} + \frac{1}{\dot{Z}_3} \tag{7–28a}$$

where $\dot{Z}_0$ is the total or equivalent impedance of all the branches in parallel; that is,

$$\dot{Z}_0 = \frac{1}{\dot{Y}_0} = \frac{1}{\dot{Y}_1 + \dot{Y}_2 + \dot{Y}_3} \tag{7–29}$$

Equation (7–27a) suggests that the line current I_0 of a parallel circuit may be determined by the admittance method. The first step is to find the in-phase and quadrature components of the branch currents. The next step is to add the in-phase components together and then to add the quadrature components together to obtain the line current I_0. This may be illustrated by considering the parallel circuit shown in Fig. 7–1. The currents in branches AA' and BB' are:

$$\dot{I}_A = \dot{E}\dot{Y}_A = \dot{E}G_A - j\dot{E}B_A = I_{A1} - jI_{A2} \tag{7–30}$$

$$\dot{I}_B = \dot{E}\dot{Y}_B = \dot{E}G_B - j\dot{E}B_B = I_{B1} - jI_{B2} \tag{7–31}$$

Since the line current I_0 is the vector sum of the two branch currents,

$$\dot{I}_0 = \dot{I}_A + \dot{I}_B = \dot{E}\,(\dot{Y}_A + \dot{Y}_B) \tag{7–32}$$

Also,

$$\dot{I}_0 = (I_{A1} + I_{B1}) - j(I_{A2} + I_{B2}) = \dot{E}(G_A + G_B) - j\dot{E}(B_A + B_B) \tag{7–32a}$$

or

$$I_0 = \dot{E}\,(G_0 - jB_0) = \dot{E}\dot{Y}_0 \tag{7–32b}$$

where $G_0 = G_A + G_B$; $B_0 = B_A + B_B$; and $\dot{Y}_0 = G_0 - jB_0$.

Thus, it may be said that, in a circuit containing several impedances in series, we add the impedances vectorially to get the equivalent impedance of the circuit; while, in a parallel circuit consisting of a number of branches in parallel, the equivalent admittance is equal to the vector sum of the admittances of the different branches.

Example 7–3.—Two impedances $\dot{Z}_A = 0.75 + j0.75$ and $\dot{Z}_B = 1.0 - j0.5$ are connected in parallel across a generator of 100 volts and 60 cycles. Find the admittances and the vector currents by using the applied emf as a reference vector. Also. find the equivalent impedance $\dot{Z}_0$.

Solution.—A simple solution to this problem is to find first the admittance of each branch and the equivalent admittance of the parallel combination. The desired current is then found by multiplying the applied voltage by the corresponding admittance. Thus,

$$G_A = \frac{R_A}{Z_A^2} = \frac{0.75}{1.06^2} = 0.666 \qquad G_B = \frac{R_B}{Z_B^2} = \frac{1}{1.118^2} = 0.8$$

$$B_A = \frac{X_A}{Z_A^2} = \frac{0.75}{1.06^2} = 0.666 \qquad B_B = \frac{X_B}{Z_B^2} = \frac{0.5}{1.118^2} = 0.4$$

$$\dot{Y}_A = 0.666 - j0.666 = 0.944\underline{/-45°} \qquad \dot{Y}_B = 0.8 + j0.4 = 0.894\underline{/26.8°}$$

Adding the admittances vectorially gives the equivalent admittance. Thus,

$$\dot{Y}_0 = \dot{Y}_A + \dot{Y}_B = 1.466 - j0.266 = 1.487\underline{/-10.3°}$$

The impedance and admittance vectors for branch AA' are shown in Fig. 7–12(a). The admittance diagram for the parallel combination is given in Fig. 7–12(b).

The currents are:

$$\dot{I}_A = \dot{E}\dot{Y}_A = 100\underline{/0°} \times 0.944\underline{/-45°} = 94.4\underline{/-45°}$$

$$\dot{I}_B = \dot{E}\dot{Y}_B = 100\underline{/0°} \times 0.894\underline{/26.6°} = 89.4\underline{/26.6°}$$

$$\dot{I}_0 = \dot{E}\dot{Y}_0 = 100\underline{/0°} \times 1.487\underline{/-10.3°} = 148.7\underline{/-10.3°}$$

The equivalent impedance is

$$\dot{Z}_0 = \frac{1}{\dot{Y}_0} = \frac{1}{1.466 - j0.266} \times \frac{1.466 + j0.266}{1.466 + j0.266} = 0.66 + j0.12$$

As far as the line current $\dot{I}_0$ and the total power dissipated are concerned, the two impedances in parallel may be replaced by a series combination of $R = 0.66$ ohms and $X_L = 0.12$ ohm.

7-8. Equivalent Series Circuit.—Sometimes it is convenient to reduce a parallel circuit, for a single frequency, to an equivalent series circuit in the solution of alternating-current circuit problems, such as the series-parallel type given in Chapter 8. An *equivalent series circuit* is defined as a simple series circuit in which the power factor (and the power consumed) and the current taken from the source are the same as those of the parallel circuit.*

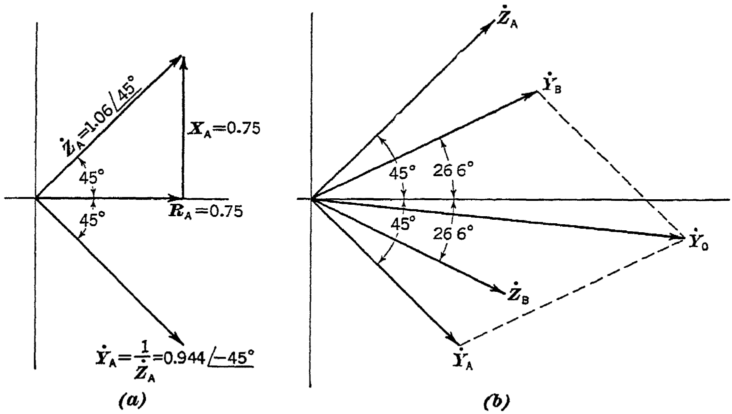


FIG. 7-12

Where resistances are in parallel in a circuit, it will be remembered that the reciprocal of the equivalent resistance is

$$\frac{1}{R_0}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}+\ \cdots \tag{7-33}$$

For two resistances,

$$R_0=\frac{R_1R_2}{R_1+R_2} \tag{7-34}$$

Likewise, for impedances in parallel, there is a similar rule. By referring to equation (7-28) it is seen that

$$\dot{Y}_0=\dot{Y}_A+\dot{Y}_B+\dot{Y}_C+\ \cdots\ +\dot{Y}_n$$

* In general, a parallel circuit cannot be reduced to an equivalent series circuit when transient or non-sinusoidal currents or currents of different frequencies are included.

From this it follows that

$$\frac{1}{\dot{Z}_0}=\frac{1}{\dot{Z}_A}+\frac{1}{\dot{Z}_B}+\frac{1}{\dot{Z}_C}+\cdots\frac{1}{\dot{Z}_n}$$

When there are only two impedances in parallel, as in Fig. 7–13(*a*), the equivalent impedance becomes

$$\dot{Z}_0=\frac{\dot{Z}_A\dot{Z}_B}{\dot{Z}_A+\dot{Z}_B} \tag{7–35}$$

which may be reduced to

$$\dot{Z}_0=R_0+jX_0 \tag{7–36}$$

as indicated in Fig. 7–13(*b*).

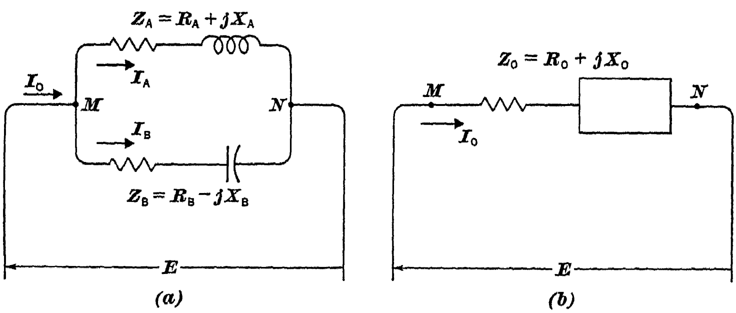


FIG. 7–13

Whether the reactance X_0 is inductive or capacitive depends on the values of the elements in the original parallel circuit.

In some problems, it is convenient to make use of expressions for the branch currents in terms of the line current. From Fig. 7–13,

$$\dot{I}_A\dot{Z}_A=\dot{I}_B\dot{Z}_B \tag{7–37}$$

Since $\dot{I}_0=\dot{I}_A+\dot{I}_B$, equation (7–37) gives

$$\dot{I}_A\dot{Z}_A=\dot{Z}_B(\dot{I}_0-\dot{I}_A)$$

or

$$\dot{I}_A(\dot{Z}_A+\dot{Z}_B)=\dot{Z}_B\dot{I}_0$$

from which

$$\dot{I}_A=\frac{\dot{Z}_B}{\dot{Z}_A+\dot{Z}_B}\dot{I}_0 \tag{7–38}$$

Similarly,

$$\dot{I}_B=\frac{\dot{Z}_A}{\dot{Z}_A+\dot{Z}_B}\dot{I}_0 \tag{7–39}$$

Example 7–4.—Two impedances $\dot{Z}_A = 3 + j4$ and $\dot{Z}_B = 8 - j6$ are connected in parallel across a 100-volt, 60-cycle source. Reduce this parallel circuit to an equivalent series circuit and find: (*a*) the line current, (*b*) the power delivered to the whole circuit, and (*c*) the power factor of the equivalent circuit. Use $\dot{E}$ as the reference vector.

Solution.—The impedance of the equivalent series circuit is

$$\dot{Z}_0 = \frac{\dot{Z}_A \dot{Z}_B}{\dot{Z}_A + \dot{Z}_B} = \frac{5\underline{/53.2^\circ} \times 10\underline{/-36.8^\circ}}{(3 + j4) + (8 - j6)}$$

$$= 4.47\underline{/26.6^\circ} = 4 + j2$$

Thus, the equivalent series circuit has an equivalent resistance $R_0 = 4$ ohms in series with an equivalent inductive reactance $X_0 = 2$ ohms. The equivalent reactance is inductive because the sign in front of the j term of $\dot{Z}_0$ is positive. If the sign had come out negative, then the equivalent reactance would have been capacitive.

The line current is

$$\dot{I}_0 = \frac{\dot{E}}{\dot{Z}_0} = \frac{100\underline{/0^\circ}}{4.47\underline{/26.6^\circ}} = 22.35\underline{/-26.6^\circ}$$

The power factor is

$$\text{P.F.}_0 = \cos\theta_0 = \frac{R_0}{Z_0} = \frac{4}{4.47} = 0.894$$

The power is

$$P = EI_0 \cos\theta_0 = 100 \times 22.35 \times 0.894 = 2000 \text{ watts}$$

or

$$P = I^2R_0 = 22.35^2 \times 4 = 2000 \text{ watts}$$

Example 7–5.—Find the branch currents in Example 7–4.

Solution.—The branch currents may be found by either of the following methods:

(*a*)

$$\dot{I}_A = \frac{\dot{E}}{\dot{Z}_A} = \frac{100\underline{/0^\circ}}{5\underline{/53.2^\circ}} = 20\underline{/-53.2^\circ}$$

$$\dot{I}_B = \frac{\dot{E}}{\dot{Z}_B} = \frac{100\underline{/0^\circ}}{10\underline{/-36.8^\circ}} = 10\underline{/36.8^\circ}$$

(*b*)

$$\dot{I}_A = \frac{\dot{Z}_B}{\dot{Z}_A + \dot{Z}_B}\dot{I}_0$$

$$= \frac{10\underline{/-36.8^\circ}}{5\underline{/53.2^\circ} + \underline{/10 - 36.8^\circ}} \times 22.35\underline{/-26.6^\circ} = 20\underline{/-53.2^\circ}$$

$$\dot{I}_B = \frac{\dot{Z}_A}{\dot{Z}_A + \dot{Z}_B}\dot{I}_0$$

$$= \frac{5\underline{/53.2^\circ}}{5\underline{/53.2^\circ} + 10\underline{/-36.8^\circ}} \times 22.35\underline{/-26.6^\circ} = 10\underline{/36.8^\circ}$$

7–9. Parallel Resonance.—A coil and a condenser are connected in parallel as shown in Fig. 7–14(*a*). When the quadrature component of the current in the inductive branch is equal to the quadrature component of the current in the capacitive branch (in this case, simply I_C), the line current I_0 is in phase with the line voltage E, as indicated in the middle vector diagram of Fig. 7-14(*b*). The circuit is then said to be in parallel resonance (or anti-resonance) on the condition of unity power factor. For frequencies less than the anti-resonant frequency f_{ar}, or when $f < f_{ar}$, the quadrature component of I_L is greater than I_C and the line current I_0 lags behind E, as shown at the left in Fig. 7-14(*b*). The parallel circuit MN of Fig. 7-14(*a*) is apparently inductive. For $f > f_{ar}$, the quadrature component of I_L is less than I_C and the line current I_0 leads E. The parallel combination MN is apparently capacitive.

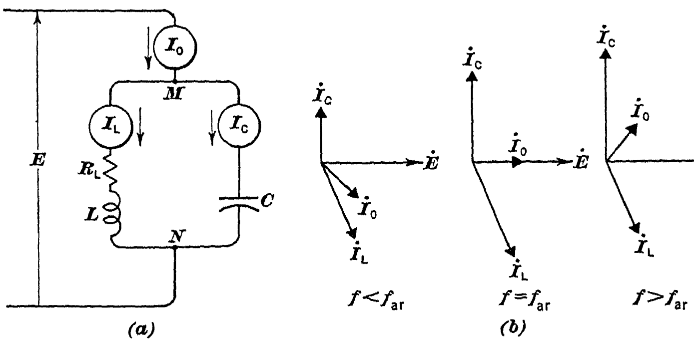


FIG. 7-14

For parallel resonance (or unity power factor resonance), the quadrature components are set equal to each other. Thus,

$$EB_L = EB_C \tag{7-40}$$

or

$$B_L = B_C \tag{7-40a}$$

7-10. Resonance When Coil Resistance Is Negligible.—In radio circuits where resistances are negligible in comparison with reactances (high Q coils and very good condensers), it is often desirable to study, as the frequency is varied, (*a*) the approximate manner in which the different reactances vary and thus the approximate variation of the branch currents; and (*b*) the change in the magnitude of the line current. For this purpose use Fig. 7-14(*a*), with R_L being *neglected.*

The total reactance X_{MN} of the parallel combination for any frequency can be determined by plotting the susceptance curve of each branch with frequency as abscissa, adding the two susceptances at the selected frequency, and then taking the reciprocal of the sum. Repetition of this procedure for different frequencies gives the variation of the total reactance with frequency. The

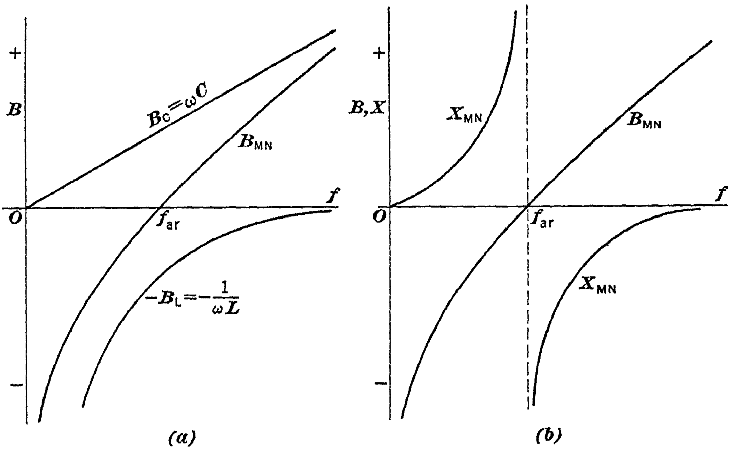


FIG. 7–15

curve marked B_{MN} in Fig. 7–15(*a*) is obtained by the addition of the two susceptance curves ωC and $\frac{-1}{\omega L}$. That is, the total susceptance is made up of two parts: (1) the inductive susceptance $B_L = \frac{1}{X_L} = \frac{1}{2\pi f L}$, which predominates at low frequencies from the standpoint of the line current $I_0 = E(B_C - B_L) = EB_{MN}$; and (2) the capacitive susceptance $B_C = \frac{1}{X_C} = 2\pi f C$, which predominates at high frequencies. The variations of B_L and B_C are of interest, since $I_L = EB_L$ and $I_C = EB_C$. The curve marked X_{MN} in Fig. 7–15(*b*) is obtained by taking the reciprocals of the points on the curve for B_{MN}. The reciprocal of a positive reactance is a negative susceptance, for

$$\frac{1}{+jX} = -j\frac{1}{X} \tag{7–41}$$

For the resistanceless combination, it is interesting to note from Fig. 7-15(*b*) how the reactance X_{MN} of the parallel circuit varies with frequency and, consequently, the manner in which the line current $I_0 = \frac{E}{X_{MN}}$ is varying. At frequencies below resonance the circuit is inductive because X_{MN} is above the frequency axis ($+X$ is inductive). Above resonance the circuit is capacitive because X_{MN} is below the frequency axis. Since the reactance of the parallel circuit is small at very low frequencies and at very high frequencies, the line current I_0 will be large at these frequencies. At the resonant frequency f_{ar}, however, both branches of the reactance curve go to infinity. This means that I_0 is zero at f_{ar}; and I_L and I_C are equal in magnitude. If resistances must be considered, then the line current will not be zero at the resonant frequency, as shown in Fig. 7-16, and its magnitude will be just large enough to take care of the I^2R losses. Thus, a parallel arrangement is used when it is desired to suppress or "tune out" a particular current or signal, as in the case of a "wave trap."

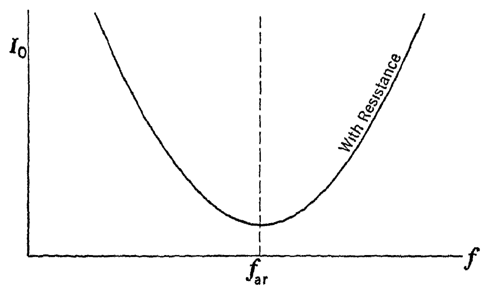


FIG. 7-16

It can be seen from Fig. 7-15 that *the slope of any reactance curve or susceptance curve is always positive at all points for positive values of L and C.*

7-11. Condition for Unity Power Factor.—In communication circuits, it is quite common to find coils connected in parallel with condensers, as indicated in Fig. 7-14(*a*), for purposes of matching resistances to obtain maximum power transfer. When a parallel combination is in parallel resonance, the power factor is unity and the circuit appears as a pure resistance across its terminals. This characteristic of parallel resonant circuits is very important in certain applications, say, where the resistance of a vacuum tube must be matched to obtain appreciable power from the tube at high frequencies (see Art. 10–8 on Maximum Power Transfer and Chapter 13 on Impedance Transformation).

It will be remembered that the criterion for unity power factor in an *RLC* series circuit was $X_L = X_C$. The resistance played no part in determining the frequency of resonance. In parallel circuits, however, the resistances are of importance in the determination of the resonant frequency, even to the extent of making it either possible or impossible to obtain resonance. It is desired to derive expressions for L, C, and f_{ar} under the condition of unity power factor. The following method is quite simple.

The equivalent impedance of the parallel circuit in Fig. 7–14(*a*) is

$$\dot{Z}_e = \frac{\dot{Z}_L \dot{Z}_C}{\dot{Z}_L + \dot{Z}_C} = \frac{(R_L + jX_L)(-jX_C)}{R_L + j(X_L - X_C)} \tag{7–42}$$

or

$$\dot{Z}_e = \frac{R_L X_C^2}{R_L^2 + (X_L - X_C)^2} + j\frac{X_C^2 X_L - X_C X_L^2 - R_L^2 X_C}{R_L^2 + (X_L - X_C)^2} \tag{7–42a}$$

For the condition of unity power factor at the terminals of the parallel combination (that is, the circuit *MN* appearing as a pure *R*), the j term is set equal to zero; thus,

$$X_C^2 X_L - X_C X_L^2 - R_L^2 X_C = 0 \tag{7–43}$$

or

$$X_L^2 - X_C X_L + R_L^2 = 0 \tag{7–43a}$$

From this last equation, expressions for X_L or L, X_C or C, and ω^2 or f_{ar} may be derived for conditions of unity power factor. These equations are:

$$\left.\begin{aligned} X_L &= \frac{X_C}{2} \pm \sqrt{\left(\frac{X_C}{2}\right)^2 - R_L^2} \\ L &= \frac{1}{2\omega^2 C} \pm \sqrt{\left(\frac{1}{2\omega^2 C}\right)^2 - \left(\frac{R_L}{\omega}\right)^2} \end{aligned}\right\} \tag{7–43b}$$

$$\left.\begin{aligned} X_C &= \frac{R_L^2 + X_L^2}{X_L} \\ C &= \frac{L}{R_L^2 + \omega^2 L^2} \end{aligned}\right\} \tag{7–43c}$$

$$\left.\begin{aligned} \omega^2 &= \frac{1}{LC} - \left(\frac{R_L}{L}\right)^2 \\ f_{ar} &= \frac{1}{2\pi\sqrt{LC}}\sqrt{1 - \frac{C}{L}R_L^2} \end{aligned}\right\} \qquad (7\text{–}43\text{d})$$

Equation (7–43b) states that there are two values of L which will give unity power factor (see Fig. 7–18). Equation (7–43c) simply expresses C in terms of R_L, L, and f. Equation (7–43d) indicates that parallel resonance is not possible if R_L is large enough to make the quantity under the radical sign negative.

If the Q of the coil is high or R_L is negligible, then the frequency for parallel resonance as given by equation (7–43d) reduces to

$$f_{ar} \approx \frac{1}{2\pi\sqrt{LC}} \qquad (7\text{–}44)$$

which is approximately the same as the relation for series resonance.

7–12. Maximum Impedance by Varying C.—When it is desirable to suppress a particular current or signal, a parallel circuit is usually tuned for maximum impedance. In the discussion in this chapter, it is to be understood that a parallel combination can be adjusted to produce maximum impedance (or minimum line current) *when considering only a single variable,* that is, when either the frequency or a single circuit element is varied. It will be shown in the articles following that unity power factor does not always occur simultaneously when the circuit is adjusted for maximum impedance. When C is the only variable, a relationship involving R_L, X_L, and X_C can be derived quite easily for the condition of maximum impedance.

For the parallel circuit shown in Fig. 7–14(a) the expression for the equivalent impedance is given by equation (7–42). The square of the absolute value is

$$Z_e^2 = \frac{(R_L^2 + X_L^2)X_C^2}{R_L^2 + (X_L - X_C)^2} \qquad (7\text{–}45)$$

When Z_e^2 is maximum, Z_e will have its maximum value. The reason for using Z_e^2 is the simplification in the mathematics. To maximize Z_e^2, differentiate equation (7–45) with respect to X_C and set the result equal to zero. Thus,

$$\frac{\partial Z_e^2}{\partial X_C} = 0 = \frac{[R_L^2 + (X_L - X_C)^2]\,[2X_C(R_L^2 + X_L^2)] + [(R_L^2 + X_L^2)X_C^2]\,[2(X_L - X_C)]}{[R_L^2 + (X_L - X_C)^2]^2}$$

From this,

$$X_L^2 - X_C X_L + R_L^2 = 0 \tag{7–46}$$

or

$$X_C = \frac{R_L^2 + X_L^2}{X_L} = X_L\left[1 + \left(\frac{R_L}{X_L}\right)^2\right] = X_L\left(1 + \frac{1}{Q^2}\right) \tag{7–46a}$$

where $Q = \frac{X_L}{R_L}$ is the Q of the coil.

It is obvious that equation (7–46) is identical with equation (7–43a), which gives the condition for unity power factor. Thus, when C is varied, the condition for maximum impedance is the same as that for unity power factor. It will be seen in Art. 7–14 that an adjustment of L in the circuit to produce maximum impedance will not produce unity power factor at the same time.

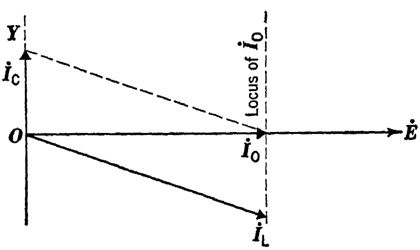


FIG. 7–17

When the Q of the coil is high or $R_L \approx 0$, then the condition for maximum impedance by varying C only is, from equation (7–46a),

$$X_C = X_L \tag{7–47}$$

or

$$C = \frac{1}{\omega^2 L} \tag{7–47a}$$

Example 7–6.—If, in Fig. 7–14(*a*), $R_L = 10$ ohms, $L = 0.01$ henry, and the applied emf $e = 141.4 \sin 1000t$, at what value should C be set for minimum line current?

Solution.—Substitution in equation (7–43*c*) gives

$$C = \frac{0.01}{10^2 + 1000^2 \times 0.01^2} = 50\mu\text{f}$$

7–13. Current Locus When C Is Varied.—The same conclusion (when C is varied, the condition for maximum impedance is the same as that for unity power factor) can be reached by studying the parallel circuit in Fig. 7–14(a) and the locus for the line current $\dot{I}_0$ in Fig. 7–17. Since the applied voltage, the frequency, and the coil remain fixed, the current $\dot{I}_L$ is constant and lags behind the voltage by a definite angle. When the condenser is varied, the current $\dot{I}_C = \frac{\dot{E}}{-jX_C}$ varies along the vertical line OY. The line current $\dot{I}_0$ is the vector sum of $\dot{I}_L$ and $\dot{I}_C$. The terminus of the line current vector follows a line that passes through the end of $\dot{I}_L$ and is parallel to OY. It can be seen that, when $\dot{I}_0$ is a minimum, it is also in phase with $\dot{E}$.

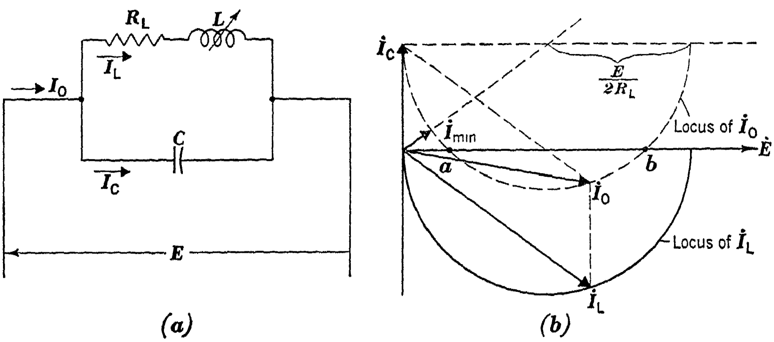


FIG. 7–18

7–14. Maximum Impedance by Varying L.—An expression similar to equation (7–46) can be derived for the case when L in Fig. 7–14(a) is made the only variable. By use of the relation $\frac{\partial Z_e^2}{\partial X_L} = 0$, that is, by differentiating equation (7–45) with respect to X_L and setting the result equal to zero, we can obtain the following expression:

$$X_L^2 - X_L X_C - R_L^2 = 0 \tag{7–48}$$

When this is compared with equation (7–43a), it will be seen that, if L alone is varied, the condition for maximum impedance is not the same as that for unity power factor.

Since only positive values for X_L or L are considered in this text, the value of X_L or L for maximum impedance may be determined by using only the plus sign before the radical in the solution of the quadratic equation (7–48). Thus,

$$X_L = \frac{X_C}{2} + \sqrt{\left(\frac{X_C}{2}\right)^2 + R_L^2} \tag{7–49}$$

$$L = \frac{1}{2\omega^2 C} + \sqrt{\left(\frac{1}{2\omega^2 C}\right)^2 + \left(\frac{R_L}{\omega}\right)^2} \tag{7–49a}$$

7–15. Current Locus When L Is Varied.—Fig. 7–18(a) shows a coil in parallel with a condenser C. It is desired to find the locus of the line current $\dot{I}_0$ when L is the only variable. Since the generator emf E is held constant, it is convenient to use $\dot{E}$ as the reference vector. The condenser current $\dot{I}_C$ is 90° ahead of the applied voltage, as shown in Fig. 7-18(b). The locus for branch current $\dot{I}_L$, when L is varied, is given by the full-line semicircle. As the line current $\dot{I}_0$ is the vector sum of $\dot{I}_C$ and $\dot{I}_L$, its locus can be found by moving the full-line semicircle up to the end of $\dot{I}_C$. The dotted semicircle is the locus of $\dot{I}_0$.

When this circuit is considered from the standpoint of parallel resonance, it is interesting to note that in equation (7–43b) and in Fig. 7–18(b) there are two values of L which will give unity power factor—at a and at b. From the standpoint of maximum impedance, the condition for minimum line current (shortest vector) is indicated by the vector marked $\dot{I}_{min.}$. The conditions for unity power factor and minimum current depend on the values of R_L and C. The following facts should be observed:

(a) The condition for maximum impedance to obtain a minimum line current is always possible.

(b) If I_C is equal to the radius $\frac{E}{2R_L}$ of the semicircle, the dotted locus will be tangent to the vector $\dot{E}$. Parallel resonance (or resonance on the condition of unity power factor) occurs at only one value of L.

(c) If I_C is greater than $\frac{E}{2R_L}$, then parallel resonance is not possible.

(d) Changing the value of C is equivalent to moving the dotted semicircle up or down with reference to the horizontal axis.

(e) Varying R_L changes the diameter of the semicircle.

7–16. Maximum Impedance by Varying f.—The condition for maximum impedance when the frequency is varied in the circuit in Fig. 7–14(a) may also be found by differentiating equation (7–45) with respect to f. When values are substituted for X_L and X_C and terms are rearranged, equation (7–45) becomes

$$Z_e^2 = \frac{R_L^2 + \omega^2 L^2}{\omega^2 C^2 R_L^2 + \omega^4 L^2 C^2 - 2\omega LC + 1} \tag{7–50}$$

By use of the relation $\frac{\partial Z_e^2}{\partial \omega} = 0$, that is, by differentiating equation (7–50) with respect to $\omega = 2\pi f$ and setting the result equal to zero, we can obtain the following expression:

$$\omega^2 = -\frac{R_L^2}{L^2} + \sqrt{\frac{1}{L^2 C^2} + \frac{2R_L^2}{L^3 C}} \tag{7–51}$$

Comparison of equation (7–51) with equation (7–43d) reveals that, when f is the only variable, maximum impedance and unity power factor do not occur at the same frequency.

If the Q of the coil is high or R_L is negligible, the frequency for maximum impedance, or $f_{Z\max}$, may be expressed as follows:

$$f_{Z\max.} = f_{ar} = \frac{1}{2\pi\sqrt{LC}} \tag{7–52}$$

7–17. Parallel Resonant Circuit With Resistance in Both Branches.—In a parallel circuit containing resistances in both branches, as in Fig. 7–19, the branch currents are:

$$\dot{I}_L = \dot{E}G_L - j\dot{E}B_L$$
$$\dot{I}_C = \dot{E}G_C + j\dot{E}B_C$$

The line current is then

$$\dot{I}_0 = \dot{E}(G_L + G_C) - j\dot{E}(B_L - B_C) \tag{7–53}$$

In order that the line current $\dot{I}_0$ may be in phase with the generator voltage $\dot{E}$ at parallel resonance, it is necessary that the quadrature components of the branch currents cancel each other or that

$$EB_L - EB_C = 0$$

from which

$$B_L = B_C \tag{7–54}$$

Substituting values for the susceptances, we obtain:

$$\frac{X_L}{R_L^2+X_L^2}=\frac{X_C}{R_C^2+X_C^2} \tag{7-55}$$

or

$$\frac{\omega L}{R_L^2+(\omega L)^2}=\frac{\frac{1}{\omega C}}{R_C^2+\left(\frac{1}{\omega C}\right)^2}$$

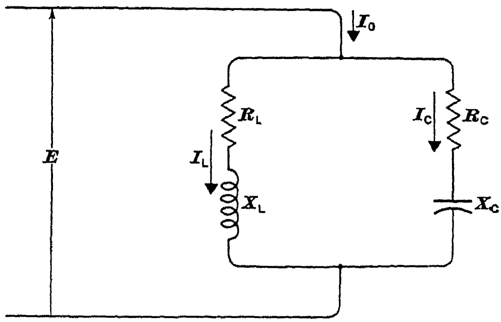


FIG. 7–19

This may be written in the form

$$\frac{\omega L}{R_L^2+(\omega L)^2}-\frac{\frac{1}{\omega C}}{R_C^2+\left(\frac{1}{\omega C}\right)^2}=0$$

from which

$$\omega\left[\frac{L}{R_L^2+\omega^2L^2}-\frac{C}{R_C^2\omega^2C^2+1}\right]=0$$

or

$$\omega\left[\omega^2LC(CR_C^2-L)-(CR_L^2-L)\right]=0 \tag{7-56}$$

The conditions that satisfy equation (7–56) are the following:

(1) $\omega=0$ (7–56a)

(2) $R_L=R_C=\sqrt{\frac{L}{C}}$ (7–56b)

(3) $\omega=\frac{1}{\sqrt{LC}}\sqrt{\frac{CR_L^2-L}{CR_C^2-L}}$ or $f_{ar}=\frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{CR_L^2-L}{CR_C^2-L}}$ (7–56c)

For condition (1), $f = 0$ and the effect is equivalent to connecting the parallel circuit to a d-c source. For condition (2), the expression within the brackets in equation (7-56) is zero and ω could have any value. Thus, the circuit is in parallel resonance for all values of frequency; that is, the line voltage $\dot{E}$ and the line current $\dot{I}_0$ will be in phase for all values of frequency Under condition (3), parallel resonance is possible if the quantity under the radical sign is positive. When $CR_L^2 < L$ and $CR_C^2 > L$, or when $CR_L^2 > L$ and $CR_C^2 < L$, the quantity under the radical sign is negative and parallel resonance cannot be attained under these conditions. It is quite obvious that both R_L and R_C are factors which must be considered in the determination of f_{ar}. Whenever R_L and R_C are equal, including zero but not equal to $\sqrt{\frac{L}{C}}$, equation (7-56c) becomes

$$f_{ar} = \frac{1}{2\pi\sqrt{LC}} \tag{7-57}$$

which is the same as the relation for a series circuit.

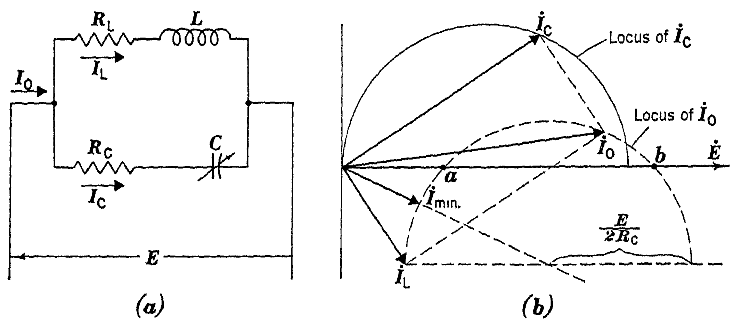


FIG. 7-20

It is conceivable that, when a parallel circuit is in parallel resonance, either I_L and I_C or both may be greater than I_0. If the branch resistances are zero, $I_0 = 0$ and $I_L = I_C$.

7-18. Current Locus in Parallel Circuit With Resistances in Both Branches.—An RC circuit is connected in parallel with an RL circuit, as in Fig. 7-20(a). To find the locus for the line current $\dot{I}_0$ when C is varied, it is convenient to draw the vector representing $\dot{I}_L$ and the locus for $\dot{I}_C$, and then to combine these two to obtain the desired locus.

The generator voltage $\dot{E}$ being used as the reference vector, the current $\dot{I}_L$ is constant and lags behind $\dot{E}$ by some definite angle, as in Fig. 7–20(*b*). When C is varied, the locus of $\dot{I}_C$ is given by the full-line semicircle. The locus of $\dot{I}_0$ is obtained by moving this full-line semicircle to the end of $\dot{I}_L$. The desired locus is the dotted semicircle.

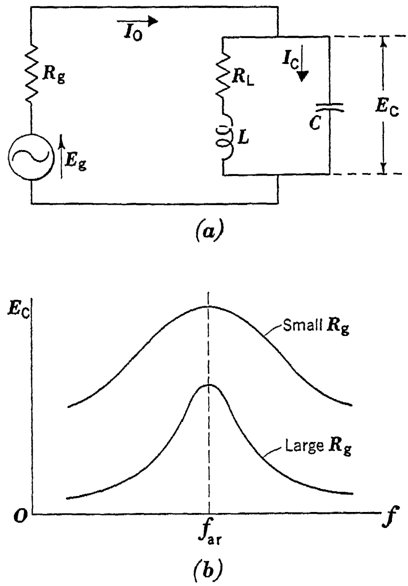


FIG. 7–21

It should be noted that, for the case indicated in Fig. 7–20(*b*), there are two values of C which will give parallel resonance or unity power factor—at *a* and at *b*. The condition for maximum impedance is given by the shortest vector, which is marked $\dot{I}_{\text{min.}}$. If the vector current $\dot{I}_L$, which is determined by R_L and L, is changed so that the dotted semicircle is tangent to $\dot{E}$, then there will be only one value of C which will make the parallel combination behave like a pure resistance. When the quadrature component of $\dot{I}_L$ is greater than $\frac{E}{2R_C}$, then unity power factor resonance is not possible. The resistance R_C determines the diameter of the locus of $\dot{I}_C$; and its value will therefore affect the conditions for unity power factor and maximum impedance.

7–19. Sharpness of Resonance in Parallel Circuits.—Consider the circuit of Fig. 7–21(a), where the condenser C is in parallel with a coil. The generator is assumed to have a constant internal resistance R_g and a negligible inductance. If the frequency is increased gradually from zero, the line current I_0 decreases to a minimum (when the impedance of the parallel combination is a maximum) and then increases.

Usually the selectivity or sharpness of resonance of a circuit such as the one shown in Fig. 7–21(a) is considered from the standpoint of the variation of E_C. A common illustration of this is in vacuum-tube circuits. Since this book is intended for the first course in circuits and it presupposes no knowledge of vacuum-tube circuits, an attempt will be made to justify the interest in the variation of E_C. Assume that the plate circuit of a vacuum tube can be simplified and may be represented by that shown in Fig. 7–21(a). If the terminals of the condenser are then connected to the grid circuit of the next tube, the output of the second tube will be controlled by the variation of E_C. This variation can be determined by studying the circuit diagram. It is obvious that

$$\dot{E}_C = \dot{E}_g - \dot{I}_0 R_g \tag{7–58}$$

If $R_g = 0$, then E_C is equal to E_g and it will remain constant as the frequency is varied. On the other hand, if R_g is large, then E_C will be maximum when the line current I_0 or $I_0 R_g$ is minimum. The magnitude of E_C falls off faster for larger values of R_g than for smaller values of R_g, as indicated in Fig. 7–21(b). In general, it may be said that to secure selectivity or sharpness with a high-impedance generator a parallel circuit should be used (and with a low-impedance generator a series circuit should be used).

When the Q of the coil is high and R_g is very large, the maximum value of the condenser current I_C, the minimum value of I_0, and the maximum value of E_C will occur at approximately the same frequency. The curves of the condenser current I_C and E_C, plotted with frequency as abscissa, will have practically the same sharpness. That is, the ratio of the difference of the frequencies $(f_2 - f_1)$ at the points where the ordinates are 70.7 per cent of the maximum ordinate to the frequency at the maximum ordinate of the I_C curve will be nearly equal to a similar ratio for the E_C curve.

Under these conditions, the sharpness of resonance* for the circuit shown in Fig. 7–21(a) is approximately

$$\frac{f_2 - f_1}{f_{ar}} = \frac{1}{Q_{\text{coil}}} + \frac{X_L}{R_g} \qquad (7\text{–}59)$$

where $Q_{\text{coil}} = \frac{X_L}{R_L}$ is the Q of the coil.

Equation (7–59) states that the selectivity or sharpness of resonance can be increased in the following ways:

(a) by increasing Q of the coil, especially by decreasing R_L

(b) by increasing R_g of the generator.

PROBLEMS

7–1. Two coils, having impedances $\dot{Z}_1 = 3 + j4$ and $\dot{Z}_2 = 4 + j3$, are connected in parallel across 100-volt, 60-cycle mains. If the expression for the applied emf is taken as $e = 141.4 \sin 377t$, what are the expressions for the currents in the coils and the current in the line?

7–2. Solve for the line current in Problem 7–1 by the use of vectors.

7–3. Sketch the circuit in Problem 7–1 and indicate positive senses for the currents. Also, draw the emf wave $e = 141.4 \sin 377t$ and the three current waves. At the instant $t = 0.01$ sec, what are the instantaneous values for the three currents? Also, give the directions of current flow.

7–4. Two impedance coils are connected in parallel across 100-volt, 60-cycle mains. Instruments indicate that the first coil takes 5 amp and absorbs 100 watts, and the second takes 10 amp and absorbs 500 watts. What is the line current?

7–5. An impedance coil having an inductance of 0.02 henry and a resistance of 2 ohms is connected in parallel with a non-inductive resistance of 5 ohms across 110-volt, 60-cycle mains. Determine: (a) the vector current in each branch; (b) the vector line current; (c) the power factor of each branch; (d) the power factor of the parallel combination. Draw the vector diagram.

7–6. Two impedance coils having $\dot{Z}_A = 3 + j4$ and $\dot{Z}_B = 7 + j3$ are connected in parallel across 220-volt mains. Find: (a) the power taken by branch A; (b) the power taken by branch B. Draw the vector diagram.

7–7. Two coils A and B are connected in parallel across 110-volt, 60-cycle mains. Coil A takes 10 amp and absorbs 200 watts, and coil B takes 15 amp and absorbs 700 watts. What is the total vector current taken from the mains? Draw the two emf triangles on one diagram, showing the currents, the RI drops and the XI drops. Use the applied emf as the reference vector.

7–8. An impedance coil having a resistance of 15 ohms and an inductance of 0.5 henry is connected in parallel with a condenser of 7 μf (7×10^{-6} farad) across 110-volt, 60-cycle mains. Find, by the admittance method, the vector

* See Appendix D and Appendix H.

current in each branch and, also, the vector line current. Draw the vector diagram, indicating the resistive and reactive drops.

7–9. Two equal impedance coils each having a resistance of 2.7 ohms take 14 amp when connected in series across 220-volt, 60-cycle mains. What current will one of these coils take from a 110-volt, 25-cycle supply? When the coils are connected in parallel, what vector current will they take from a 110-volt, 25-cycle source?

7–10. A non-inductive resistance and a condenser are connected in parallel across 110-volt, 60-cycle mains. If the current flowing through the resistance is 15 amp and that taken by the condenser is 10 amp, what is the current taken from the mains? What power is supplied to the parallel combination?

7–11. A non-inductive resistance and a condenser are connected in parallel. The parallel combination takes a current of 17 amp and absorbs 1400 watts from a 220-volt, 60-cycle source. What are the vector currents in the branches? Draw the vector diagram.

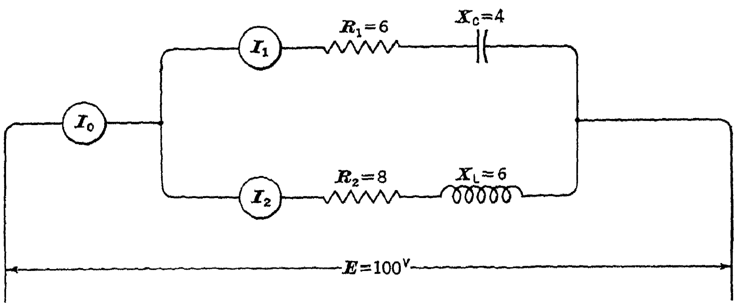


FIG. 7–22

7–12. A circuit containing a non-inductive resistance and a condenser takes 800 watts and a current of 25 amp from 220-volt, 60-cycle mains. If the resistance and the condenser are connected in series, what are their values? If they are in parallel, what are their values?

7–13. Two admittances, $\dot{Y}_A = 0.04 + j0.06$ and $\dot{Y}_B = 0.16 - j0.12$, are connected in parallel across an alternating-current circuit. The current in branch B is 10 amp. Using $\dot{I}_B$ as a reference vector, find $\dot{I}_A$ and the applied emf $\dot{E}_0$. Draw the vector diagram.

7–14. Two parallel branches, one of which consists of 5 ohms resistance and 0.1 henry inductance and the other 3 ohms resistance and 100 μf capacitance, take 30 amp from 60-cycle mains. What is the current in each branch? What should be the instantaneous values of the currents when the current in the mains has a value of 25 amp?

7–15. An impedance coil and a condensive impedance unit are connected in parallel across 110-volt, 60-cycle mains. The currents in the branches are 0 and 5 amp, respectively, and the powers taken by them are 1000 watts and 100 watts, respectively. What is the line current?

7–16. In the circuit shown in Fig. 7–22, the values for the reactances were obtained with a 60-cycle source. Find: (*a*) the values of L and C; (*b*) the resonant frequency, on the basis of unity power factor; (c) the impedance of the parallel combination for resonant condition.

7–17. Plot a curve of I_0 versus frequency over a range from 25 to 500 cycles for the circuit in Problem 7–16. Reduce the parallel combination to an equivalent series circuit of two elements for the frequency of 60 cycles per second.

7–18. Refer to Fig. 7–13(a). Let

$$\dot{Y}_A = \frac{1}{\dot{Z}_A} = G_A - jB_A$$

$$\dot{Y}_B = \frac{1}{\dot{Z}_B} = G_B + jB_B$$

Show that the equivalent impedance is

$$\dot{Z}_0 = \frac{G_A + G_B}{(G_A+G_B)^2 + (B_A - B_B)^2} + j\frac{B_A - B_B}{(G_A+G_B)^2 + (B_A - B_B)^2}$$

7–19. With the use of the R's and X's, show that: (a) the two currents in the two circuits in Fig. 7–13 are equal and (b) the power factor angles have the same value.

7–20. Sketch two equivalent series circuits for Fig. 7–13(a): one for the case when B_A is greater than B_B, and another when B_A is less than B_B.

7–21. An impedance coil, having $R = 200$ ohms and $L = 2$ henrys, is connected in parallel with a capacitive circuit, for which $R = 200$ ohms and $C = 50\mu f$. The applied emf is $e = 1000\sqrt{2} \sin \omega t$. Find the line current for: (a) $\omega = 0$; (b) $\omega = 800$.

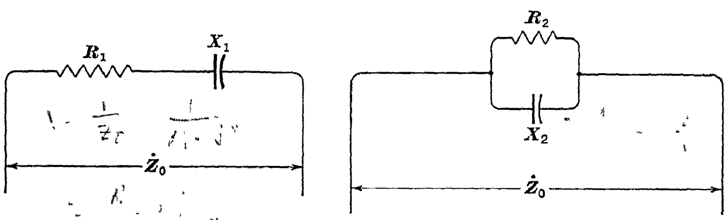


FIG. 7–23

7–22. It is desired to replace a series circuit by a parallel circuit, as shown in Fig. 7–23. The impedance $\dot{Z}_0$ is to remain unchanged. If R_1 and X_1 are given, derive a relation for calculating R_2 and X_2 in terms of R_1 and X_1.

7–23. If, in Problem 7–22, $R_1 = 500$ ohms, $C_1 = 10\ \mu f$, and $\omega = 5000$, find R_2 and C_2.

7–24. An impedance coil and a condenser are connected in parallel across 220-volt, 60-cycle mains. The coil has a resistance of 2 ohms and takes 10 amp. Calculate the value of capacitance for minimum line current.

7–25. An impedance coil having $R = 200$ ohms and $L = 2$ henrys is connected in parallel with a capacitive circuit having $R = 200$ ohms and $C = 50$

μf. If the applied voltage is $E = 500$ volts, find the total line current for: (a) $f = 0$; (b) $f = 60$ cps, and (c) $f = 1000$ cps.

7-26. A coil having a resistance of 3 ohms and an inductance of 0.02 henry is placed in parallel with a variable condenser. For what value of capacitance (in farads) will the current in the mains be minimum when the parallel combination is connected across 110-volt, 60-cycle mains?

7-27. Two circuits A and B are to be connected to a variable frequency source. Circuit A contains a pure inductance L_1 and circuit B contains a pure inductance L_2 and a capacitance of 2 μf in series. When A and B are connected in series, the resonant frequency is 1000 cps. When they are connected in parallel, the resonant frequency of circuit B is 2000 cps. Determine the values of L_1 and L_2.

7-28. A coil having a constant $R = 20$ ohms and $L = 159$ microhenrys is connected in parallel with a condenser $C = 159$ $\mu\mu$f. The parallel combination is connected across a generator of 1 volt with a constant internal R_g and negligible inductance. Draw resonance curves for the line current when: (a) $R_g = 0$; (b) $R_g = 10{,}000$ ohms; and (c) $R_g = 100{,}000$ ohms. Use the frequency range from 800 kilocycles to 1200 kilocycles.

7-29. Repeat Problem 7-28 if the frequency is held constant at 1000 kilocycles and the capacitance is varied.

7-30. Two impedances $\dot{Z}_1$ and $\dot{Z}_2$ are connected in parallel. Each impedance is a pure R, a pure L, or a pure C. When the parallel combination is connected to a direct-current source, the following readings are obtained:

$$E_{d-c} = 30 \text{ volts and } I_{\text{line}} = 10 \text{ amp}$$

Determine the types of elements needed for $\dot{Z}_1$ and $\dot{Z}_2$. Is there more than one possible combination?

CHAPTER 8

SERIES-PARALLEL COMBINATION OF ELEMENTS

In many electric circuits, such as power networks, communication networks, electrical apparatus, etc., there exist series-parallel combinations of circuit elements. A power line with resistance and inductance supplying power to a lighting load and a motor load connected in parallel gives a series-parallel circuit. Another example of a series-parallel circuit is the equivalent cir-

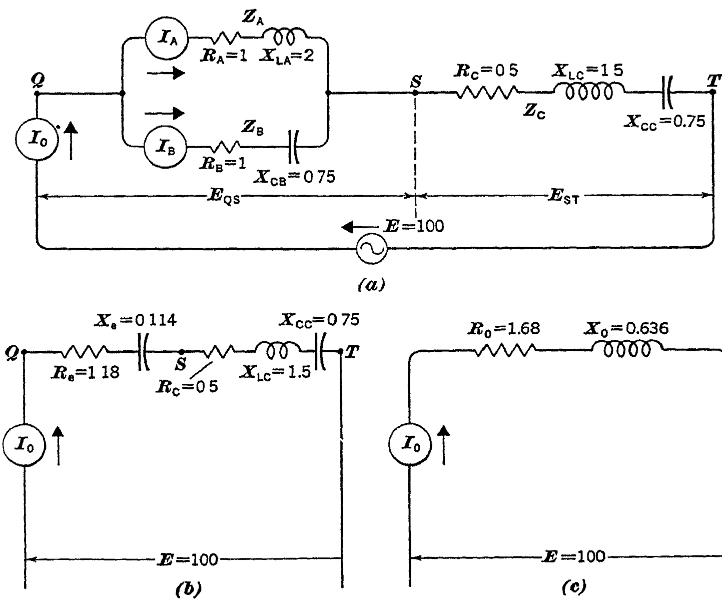


FIG. 8–1

cuit of a transformer (see Chapter 11). In the general case, a series-parallel circuit may consist of portions with several branches in parallel and these portions are connected in series with other portions consisting of single or branched circuits.

8–1. Series-Parallel Branches.—A simple series-parallel circuit is shown in Fig. 8–1(a). For the case where the applied emf E is known, two methods of determining the currents will be given.

Method A: (*a*) The impedance of the combination is

$$\dot{Z}_0 = \dot{Z}_C + \frac{\dot{Z}_A \dot{Z}_B}{\dot{Z}_A + \dot{Z}_B} \tag{8-1}$$

(*b*) The line current is

$$\dot{I}_0 = \frac{\dot{E}}{\dot{Z}_0} \tag{8-2}$$

(*c*) The branch currents are

$$\dot{I}_A = \frac{\dot{Z}_B}{\dot{Z}_A + \dot{Z}_B} \dot{I}_0 \tag{8-3}$$

$$\dot{I}_B = \frac{\dot{Z}_A}{\dot{Z}_A + \dot{Z}_B} \dot{I}_0 = \dot{I}_0 - \dot{I}_A \tag{8-4}$$

Method B: (*a*) Reduce the parallel part QS of the circuit to an equivalent series circuit with

$$\dot{Z}_e = R_e + jX_e$$

as shown in Fig. 8-1(*b*).

(*b*) Combine this equivalent series circuit with the series part ST into a simple series circuit with two elements for which $\dot{Z}_0 = R_0 + jX_0$, as indicated in Fig. 8-1(*c*).

(*c*) The given applied emf $\dot{E}$ divided by the now known $\dot{Z}_0$ will give the line current $\dot{I}_0$.

(*d*) The voltage drops $\dot{E}_{QS}$ and $\dot{E}_{ST}$ can easily be found because $\dot{E}_{QS} = \dot{Z}_e \dot{I}_0$ and

$$\dot{E}_{ST} = [R_C + j(X_{LC} - X_{CC})]\dot{I}_0 = \dot{Z}_C \dot{I}_0$$

(*e*) For a known $\dot{E}_{QS}$, the branch currents are

$$\dot{I}_A = \frac{\dot{E}_{QS}}{\dot{Z}_A} \text{ and } \dot{I}_B = \frac{\dot{E}_{QS}}{\dot{Z}_B}$$

(*f*) The power factor of the entire circuit is equal to $\dfrac{R_0}{Z_0}$.

Example 8-1.—Fig. 8-1(*a*) shows a simple series-parallel combination of circuit elements. The applied emf and the circuit elements are given. It is required to find the total line current $\dot{I}_0$, the branch currents $\dot{I}_A$ and $\dot{I}_B$, and the power factor of the entire circuit. Draw the vector diagram.

Solution.—Method B will be followed in the solution of this example. The equivalent impedance for the parallel part QS is

$$\dot{Z}_e = \frac{\dot{Z}_A \dot{Z}_B}{\dot{Z}_A + \dot{Z}_B} = \frac{2.24\underline{/63.5^\circ} \times 1.25\underline{/-36.8^\circ}}{2.24\underline{/63.5^\circ} + 1.25\underline{/-36.8^\circ}}$$

$$= 1.18 - j0.114 = 1.19\underline{/-5.5^\circ}$$

The impedance for the entire circuit is

$$\dot{Z}_0 = \dot{Z}_e + \dot{Z}_C = [1.18 - j0.114] + [0.5 + j(1.5 - 0.75)]$$
$$= 1.68 + j0.636 = 1.8\underline{/20.7^\circ}$$

If the applied voltage $\dot{E}$ is selected as the reference vector, the line current is

$$\dot{I}_0 = \frac{\dot{E}}{\dot{Z}_0} = \frac{100\underline{/0^\circ}}{1.8\underline{/20.7^\circ}} = 55.5\underline{/-20.7^\circ}$$

The voltage drops across QS and ST are

$$\dot{E}_{QS} = \dot{Z}_e \dot{I}_0 = 1.19\underline{/-5.5^\circ} \times 55.5\underline{/-20.7^\circ} = 66.0\underline{/-26.2^\circ}$$
$$\dot{E}_{ST} = \dot{Z}_C \dot{I}_0 = 0.902\underline{/56.3^\circ} \times 55.5\underline{/-20.7^\circ} = 50.1\underline{/35.6^\circ}$$

The branch currents are

$$\dot{I}_A = \frac{\dot{E}_{QS}}{\dot{Z}_A} = \frac{66.0\underline{/-26.2^\circ}}{2.24\underline{/63.5^\circ}} = 29.4\underline{/-89.7^\circ}$$

$$\dot{I}_B = \frac{\dot{E}_{QS}}{\dot{Z}_B} = \frac{66.0\underline{/-26.2^\circ}}{1.25\underline{/-36.8^\circ}} = 52.8\underline{/10.7^\circ}$$

The power factor of the entire circuit is

$$\text{P.F.} = \frac{R_0}{Z_0} = \frac{1.68}{1.80} = 0.934 \text{ (lagging)}$$

The vector diagram is shown in Fig. 8–2.

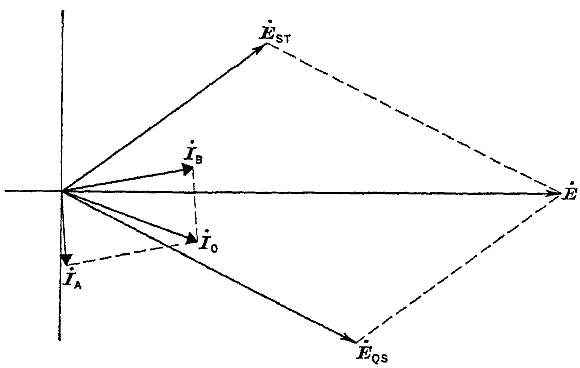
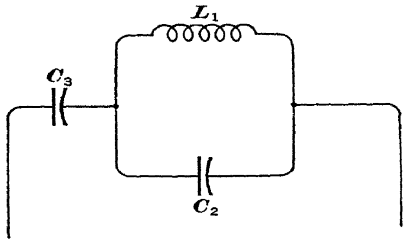


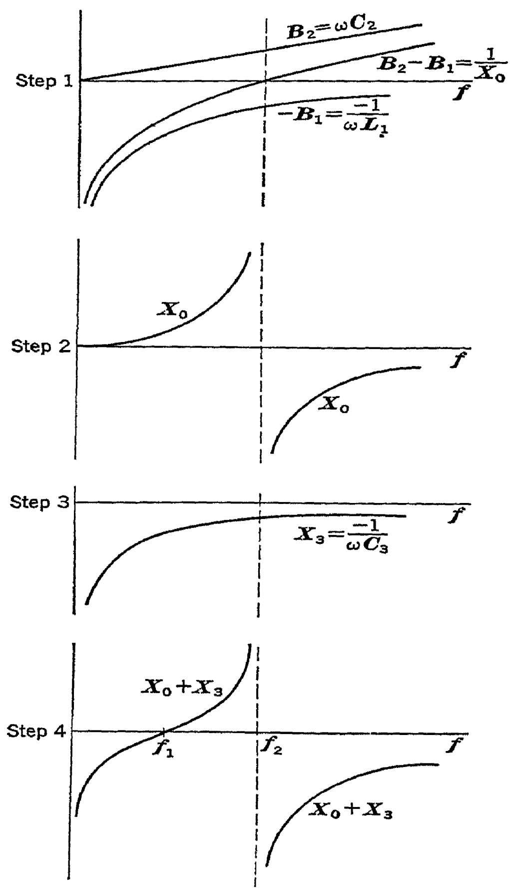
FIG. 8–2

8–2. Multiple Resonance.—In radio circuits, it is quite often desirable to have a network which will be resonant at one frequency and anti-resonant at another frequency. The series-parallel combination shown in Fig. 8–3 will be resonant at some frequency f_1 and anti-resonant at a higher frequency f_2. That is, the parallel combination of L_1 and C_2 will be anti-resonant at f_2 and present a very high impedance to a wave or signal of that



L_1
C_3
C_2

FIG. 8-3



Step 1
$B_2 = \omega C_2$
$B_2 - B_1 = \frac{1}{X_0}$
f
$-B_1 = \frac{-1}{\omega L_1}$
Step 2
X_0
f
X_0
Step 3
f
$X_3 = \frac{-1}{\omega C_3}$
Step 4
$X_0 + X_3$
f_1
f_2
f
$X_0 + X_3$

FIG. 8-4

frequency. At f_1, the parallel combination acts like an inductance in series with C_3. This series combination is resonant at f_1 and allows a wave or signal of this frequency to go through without opposition since the circuit contains no resistance. An example of such a combination is the "wave trap" in a receiving circuit. The undesirable frequency f_2 can be tuned out by varying C_2, and at the same time the circuit can be adjusted by varying C_3 to accept the desired frequency f_1.

The phenomenon of multiple resonance can be studied by the use of reactance curves. For the circuit shown in Fig. 8–3, the reactance curves are given in Fig. 8–4. The following procedure gives the steps in obtaining the different curves.

Step 1. Sketch the susceptance curves of L_1 and C_2, and add the two to obtain the total susceptance curve for the parallel combination.

Step 2. Take the reciprocal of the total susceptance curve to obtain the reactance curve X_0 of the parallel combination.

Step 3. Sketch the reactance curve X_3 for C_3.

Step 4. Add the reactance curves X_0 and X_3. Then, f_1 is the resonant frequency, and f_2 is the anti-resonant frequency.

More complicated circuits consisting of reactances only can be analyzed by following a procedure similar to that just outlined.

Example 8–2.—The series-parallel combination in Fig. 8–3 is used as a tuning circuit to pass 10^6 cycles and, at the same time, to suppress 2×10^6 cycles. If $L_1 = 200$ microhenrys, find C_2 and C_3.

Solution.—Since the anti-resonant frequency f_{ar} and L_1 are given, it is a simple matter to find C_2. Thus,

$$2\pi f_{ar}L_1 = \frac{1}{2\pi f_{ar}C_2}$$

$$C_2 = \frac{1}{4\pi^2 f^2_{ar}L_1} = \frac{1}{4\pi^2 \times (2\times10^6)^2 \times 200\times10^{-6}} = 31.7\times10^{-12} \text{ farad}$$

At the resonant frequency $f_r = 10^6$ cps, the reactance of the parallel combination of L_1 and C_2 must be inductive and equal to the reactance of C_3. Thus,

$$-jX_3 = \frac{j2\pi f_r L_1\left(-j\frac{1}{2\pi f_r C_2}\right)}{j\left(2\pi f_r L_1 - \frac{1}{2\pi f_r C_2}\right)} = -j1670$$

from which

$$C_3 = \frac{1}{2\pi\times10^6\times1670} = 95.5\times10^{-12} \text{ farad}$$

PROBLEMS

8–1. Two coils, $\dot{Z}_A = 0.5 + j0.75$ and $\dot{Z}_B = 2 + j1$, in parallel are connected in series with an impedance $\dot{Z}_C = 4 - j2$. If the circuit is connected to a 100-volt alternating-current circuit, what is the vector current flowing in each of the impedances? Draw the vector diagram, using the applied emf as the reference.

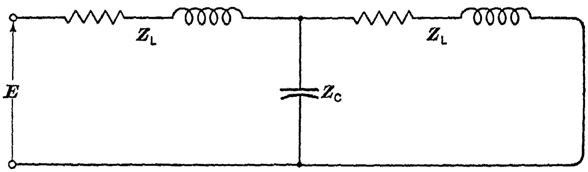


FIG. 8–5

8–2. Two admittances, $\dot{Y}_A = 0.2 + j0.1$ and $\dot{Y}_B = 0.3 - j0.4$, in parallel are connected in series with a coil having a resistance of 3 ohms and an inductive reactance of 5 ohms. The circuit is connected to a 60-cycle source, and a voltmeter across the coil reads 50. Determine the vector current flowing in: (*a*) branch *A*; (*b*) branch *B*; and (*c*) the coil. Using the emf across the coil as a reference, draw the vector diagram.

8–3. A lamp of *R* ohms is connected in parallel with a coil having a reactance of *X* ohms and a negligible resistance. This parallel circuit is connected in series with a variable condenser For what value of capacitive reactance will this entire circuit be in series resonance?

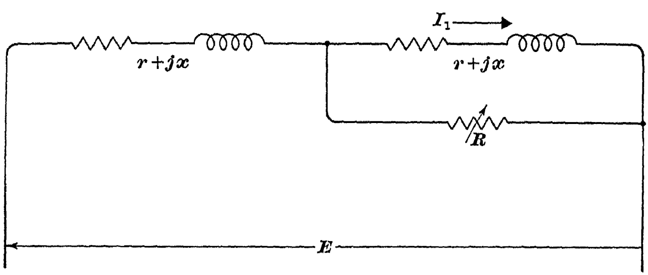


FIG. 8–6

8–4. Two equal coils, each having $\dot{Z}_L = 3 + j4$, are in series, and one of the coils is shunted by a condenser $\dot{Z}_C = -j2$, as shown in Fig. 8–5. If a voltmeter across the condenser reads 40 volts, what is the applied emf?

8–5. Two impedances $\dot{Z}_1$ and $\dot{Z}_2$ in parallel are connected in series with a non-inductive resistance of 10 ohms. The combination takes 500 watts at unity power factor from a 100-volt, 60-cycle source. If the magnitudes of the line and branch currents are equal, what are the two possible values for Z_1 and Z_2?

8–6. A non-inductive resistance of R ohms is shunted across one of two equal impedances, as shown in Fig. 8–6. For what value of R will the current I_1 be in quadrature with the applied emf E? Determine R in terms of r and x.

8–7. Fig. 8–7 shows a T-section considered very often in communication circuits. A non-inductive load R is connected across the output terminals 3 and 4. What must be its value to make the resistance measured at the sending-end terminals 1 and 2 equal to R?

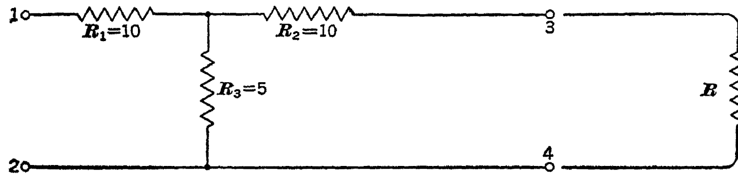


FIG. 8–7

8–8. The circuit shown in Fig. 8–8 is connected to a 60-cycle source. The voltage cross the parallel portion is 35.3 volts. Using this voltage drop as a reference vector, determine the vector line current. Draw the vector diagram.

8–9. If in Problem 8–8 E_{ST} (instead of E_{QS}) is given as 35.5 volts, determine the three currents. Draw the vector diagram with $\dot{E}_{ST}$ as reference.

8–10. The current flowing in the line of Problem 8–8 is 20 amp. Calculate the three voltages E_{QS}, E_{ST}, and E. Using $\dot{I}_0$ as a reference vector, draw the vector diagram.

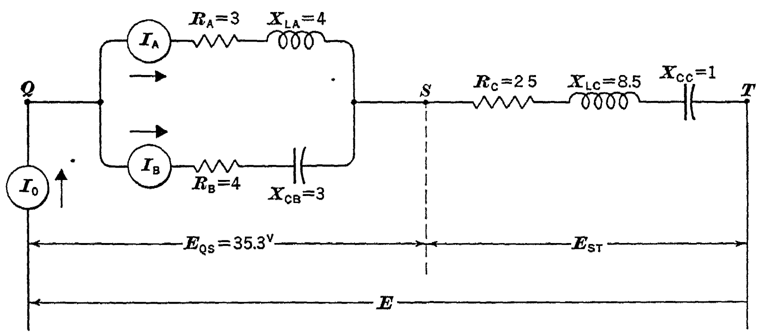


FIG. 8–8

8–11. The circuit constants of the circuit in Problem 8–8 are measured with a 60-cycle source. Repeat the problem when the circuit is connected to a 25-cycle source.

8–12. An infinite bus or a generator of infinite capacity or a constant potential generator is a source which can deliver any amount of current without changing the terminal voltage. Such a source may be pictured as one without internal impedance. The circuit shown in Fig. 8–9 is connected to a constant potential source E_g. If $\dot{Z}_1 = jX_L$ and $\dot{Z}_2 = -jX_C$, and they are equal in magnitude, show that I_R is independent of Z_R.

8–13. Explain the variation (if any) of the drops across the impedances Z_1 and Z_2 in Problem 8–12 as Z_R is varied from 0 to ∞.

8–14. Will I_R be independent of Z_R in Problem 8–12 when $\dot{Z}_1 = R_1 + jX_1$ and $\dot{Z}_2 = R_2 - jX_2$? Equal magnitudes for the impedances may be assumed.

8–15. Sketch the reactance curves for the circuit of Fig. 8–10. Indicate the resonant and anti-resonant frequencies.

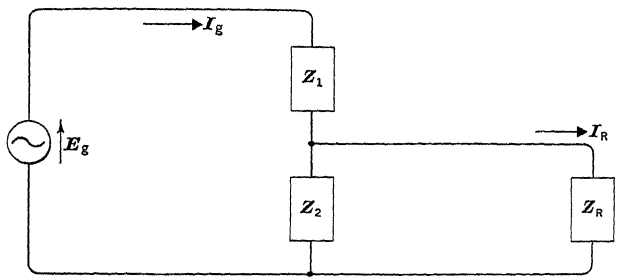


FIG. 8–9

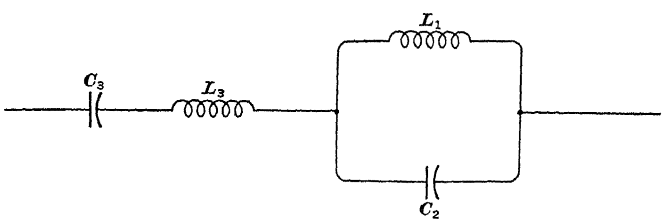


FIG. 8–10

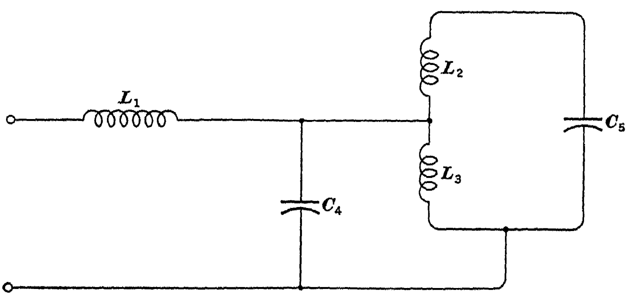


FIG. 8–11

8–16. Sketch the reactance curves for the circuit shown in Fig. 8–11. Indicate the resonant and anti-resonant frequencies.

8–17. The coil L_1 in Fig. 8–3 has an inductance of 200 microhenrys and negligible resistance. If C_3 is replaced by L_3, what are the values of C_2 and L_3 such that the combination is resonant at 2,000,000 cycles and anti-resonant at 1,000,000 cycles?

CHAPTER 9

REAL AND APPARENT POWER

The product of effective voltage and effective current in an alternating-current circuit is called volt-amperes (abbreviated va) or apparent power. The real power or average power, or simply power, which may be determined by a wattmeter, is expressed in watts. For sinusoidal voltage and sinusoidal current, as considered in Arts. 1–16 and 1–17, the average power may be taken equal to

$$P = EI \cos \theta \text{ watts}$$

or

$$P = EI \times (\text{P.F.}) \text{ watts}$$

In general, effective volt-amperes do not represent real power. The power factor must be considered. In a purely resistive circuit, the power factor is unity and the real power in watts is equal to the volt-amperes. In a purely inductive circuit, where the power factor is zero, the real power is zero; but the apparent power can have a definite value.

A knowledge of volt-amperes is sometimes of considerable importance, since volt-amperes and not watts limit the output of a machine or a transformer. For the same voltage and current, the average power $P = EI \cos \theta$ may vary, the value depending on the power factor $\cos \theta$. It is obvious that, for the same power and voltage, the current can be made a large quantity if the factor $\cos \theta$ is reduced to a small value. A larger current means a larger I^2R loss and this, in turn, means a greater temperature rise. The limit of the continuous output of generators, motors, or transformers is usually determined by the temperature rise in the windings.

Synchronous machines and transformers are usually rated in kilovolt-amperes (abbreviated kva and equal to 1000 volt-amperes) at a given power factor. It is the purpose of this chapter to show how power, power factor, reactive volt-amperes, and volt-amperes are considered in electrical problems involving rotating machines and static transformers.

9–1. The Power Diagram.—Fig. 9–1(*a*) shows a vector diagram for voltage and current in an *RL* circuit. The applied voltage E is used as the reference vector, and the components of the current are:

$$I_r = I \cos\theta \qquad \text{and} \qquad I_x = I \sin\theta \tag{9–1}$$

If the current and its components are multiplied by E, the diagram shown in Fig. 9–1(*b*) results. Since the power is the product of E and the in-phase component of the current, it is

$$P_r = EI \cos\theta = EI_r \text{ watts} \tag{9–2}$$

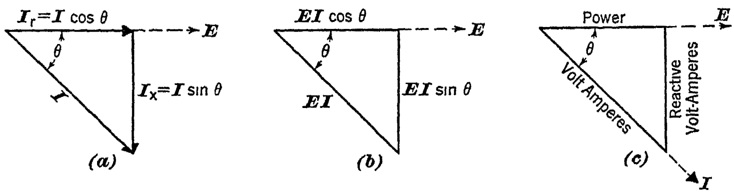


FIG. 9–1

Power may be expressed in watts, but it is also expressed in kilowatts (abbreviated kw and equal to 1000 watts).

For sinusoidal waves $e = E_m \sin\omega t$ and $i = I_m \sin(\omega t + \theta)$, the power factor is

$$\text{P.F.} = \cos\theta = \frac{\text{Power}}{\text{Volt-amperes}} = \frac{P_r}{EI} \tag{9–3}$$

The reactive volt-amperes are

$$P_x = EI \sin\theta = EI_x \text{ vars} \tag{9–4}$$

They are the product of E and the quadrature component of the current. The unit of reactive volt-amperes is the "var" (coined from the first letters of the words volts, amperes, and reactive). The larger unit is the kilovar (abbreviated kvar and equal to 1000 vars). For sinusoidal waves, $\sin\theta$ is called the reactive factor (R.F.). Thus,

$$\text{R.F.} = \sin\theta = \frac{\text{Reactive volt-amperes}}{\text{Volt-amperes}} = \frac{P_x}{EI} \tag{9–5}$$

or

$$\text{R.F.} = \sqrt{1 - (\cos\theta)^2} = \sqrt{1 - (\text{P.F.})^2} \tag{9–5a}$$

The total volt-amperes are equal to the square root of the sum of the square of the power and the square of the reactive volt-amperes, or

$$EI = \sqrt{(EI\cos\theta)^2 + (EI\sin\theta)^2}$$

or

$$EI = \sqrt{\text{watts}^2 + \text{vars}^2} = \text{volt-amperes} \qquad (9\text{–}6)$$

A relationship between power, reactive volt-amperes, and volt-amperes is shown in Fig. 9–1(*c*).

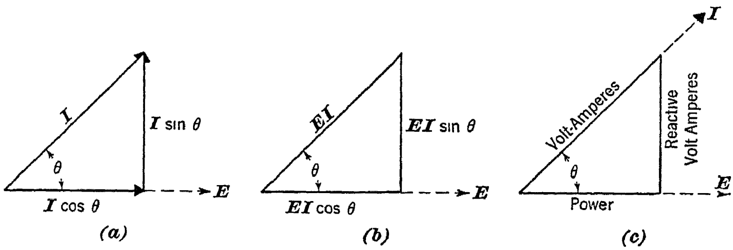


FIG. 9–2

For a leading current, the diagrams in Fig. 9–1 become those shown in Fig. 9–2.

Example 9–1.—A circuit consisting of $R = 3$ ohms and $X_L = 4$ ohms in series is connected to a 100-volt, 60-cycle source. Calculate the power, the reactive volt-amperes, the volt-amperes, the power factor, and the reactive factor. Draw the power diagram.

Solution.—The impedance is

$$\dot{Z} = 3 + j4 = 5\underline{/53.2°}$$

If $\dot{E} = 100\underline{/0°}$, the current is

$$\dot{I} = \frac{100\underline{/0°}}{5\underline{/53.2°}} = 20\underline{/-53.2°} = 12 - j16$$

from which

$$I_r = 12 \qquad \text{and} \qquad I_x = 16$$

The power is

$$P_r = EI\cos\theta = EI_r = 100 \times 12 = 1200 \text{ watts}$$

The reactive volt-amperes are

$$P_x = EI\sin\theta = EI_x = 100 \times 16 = 1600 \text{ vars}$$

The volt-amperes are

$$\text{Volt-amperes} = EI = 100 \times 20 = 2000 \text{ va}$$

$$= \sqrt{P_r^2 + P_x^2} = \sqrt{1200^2 + 1600^2} = 2000 \text{ va}$$

The power factor is

$$\text{P.F.} = \cos\theta = \frac{\text{Power}}{\text{Volt-amperes}} = \frac{1200}{2000} = 0.6$$

The reactive factor is

$$\text{R.F.} = \sin\theta = \frac{\text{Reactive volt-amperes}}{\text{Volt-amperes}} = \frac{1600}{2000} = 0.8$$

The power diagram is shown in Fig. 9–3.

9–2. Reactive Volt-Amperes.—In an alternating-current circuit with sinusoidal waves, the reactive volt-amperes are equal to the product of the voltage and the quadrature component of the current, or

$$P_x = EI \sin\theta \text{ vars} \tag{9–4}$$

The reactive volt-amperes in the electrical power industry are caused primarily by inductive loads, such as induction motors. It is important to have a clear picture of the effect of reactive volt-amperes on the entire electrical system.

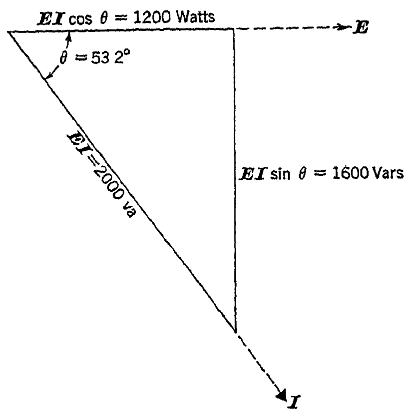


FIG. 9–3

The reactive volt-amperes represent an exchange of energy between a part of the circuit and the source during each cycle. In the case of an RL circuit, there is a continuous exchange of energy between the magnetic field of the coil and the generator. Energy is stored in the magnetic field of the coil while the current increases, and this energy is released when the current decreases. This energy does not leave the system, and so it does not appear in the real power delivered from the source to the load. It does increase the generator current for the same power delivered to the load. This also can be seen from Fig. 9–1(c) and equation (9–2). When,

θ becomes larger because of an increase in reactive volt-amperes, $\cos\theta$ becomes smaller and I is increased for the same power and voltage. Thus,

$$I = \frac{P_r}{E \cos\theta} \tag{9-2a}$$

A larger current means a larger capacity for the generator and usually a greater I^2R loss in the system. This is illustrated in Example 9-2.

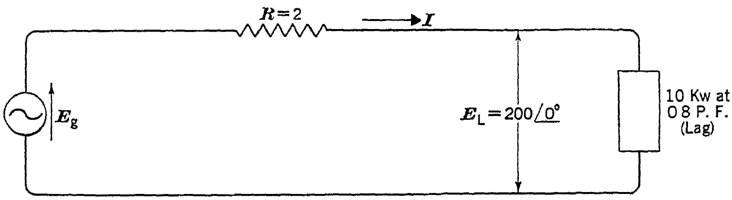


FIG. 9-4

Example 9-2.—An impedance load takes 10 kw at a constant alternating voltage of 200 volts, as shown in Fig. 9-4. The lagging power factor of the load is 0.8 (or 80 per cent). The transmission line has a total resistance $R = 2$ ohms for the two conductors and negligible inductance. Calculate: (*a*) the load volt-amperes, (*b*) the load reactive volt-amperes, (*c*) the line current, (*d*) the line loss, and (*e*) the generator capacity.

Repeat the problem for a load of 10 kw at unity power factor, and compare the capacities of the two generators.

Solution.—(1) *For 10 kw at 0.8 P.F. lagging:* The volt-amperes are, from equation (9-3),

$$\text{Volt-amperes} = \frac{\text{Watts}}{\text{P.F.}} = \frac{10{,}000}{0.8} = 12{,}500 \text{ va}$$

The reactive volt-amperes are

$$P_x = \sqrt{\text{va}^2 - \text{watts}^2} = \sqrt{12{,}500^2 - 10{,}000^2} = 7{,}500 \text{ vars}$$

The magnitude of the line current is

$$I = \frac{\text{Volt-amperes}}{\text{Volts}} = \frac{12{,}500}{200} = 62.5 \text{ amp}$$

With reference to the load voltage drop $\dot{E}_L = 200\underline{/0°}$ (at 0.8 P.F. lagging), the vector current is

$$\dot{I} = 62.5\underline{/-36.8°}$$

The transmission line loss is

$$I^2R = 62.5^2 \times 2 = 7812.5 \text{ watts}$$

The transmission line drop is

$$R\dot{I} = 2 \times 62.5\underline{/-36.8°} = 100 - j75$$

The generator voltage is

$$\dot{E}_g = R\dot{I} + \dot{E}_L = (100 - j75) + 200 = 300 - j75 = 309\underline{/-14.1^\circ}$$

The generator capacity is

$$E_g I = 309 \times 62.5 = 19{,}200 \text{ va}$$

(2) *For 10 kw at unity power factor:* The volt-amperes are

$$\text{Volt-amperes} = \text{watts} = 10{,}000$$

The reactive volt-amperes are

$$P_x = 0$$

The magnitude of the line current is

$$I = \frac{10{,}000}{200} = 50 \text{ amp}$$

With reference to the load voltage drop $\dot{E}_L$ (at P.F. = 1), the vector current is

$$\dot{I} = 50\underline{/0^\circ}$$

The transmission line loss is

$$I^2R = 50^2 \times 2 = 5000 \text{ watts}$$

The transmission line drop is

$$R\dot{I} = 2 \times 50\underline{/0^\circ} = 100 + j0$$

The generator voltage is

$$\dot{E}_g = R\dot{I} + \dot{E}_L = 100 + 200 = 300 + j0$$

The generator capacity is

$$E_g I = 300 \times 50 = 15{,}000 \text{ va}$$

It can be seen that for the same kw load the line current, the I^2R loss in the transmission line, and the generator capacity are larger for the condition when the power factor is less than unity.

Example 9–3.—An induction motor takes 20 kw from a 100-volt line at 0.8 power factor. Under normal operation, the power factor of an induction motor, like that of an RL circuit, is lagging. Calculate the kva, the kvars, the reactive factor, and the line current. Draw the power diagram.

Solution.—The kva are

$$\text{kva} = \frac{\text{kw}}{\text{P.F.}} = \frac{20}{0.8} = 25$$

The kvars are

$$\text{kvars} = \sqrt{\text{kva}^2 - \text{kw}^2} = \sqrt{25^2 - 20^2} = 15$$

The reactive factor is

$$\text{R.F.} = \frac{\text{kvars}}{\text{kva}} = \frac{15}{25} = 0.6$$

The line current is

$$I = \frac{\text{Volt-amperes}}{\text{Volts}} = \frac{25{,}000}{100} = 250 \text{ amp}$$

This lags behind the applied voltage by an angle whose cosine is 0.8. The power diagram is shown in Fig. 9–5.

Example 9–4.—An over-excited synchronous motor takes 10 kw at 0.8 power factor from a 200-volt line. The power factor of an over-excited synchronous motor, like that of an *RC* circuit, is leading. Calculate the kva, the kvars, the reactive factor, and the current taken by the motor. Draw the power diagram.

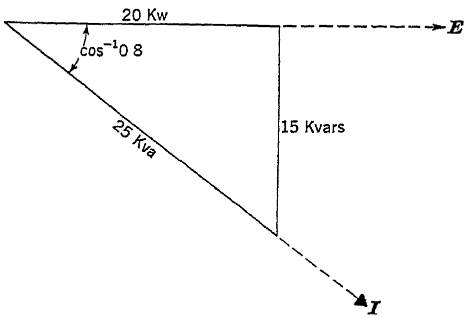


FIG. 9–5

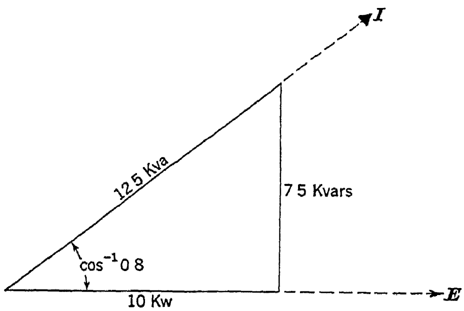


FIG. 9–6

Solution.—The kva are

$$\text{kva} = \frac{\text{kw}}{\text{P.F.}} = \frac{10}{0.8} = 12.5$$

The kvars are

$$\text{kvars} = \sqrt{\text{kva}^2 - \text{kw}^2} = \sqrt{12.5^2 - 10^2} = 7.5$$

The reactive factor is

$$\text{R.F.} = \frac{\text{kvars}}{\text{kva}} = \frac{7.5}{12.5} = 0.6$$

The motor current is

$$I = \frac{\text{Volt-amperes}}{\text{Volts}} = \frac{12{,}500}{200} = 62.5 \text{ amp}$$

This leads the terminal voltage by an angle whose cosine is 0.8. The power diagram is drawn in Fig. 9–6.

PROBLEMS

9–1. A lamp load takes 5000 watts (the power factor is considered to be unity) from a 200-volt, 60-cycle supply. Calculate the volt-amperes and the line current. What are the reactive volt-amperes?

9–2. An electric flatiron (assume P.F. = 1) is connected to a 100-volt, 60-cycle line. The current is 5 amp. What are the volt-amperes and the real power?

9–3. A capacitive circuit consisting of $R = 2$ ohms and $X_C = 2$ ohms in series is connected to a 100-volt, 60-cycle source. Calculate: (*a*) the current, (*b*) the real power; (*c*) the reactive volt-amperes, and (*d*) the volt-amperes. Draw the power diagram.

9–4. An inductive coil takes 200 watts at 0.8 power factor from a 100-volt, 60-cycle source. Calculate: (*a*) the volt-amperes, (*b*) the reactive volt-amperes, and (*c*) the current. Draw the power diagram.

9–5. An impedance coil consisting of $R = 3$ ohms and $X_L = 4$ ohms is connected to a 60-cycle source. The current flowing through the coil is $I = 10$ amp. Using the current as a reference vector, draw the power diagram.

9–6. A rheostat of $R = 10$ ohms and a condenser of $C = 265$ μf are connected in series to a 60-cycle supply. The current flowing through the circuit is 5 amp. Using the current as a reference vector, draw the power diagram.

9–7. An induction motor takes a lagging current from a 200-volt, 60-cycle line. The load is 5000 watts at 0.6 power factor. Calculate the volt-amperes, the reactive volt-amperes, and the reactive factor. Draw the power diagram, using the current as the reference vector.

9–8. A 50-kva, 60-cycle transformer takes 0.5 amp at no load from a 2200-volt source. The power factor is 0.7 lagging. What real power does it take at no load? Draw the power diagram.

9–9. The d-c field of a synchronous motor is adjusted for a leading current from a 200-volt, 60-cycle line. The field current is greater than that required for unity power factor and the machine is said to be over-excited. The line current is 100 amp and it leads the applied emf by an angle of 30°. Calculate: (*a*) the volt-amperes, (*b*) the power factor, (*c*) the real power, (*d*) the reactive factor, and (*e*) the reactive volt-amperes. Draw the power diagram, using the voltage as the reference vector.

9–10. A lighting load of 1 kw (unity P.F.) is connected in parallel with the induction motor of Problem 9–7 to a 200-volt, 60-cycle line. Calculate: (*a*) the total real power, (*b*) the total reactive volt-amperes, (*c*) the total volt-amperes, (*d*) the combined power factor, and (*e*) the combined reactive factor. What is the total line current? Draw a vector diagram for voltage and currents and also a power diagram.

9–11. A power transformer is feeding a load on its secondary side. The primary coil is taking 50 kw at 0.7 power factor from a 2300-volt, 60-cycle supply. Consider this to be equivalent to an *RL* circuit as far as the supply terminals are concerned. Calculate the volt-amperes, the reactive volt-amperes, and the supply line current. Draw the power diagram.

9–12. An over-excited synchronous motor with a constant loss of 3 kw and taking a leading current is connected in parallel with the transformer of Problem 9–11 for power factor correction. What should be the kva input to the synchronous motor to bring the combined power factor to unity?

9–13. Two alternators operating in parallel are delivering power at 220 volts. The first supplies a current of 20 amp at a power factor of 0.85 lagging with respect to the terminal emf. The second supplies a current of 50 amp which leads the terminal emf by 30°. What per cent of the total power is each machine delivering? What is the power factor of the total load? Draw the vector diagram.

9–14. An inductive load takes 10 kw at a power factor of 0.8 from a 220-volt bus, which is fed by two alternators operating in parallel. If the machine currents are equal and one alternator is operating at unity power factor, what is the power supplied by each machine? At what power factor is the other alternator operating?

9–15. A factory takes from 2300-volt, 60-cycle, single-phase mains the following loads: 75 kw at 0.9 power factor lagging and 200 kw at 0.8 power factor lagging. Find the current in the mains and the combined power factor.

9–16. A synchronous motor and an induction motor are operating in parallel. The total load is 500 kw. The power factor of the induction motor is 0.8 lagging, the power factor of the synchronous motor is 0.7 leading, and the power factor at the bus-bars is unity. Calculate the load in kw and kva for each motor.

9–17. Two a-c generators operating in parallel are supplying power to an inductive load at 220 volts and 60 cps. The power delivered is 50 kw at 0.85 power factor. The machine currents are equal in magnitude, and one alternator is operating at unity power factor. (*a*) What power does each machine supply? (*b*) At what power factor is the other alternator operating? (*c*) Draw the power diagram.

9–18. A constant potential is maintained at the end of a transmission line, and 75 per cent of a total load of 200 kw is carried by an induction motor at 0.85 power factor lagging. Determine the kva input to a synchronous motor which will carry the remainder of the load and correct the power factor to 0.95 lagging.

CHAPTER 10

NETWORK THEOREMS

In the solution of many specific problems, much time and labor can often be saved by making use of certain theorems instead of starting from the beginning with Ohm's Law and Kirchhoff's Laws. It is only necessary to recognize certain fundamental similarities between new complicated structures and other established simpler networks. It is the purpose of this chapter to present theorems which will establish these similarities.

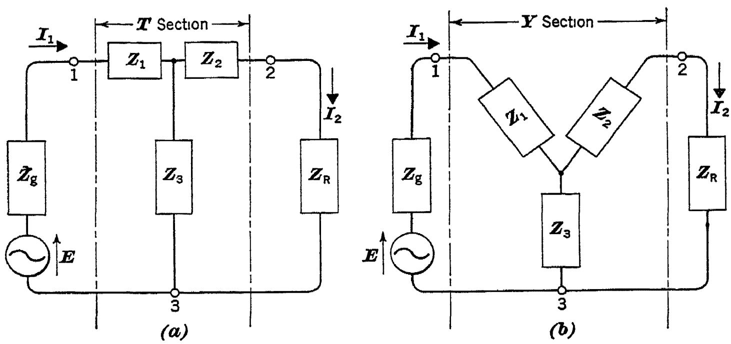


FIG. 10–1

10–1. Conversion of a π-Section Into an Equivalent T-Section. It is quite frequently desirable to make the conversion of a π-section or a Δ-section into an equivalent T-section or Y-section for purposes of solution of communication and power circuits. The branch impedances of the equivalent wye connection must have definite values in terms of the branch impedances of the delta connection.

Fig. 10–1(a) shows a T-section, so called because of its similarity to the letter T. In power circuits, it is more common to see the same section arranged in wye, as in Fig. 10–1(b).

Fig. 10–2(a) gives a π-section, so called because of its resemblance to the Greek letter of that name; while Fig. 10–2(b) shows the π-section as a delta-connection in power circuits.

These sections are connected between a generator, of generated voltage $\dot{E}$ and internal impedance $\dot{Z}_g$, and a load impedance $\dot{Z}_R$. To convert a π-section into an equivalent T-section, from the standpoint of $\dot{I}_1$ and $\dot{I}_2$ for the given generator and load, the vector impedances across corresponding terminals 1 and 2, 2 and 3, and 3 and 1 of the T-section must be equal to those of the π-section; that is,

$$(\dot{Z}_{12})_Y = (\dot{Z}_{12})_\Delta \tag{10-1}$$

$$(\dot{Z}_{23})_Y = (\dot{Z}_{23})_\Delta \tag{10-2}$$

$$(\dot{Z}_{31})_Y = (\dot{Z}_{31})_\Delta \tag{10-3}$$

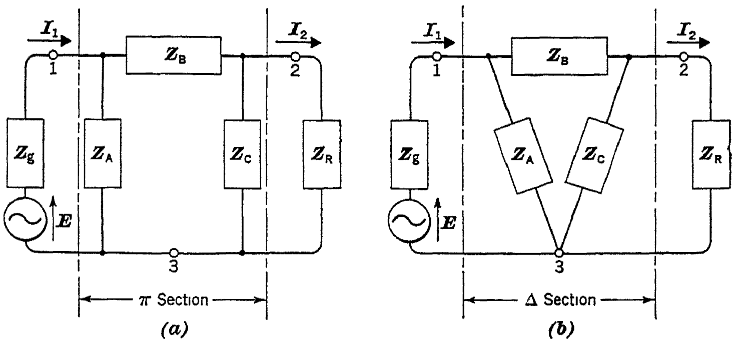


FIG. 10–2

Consider only the T-section and π-section, Figs. 10–1 and 10–2, and leave out the generator and load impedance for the present. The impedances across the terminals 1 and 2 are:

$$(\dot{Z}_{12})_Y = \dot{Z}_1 + \dot{Z}_2 \tag{10-4}$$

$$(\dot{Z}_{12})_\Delta = \frac{\dot{Z}_B(\dot{Z}_A + \dot{Z}_C)}{\dot{Z}_A + \dot{Z}_B + \dot{Z}_C} \tag{10-5}$$

Setting them equal to each other gives

$$\dot{Z}_1 + \dot{Z}_2 = \frac{\dot{Z}_B(\dot{Z}_A + \dot{Z}_C)}{\dot{Z}_A + \dot{Z}_B + \dot{Z}_C} \tag{10-6}$$

The impedances across the terminals 2 and 3 are

$$(\dot{Z}_{23})_Y = \dot{Z}_2 + \dot{Z}_3 \tag{10-7}$$

$$(\dot{Z}_{23})_\Delta = \frac{\dot{Z}_C(\dot{Z}_A + \dot{Z}_B)}{\dot{Z}_A + \dot{Z}_B + \dot{Z}_C} \tag{10-8}$$

Since they are to be equal to each other,

$$\dot{Z}_2+\dot{Z}_3=\frac{\dot{Z}_C(\dot{Z}_A+\dot{Z}_B)}{\dot{Z}_A+\dot{Z}_B+\dot{Z}_C} \tag{10–9}$$

Similarly, the impedances across the terminals 3 and 1 are

$$(\dot{Z}_{31})_Y=\dot{Z}_1+\dot{Z}_3 \tag{10–10}$$

$$(\dot{Z}_{31})_\Delta=\frac{\dot{Z}_A(\dot{Z}_B+\dot{Z}_C)}{\dot{Z}_A+\dot{Z}_B+\dot{Z}_C} \tag{10–11}$$

Hence,

$$\dot{Z}_1+\dot{Z}_3=\frac{\dot{Z}_A(\dot{Z}_B+\dot{Z}_C)}{\dot{Z}_A+\dot{Z}_B+\dot{Z}_C} \tag{10–12}$$

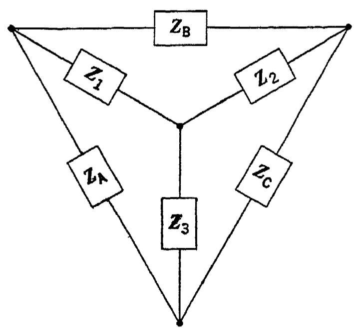


FIG. 10–3

If we subtract equation (10–9) from equation (10–6) and add equation (10-12), we obtain:

$$\dot{Z}_1=\frac{\dot{Z}_A\dot{Z}_B}{\dot{Z}_A+\dot{Z}_B+\dot{Z}_C} \tag{10–13}$$

Similarly, subtracting equation (10–12) from (10–6) and adding equation (10–9) gives:

$$\dot{Z}_2=\frac{\dot{Z}_B\dot{Z}_C}{\dot{Z}_A+\dot{Z}_B+\dot{Z}_C} \tag{10–14}$$

Also, by subtracting equation (10–6) from (10–9) and adding equation (10–12), we get:

$$\dot{Z}_3=\frac{\dot{Z}_A\dot{Z}_C}{\dot{Z}_A+\dot{Z}_B+\dot{Z}_C} \tag{10–15}$$

Equations (10–13), (10–14), and (10–15) express the vector impedance values of the equivalent wye in terms of the vector impedances of the delta. The calculated elements may not be physically realizable, such as negative resistances, but the relations hold. This equivalence, as far as $\dot{I}_1$ and $\dot{I}_2$ are concerned, is general, irrespective of the characteristics of the generator or the load in Figs. 10–1 and 10–2.

When equations (10–13), (10–14), and (10–15) are applied, it must be understood that they hold only for the notation and arrangement of the impedances given in Figs. 10–1 and 10–2. The following "rule," with the aid of Fig. 10–3, might be helpful in determining the proper relations in the circuit for the new Z's when transforming from a π-section to a T-section. Let $\dot{Z}_A$ and $\dot{Z}_B$ be called the "adjacents" of $\dot{Z}_1$; $\dot{Z}_B$ and $\dot{Z}_C$, the "adjacents" of $\dot{Z}_2$; and $\dot{Z}_C$ and $\dot{Z}_A$, the "adjacents" of $\dot{Z}_3$. Also, let $\dot{Z}_A+\dot{Z}_B+\dot{Z}_C$ be designated as the "sum." With this new terminology, equations (10–13), (10–14), and (10–15) state that each arm of the equivalent T is equal to the product of the "adjacents" divided by the "sum." For example,

$$\dot{Z}_1=\frac{\dot{Z}_A\dot{Z}_B}{\dot{Z}_A+\dot{Z}_B+\dot{Z}_C}=\frac{\text{Product of ``adjacents''}}{\text{``Sum''}} \tag{10–13a}$$

Theorem 1.—At a single frequency a π- (or Δ-) section of a network can be replaced by a T- (or Y-) section, provided the relations in equations (10–13), (10–14), and (10–15) are maintained.

If $\dot{Z}_A=\dot{Z}_B=\dot{Z}_C=\dot{Z}_\Delta$, or the delta circuit is balanced,

$$\dot{Z}_1=\dot{Z}_2=\dot{Z}_3=\frac{\dot{Z}_\Delta}{3}=\dot{Z}_Y \tag{10–16}$$

An application of this theorem is given in Art. 14–24 in the chapter on Polyphase Circuits.

10–2. Conversion of a T-Section Into an Equivalent π-Section. By adding product terms of $\dot{Z}_1$, $\dot{Z}_2$, and $\dot{Z}_3$ in equations (10–13), (10–14), and (10–15), and rearranging terms, we obtain:

$$\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3=\frac{\dot{Z}_A\dot{Z}_B^2\dot{Z}_C+\dot{Z}_A\dot{Z}_B\dot{Z}_C^2+\dot{Z}_A^2\dot{Z}_B\dot{Z}_C}{(\dot{Z}_A+\dot{Z}_B+\dot{Z}_C)^2}$$

or

$$\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3=\frac{\dot{Z}_A\dot{Z}_B\dot{Z}_C}{\dot{Z}_A+\dot{Z}_B+\dot{Z}_C} \tag{10–17}$$

Dividing equation (10–17) by equation (10–14) gives:

$$\dot{Z}_A = \frac{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3}{\dot{Z}_2} \tag{10–18}$$

Similarly, dividing equation (10–17) by (10–15) gives:

$$\dot{Z}_B = \frac{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3}{\dot{Z}_3} \tag{10–19}$$

and dividing equation (10–17) by (10–13) gives:

$$\dot{Z}_C = \frac{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3}{\dot{Z}_1} \tag{10–20}$$

The three equations (10–18), (10–19), and (10–20) give the vector impedance values of the equivalent delta connection (not necessarily physically realizable) in terms of the impedances of the wye connection. They hold only for the notation and arrangement given in Figs. 10–1 and 10–2. The following "rule," with the aid of Fig. 10–3, will be useful in transforming from a T-section to a π-section. Let $\dot{Z}_1$ be designated as the "opposite" of $\dot{Z}_C$; $\dot{Z}_2$, the "opposite" of $\dot{Z}_A$; and $\dot{Z}_3$, the "opposite" of $\dot{Z}_B$. Also, let $\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3$ be called the "sum of the products." With this notation, equations (10–18), (10–19), and (10–20) state that each arm of the equivalent π-section is equal to the "sum of the products" divided by the "opposite." For example,

$$\dot{Z}_A = \frac{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3}{\dot{Z}_2} = \frac{\text{“Sum of products”}}{\text{“Opposite”}} \tag{10–18a}$$

Theorem 2.—At a single frequency a T- (or Y-) section of a network can be replaced by a π- (or Δ-) section, provided the relations in equations (10–18), (10–19), and (10–20) are maintained.

10–3. Elements of Networks.—The circuit elements composing a network may be *bilateral* or *unilateral.* Resistance, inductance, and capacitance are bilateral because they transmit energy equally in either direction in a circuit. Vacuum tubes are unilateral because they transmit energy effectively in only one direction.

A network may be *linear* or *non-linear.* A network is linear when it contains elements that do not change with the magnitude of the applied voltage or with current strength. For example, when the applied voltage in a linear network is doubled, the current is also doubled. Most resistances, air-core inductances, and capacitances are linear. A non-linear circuit contains elements which vary in value with the impressed voltage or with the current strength. Examples of such circuits are those containing iron-core coils or vacuum tubes.

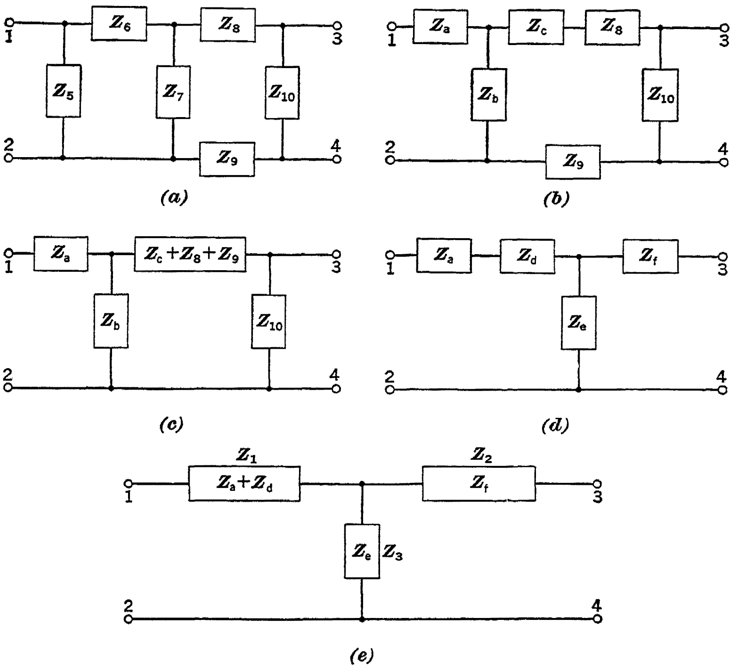


FIG. 10-4

From the definitions just given, resistances, inductances, and capacitances may be called linear bilateral elements. Impedances containing resistances, air-core inductances, and (or) capacitances may be called linear bilateral impedances.

A network containing no internal sources of power is said to be a *passive* network.

10-4. Equivalence of Complex Networks to Simple T or π Circuits.—At a single frequency, any complex network consisting of linear bilateral impedances and having two input terminals 1 and 2 and two output terminals 3 and 4 can be replaced by a simple T-section or a simple π-section, as far as the currents at the input and output terminals are concerned. This follows from theorems 1 and 2 and the experimental method of replacing a network by an equivalent T-section or π-section (Art. 10–5), because, by successive conversions from a π-section to a T-section and from a T-section to a π-section, any complex network can be reduced to a single equivalent section. Thus, the circuit with two input terminals and two output terminals arranged as shown in Fig. 10–4(*a*) can be reduced to the simple T-section in Fig. 10–4(*e*).

The π-section in Fig. 10–4(*a*) consisting of ($\dot{Z}_5$, $\dot{Z}_6$, $\dot{Z}_7$) is converted into the T-section ($\dot{Z}_a$, $\dot{Z}_b$, $\dot{Z}_c$) in Fig. 10–4(*b*). Since the current flowing in $\dot{Z}_9$ is the same as that flowing in $\dot{Z}_c$ and $\dot{Z}_8$, the impedance $\dot{Z}_9$ may be moved to the upper wire and combined with $\dot{Z}_c$ and $\dot{Z}_8$ into a single impedance, as in Fig. 10–4(*c*). Then the π-section ($\dot{Z}_b$, $\dot{Z}_c+\dot{Z}_8+\dot{Z}_9$, $\dot{Z}_{10}$) is changed to the T-section ($\dot{Z}_d$, $\dot{Z}_e$, $\dot{Z}_f$) in Fig. 10–4(*d*). Finally, $\dot{Z}_a$ and $\dot{Z}_d$ are combined into one impedance, giving a T-section with $\dot{Z}_1$, $\dot{Z}_2$, and $\dot{Z}_3$, as shown in Fig. 10–4(*e*). By Theorem 2, the T-section can be converted to a π-section if desired.

10–5. Experimental Determination of Impedances of Equivalent T-Section or π-Section.—Since a network, such as that shown in Fig. 10–4(*a*), can be represented by a simple T-section, as given in Fig. 10–4(*e*), it would be of interest to know how the three impedances $\dot{Z}_1$, $\dot{Z}_2$, and $\dot{Z}_3$ can be determined experimentally. With three unknown impedances, at least three independent measurements must be made on the original network to determine their values. The most convenient ones would be the following:

(1) With the output terminals 3 and 4 open-circuited, measure the impedance at the input terminals 1 and 2. Let this impedance be designated by $\dot{Z}_{O1}$.

(2) With the output terminals 3 and 4 short-circuited, measure the impedance at the input terminals 1 and 2. Let this be $\dot{Z}_{S1}$.

(3) With the input terminals 1 and 2 open-circuited, measure the impedance at the output terminals, 3 and 4. Call this $\dot{Z}_{O2}$.

(4) With the input terminals 1 and 2 short-circuited, measure the impedance at the output terminals 3 and 4. Designate it by $\dot{Z}_{S2}$.

It will be shown that only three of the four readings are independent. If the four readings were made on the T-section in Fig. 10–4(*e*), and in the order listed, we would have:

$$\dot{Z}_{O1}=\dot{Z}_1+\dot{Z}_3 \tag{10–21}$$

$$\dot{Z}_{S1}=\dot{Z}_1+\frac{\dot{Z}_2\dot{Z}_3}{\dot{Z}_2+\dot{Z}_3}=\frac{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_3\dot{Z}_1}{\dot{Z}_2+\dot{Z}_3} \tag{10–22}$$

$$\dot{Z}_{O2}=\dot{Z}_2+\dot{Z}_3 \tag{10–23}$$

$$\dot{Z}_{S2}=\dot{Z}_2+\frac{\dot{Z}_1\dot{Z}_3}{\dot{Z}_1+\dot{Z}_3}=\frac{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_3\dot{Z}_1}{\dot{Z}_1+\dot{Z}_3} \tag{10–24}$$

Multiplying equation (10–21) by equation (10–24) and equation (10–22) by equation (10–23), we obtain

$$\dot{Z}_{O1}\dot{Z}_{S2}=\dot{Z}_{O2}\dot{Z}_{S1}=\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_3\dot{Z}_1 \tag{10–25}$$

This result shows that, if any three of the four readings were made, the fourth could be calculated.

It can be seen from equations (10–21) to (10–24) that the impedances $\dot{Z}_1$, $\dot{Z}_2$, and $\dot{Z}_3$ can be expressed in terms of the four readings. From equations (10–21) and (10–23),

$$\dot{Z}_1=\dot{Z}_{O1}-\dot{Z}_3 \tag{10–26}$$

$$\dot{Z}_2=\dot{Z}_{O2}-\dot{Z}_3 \tag{10–27}$$

Substituting these values in equation (10–22) gives:

$$\dot{Z}_{S1}=\frac{\dot{Z}_{O1}\dot{Z}_{O2}-\dot{Z}_3^2}{\dot{Z}_{O2}} \tag{10–28}$$

from which

$$\dot{Z}_3=\sqrt{\dot{Z}_{O2}(\dot{Z}_{O1}-\dot{Z}_{S1})} \tag{10–29}$$

Substituting this value for $\dot{Z}_3$ in equations (10–26) and (10–27), we obtain:

$$\dot{Z}_1=\dot{Z}_{O1}-\sqrt{\dot{Z}_{O2}(\dot{Z}_{O1}-\dot{Z}_{S1})} \tag{10–30}$$

$$\dot{Z}_2=\dot{Z}_{O2}-\sqrt{\dot{Z}_{O2}(\dot{Z}_{O1}-\dot{Z}_{S1})} \tag{10–31}$$

Thus, equations (10–29) to (10–31) will give the values of the three impedances of a T-section in terms of the impedances measured from the original network. Since the impedances of a π-section can be obtained from the known impedances of a T-section, by application of equations (10–18) to (10–20), it is obvious that $\dot{Z}_A$, $\dot{Z}_B$, and $\dot{Z}_C$ of a π-section can be expressed in terms of the measured values $\dot{Z}_{O1}$, $\dot{Z}_{O2}$, and $\dot{Z}_{S1}$, as follows:

$$\dot{Z}_A = \frac{\dot{Z}_{S1}\dot{Z}_{O2}}{\dot{Z}_{O2} - \sqrt{\dot{Z}_{O2}(\dot{Z}_{O1} - \dot{Z}_{S1})}} \tag{10–32}$$

$$\dot{Z}_B = \frac{\dot{Z}_{S1}\dot{Z}_{O2}}{\sqrt{\dot{Z}_{O2}(\dot{Z}_{O1} - \dot{Z}_{S1})}} \tag{10–33}$$

$$\dot{Z}_C = \frac{\dot{Z}_{S1}\dot{Z}_{O2}}{\dot{Z}_{O1} - \sqrt{\dot{Z}_{O2}(\dot{Z}_{O1} - \dot{Z}_{S1})}} \tag{10–34}$$

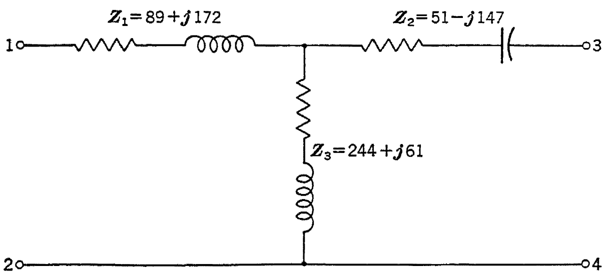


FIG. 10–5

Example 10–1.—A network has two input terminals and two output terminals, such as shown in Fig. 10–4(*a*). When three measurements were made as outlined in the foregoing explanations, the impedances were:

$$\dot{Z}_{O1} = 333 + j233$$
$$\dot{Z}_{S1} = 186 + j88$$
$$\dot{Z}_{O2} = 295 - j86$$

Determine the equivalent T-section of the network.

Solution.—From equation (10–29),

$$\dot{Z}_3 = \sqrt{(295 - j86)[(333 + j233) - (186 + j88)]}$$
$$= 252\underline{/14.2°} = 244 + j61$$

From equations (10–26) and (10–27),

$$\dot{Z}_1 = (333 + j233) - (244 + j61) = 89 + j172$$
$$\dot{Z}_2 = (295 - j86) - (244 + j61) = 51 - j147$$

The equivalent T-section of the original network is shown in Fig. 10–5.

10–6. Thévenin's Theorem.—*At a single frequency, the current flowing in any impedance $\dot{Z}_R$, connected to the terminals of a linear* *network containing bilateral impedances and generators, is equal to the current flowing in the same* $\dot{Z}_R$ *when it is connected to a simple generator whose generated voltage is the open-circuited voltage at the two terminals in question and whose impedance is the impedance of the network looking back from the terminals into the network with all generators replaced by their internal impedances.*

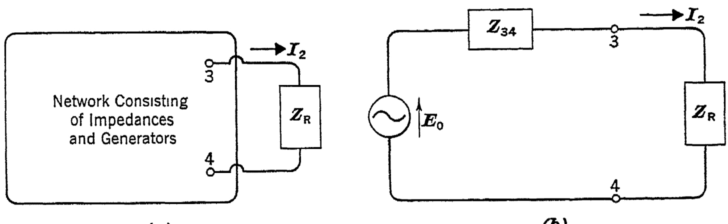


FIG. 10–6

Fig. 10–6(a) shows a network consisting of linear bilateral impedances and generators with a load impedance $\dot{Z}_R$ connected to the terminals 3 and 4. As far as the current $\dot{I}_2$ flowing through $\dot{Z}_R$ is concerned, it follows from Thévenin's Theorem that this circuit can be represented by that given in Fig. 10–6(b), where

$\dot{E}_0$ = voltage measured at terminals 3 and 4 with $\dot{Z}_R$ removed;

$\dot{Z}_{34}$ = the impedance measured at the terminals 3 and 4 with $\dot{Z}_R$ removed and the generators replaced by their internal impedances.

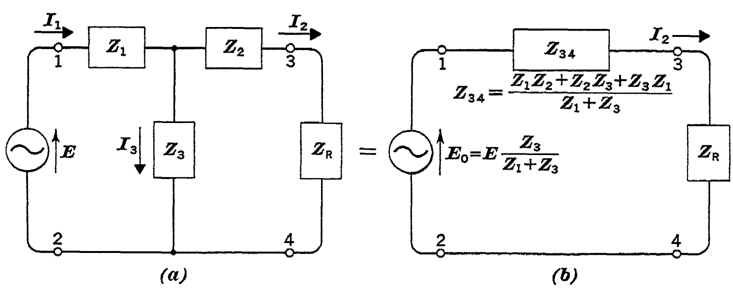


FIG. 10–7

To demonstrate the truth of Thévenin's Theorem, the problem of determining the current through a load impedance $\dot{Z}_R$ connected to a network consisting of a T-section and a generator as shown in Fig. 10–7(a) may be considered. It will be shown that this network can be represented by the circuit given in Fig. 10–7(*b*) as far as the determination of $\dot{I}_2$ is concerned. This demonstration holds for any linear bilateral network, since it can be reduced to a T-section.

In this demonstration it is assumed that $\dot{Z}_1$ *includes the internal impedance of the generator.* The impedance at terminals 1 and 2 looking into the network which includes the internal impedance of the generator, the load impedance $\dot{Z}_R$, and the T-section is

$$\dot{Z}_{12} = \dot{Z}_1 + \frac{\dot{Z}_3(\dot{Z}_2 + \dot{Z}_R)}{\dot{Z}_2 + \dot{Z}_3 + \dot{Z}_R} \tag{10–35}$$

The generator current is

$$\dot{I}_1 = \frac{\dot{E}}{\dot{Z}_{12}} = \frac{\dot{E}}{\dfrac{\dot{Z}_3(\dot{Z}_2 + \dot{Z}_R)}{\dot{Z}_2 + \dot{Z}_3 + \dot{Z}_R} + \dot{Z}_1} \tag{10–36}$$

where $\dot{E}$ is the generated voltage of the generator.

From Fig. 10–7(*a*), it can be seen that the voltage drop across $\dot{Z}_3$ is equal to the drop across $\dot{Z}_2$ and $\dot{Z}_R$. Thus,

$$(\dot{I}_1 - \dot{I}_2)\dot{Z}_3 = \dot{I}_2(\dot{Z}_2 + \dot{Z}_R)$$

or

$$\dot{I}_2 = \frac{\dot{I}_1\dot{Z}_3}{\dot{Z}_2 + \dot{Z}_3 + \dot{Z}_R} \tag{10–37}$$

Substituting the value from equation (10–36) for $\dot{I}_1$ in equation (10–37), we get

$$\dot{I}_2 = \frac{\dot{E}\dot{Z}_3}{\dot{Z}_1\dot{Z}_2 + \dot{Z}_2\dot{Z}_3 + \dot{Z}_3\dot{Z}_1 + \dot{Z}_R(\dot{Z}_1 + \dot{Z}_3)} \tag{10–38}$$

This can be rearranged to have $\dot{Z}_R$ appear alone in the denominator, as follows:

$$\dot{I}_2 = \frac{\dot{E}\,\dfrac{\dot{Z}_3}{\dot{Z}_1 + \dot{Z}_3}}{\dfrac{\dot{Z}_1\dot{Z}_2 + \dot{Z}_2\dot{Z}_3 + \dot{Z}_3\dot{Z}_1}{\dot{Z}_1 + \dot{Z}_3} + \dot{Z}_R} \tag{10–39}$$

or

$$\dot{I}_2 = \frac{\dot{E}_0}{\dot{Z}_{34} + \dot{Z}_R} \tag{10–40}$$

where

$$\dot{E}_0 = \dot{E}\,\frac{\dot{Z}_3}{\dot{Z}_1 + \dot{Z}_3} \tag{10–41}$$

$$\dot{Z}_{34}=\frac{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_3\dot{Z}_1}{\dot{Z}_1+\dot{Z}_3} \tag{10–42}$$

It is obvious from Fig. 10–7(a) that equation (10–41) is the voltage across terminals 3 and 4 with $\dot{Z}_R$ removed because, under this condition,

$$\dot{I}_2=0 \text{ and } \dot{I}_1=\dot{I}_3=\frac{\dot{E}}{\dot{Z}_1+\dot{Z}_3}$$

Also, the voltage across $\dot{Z}_3$ is $\dot{I}_3\dot{Z}_3=\dot{E}_0$. Equation (10–42) is the impedance measured at terminals 3 and 4 with $\dot{Z}_R$ removed and the generator replaced by its internal impedance $\dot{Z}_{34}=\dot{Z}_2+\dfrac{\dot{Z}_1\dot{Z}_3}{\dot{Z}_1+\dot{Z}_3}$.

The current $\dot{I}_2$ flowing in the load impedance $\dot{Z}_R$, Fig. 10–7(a), is the same as the current flowing in $\dot{Z}_R$, Fig. 10–7(b), if the load impedance is connected to the generator whose generated voltage is $\dot{E}_0$ and whose internal impedance is $\dot{Z}_{34}$. Thévenin's Theorem is thereby demonstrated for any linear bilateral network with only one generator. If there are other generators, the demonstration must be carried out by the principle of superposition. See Art. 10–9.

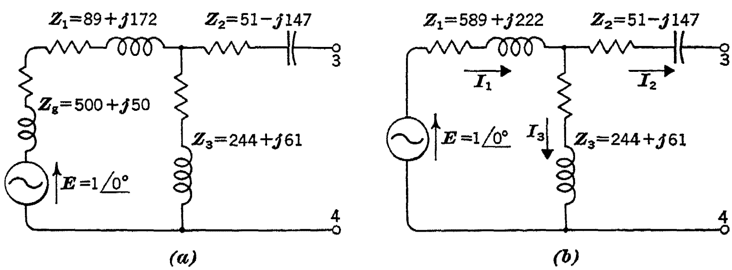


FIG. 10–8

Example 10–2.*—A generator having an internal impedance of $500+j50$ and a generated voltage of 1 volt is connected to the terminals 1 and 2 of the circuit shown in Fig. 10–5. By Thévenin's Theorem, determine the equivalent generator which can replace the actual generator and the network as far as the current flowing in the load connected at terminals 3 and 4 is concerned. The complete circuit is shown in Fig. 10–8(a).

Solution.—In the derivation of the equation for $\dot{E}_0$ and $\dot{Z}_{34}$ it will be remembered that $\dot{Z}_1$ includes the internal impedance of the actual generator. Thus, the new $\dot{Z}_1$, Fig. 10–8(b), is

$$\dot{Z}_1=(89+j172)+(500+j50)=589+j222$$

* See Problem 11–36 for another application of this theorem. Also see Appendix D for an important example.

With no load at terminals 3 and 4, the current $\dot{I}_2$ is zero and $\dot{I}_1 = \dot{I}_3$. The open-circuited voltage $\dot{E}_0$ measured at terminals 3 and 4 is equal to the drop $\dot{I}_3\dot{Z}_3$ or the value given by equation (10–41). Thus,

$$\dot{E}_0 = \dot{E}\frac{\dot{Z}_3}{\dot{Z}_1+\dot{Z}_3} = 1\times\frac{252\underline{/14.2°}}{(589+j222)+(244+j61)} = 0.286\underline{/-4.6°}$$

The impedance $\dot{Z}_{34}$ measured at 3 and 4 in Fig. 10–8(b), with the generator short-circuited, is

$$\dot{Z}_{34} = \dot{Z}_2 + \frac{\dot{Z}_1\dot{Z}_3}{\dot{Z}_1+\dot{Z}_3} = 156\underline{/-70.8°} + \frac{630\underline{/20.7°}\times 252\underline{/14.2°}}{630\underline{/20.7°}+252\underline{/14.2°}}$$

$$= 244\underline{/-23.34°} = 225.75 - j96.8$$

As far as the load current is concerned, the network and generator can be replaced by an equivalent generator whose generated voltage is $\dot{E}_0 = 0.286\underline{/-4.6°}$ and whose internal impedance is equal to $\dot{Z}_{34} = 225.75 - j96.8$.

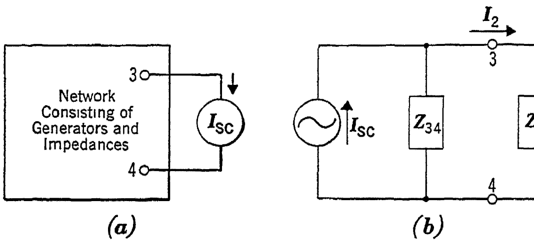


FIG. 10–9

10–7. Norton's Theorem.—In many respects, Norton's Theorem is similar to Thévenin's Theorem. The theorem may be stated as follows: *At a single frequency, the current flowing in any impedance $\dot{Z}_R$, connected to two terminals of a linear network containing bilateral impedances and generators, is equal to the current flowing in the same $\dot{Z}_R$ when it is connected to a constant-current generator whose generated current $\dot{I}_{SC}$ is that current which flows through the two terminals when these terminals are short-circuited, the constant-current generator being in parallel with an impedance $\dot{Z}_{34}$ which is the impedance of the network looking back from the terminals into the network with all generators replaced by their internal impedances.*

The truth of Norton's Theorem may be demonstrated for the circuit of Fig. 10–6(a) with the aid of Thévenin's Theorem. The actual current flowing through $\dot{Z}_R$ in Fig. 10–6(b) is, by equation (10–40),

$$\dot{I}_2 = \frac{\dot{E}_0}{\dot{Z}_{34}+\dot{Z}_R}$$

It will be shown that the equivalent constant-current generator to the left of the terminals 3 and 4, Fig. 10–9(*b*), will give the same current $\dot{I}_2$ through $\dot{Z}_R$.

If the terminals 3 and 4 of the original network are short-circuited, as shown in Fig. 10–9(*a*), then, with the aid of Fig. 10–6(*b*),

$$\dot{I}_{SC}=\frac{\dot{E}_0}{\dot{Z}_{34}} \tag{10–43}$$

In Fig. 10–9(*b*), the current through $\dot{Z}_R$ is

$$\dot{I}_2=\frac{\dot{Z}_{34}\dot{I}_{SC}}{\dot{Z}_{34}+\dot{Z}_R} \tag{10–44}$$

where $\dot{Z}_{34}$ is determined exactly as in the case of Thévenin's Theorem.

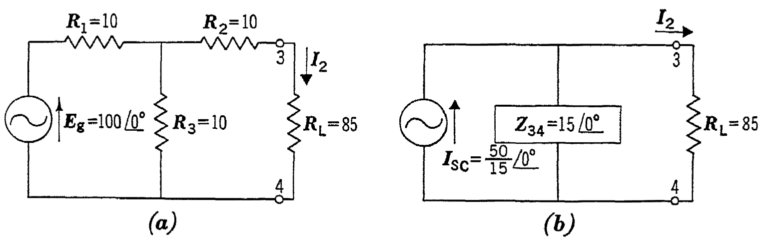


FIG. 10–10

Substituting the value of $\dot{I}_{SC}$ from equation (10–43) in equation (10–44) gives

$$\dot{I}_2=\frac{\dot{E}_0}{\dot{Z}_{34}+\dot{Z}_R} \tag{10–45}$$

which is the same as equation (10–40). Hence, it may be said that, as far as the current $\dot{I}_2$ is concerned, Fig. 10–9(*b*) is equivalent to the original network in Fig. 10–6(*a*), just as Fig. 10–6(*b*) is equivalent to Fig. 10–6(*a*).

In the equivalent circuit of some vacuum tubes, such as the triode, that portion representing the plate circuit may be replaced either by a "constant-voltage generator" equivalent circuit, to which Thévenin's Theorem is applied, or by a "constant-current generator" equivalent circuit, to which Norton's Theorem is applied. The choice depends on the purpose and the result desired.

Example 10–3.—Find the load current $\dot{I}_2$ in the circuit shown in Fig. 10–10(*a*) by using Norton's Theorem.

Solution.—To find $\dot{I}_{SC}$ it is convenient to obtain first the generator current $\dot{I}'_g$ with the terminals 3 and 4 in Fig. 10–10(*a*) short-circuited. Thus,

$$\dot{I}'_g = \frac{\dot{E}_g}{R_1 + \dfrac{R_2 R_3}{R_2 + R_3}} = \frac{100\underline{/0^\circ}}{15\underline{/0^\circ}} = \frac{100}{15}\underline{/0^\circ}$$

The corresponding short-circuited current flowing through terminals 3 and 4 is

$$\dot{I}_{SC} = \frac{R_3}{R_2 + R_3}\dot{I}'_g = \frac{10}{10+10} \times \frac{100}{15}\underline{/0^\circ} = \frac{50}{15}\underline{/0^\circ}$$

The impedance $\dot{Z}_{34}$ is determined at the terminals 3 and 4 with the generator replaced by its internal impedance (which is zero in this case). Hence,

$$\dot{Z}_{34} = R_2 + \frac{R_1 R_3}{R_1 + R_3} = 15\underline{/0^\circ}$$

The constant-current generator replacing the network to the left of terminals 3 and 4 is given in Fig. 10–10(*b*), and the actual load current is

$$\dot{I}_2 = \frac{\dot{Z}_{34}}{\dot{Z}_{34} + R_L}\dot{I}_{SC} = \frac{15}{15+85} \times \frac{50}{15}\underline{/0^\circ} = 0.5\underline{/0^\circ}$$

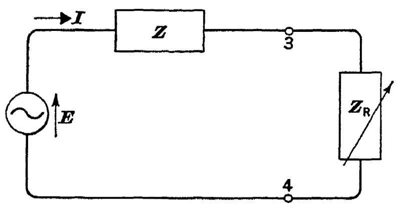


FIG. 10–11

10–8. Maximum Power Transfer.—The voltage of a power system is maintained substantially constant between wide limits of load variation. This is seldom true in communication circuits. The power transferred from the generating station to the load in a power system is usually large; and, hence, high efficiency is desirable for such a system. On the other hand, the amount of power available in communication networks is low; and the maximum power transfer from one circuit to another, rather than the efficiency of the system, is a criterion of good design. A few analyses will serve to indicate the relation that must exist between two circuits to make the power transferred from one to the other a maximum.

Since a network consisting of impedances and a generator can be represented, from Thévenin's Theorem, by the circuit shown in Fig. 10–7(*b*), the problem of transferring power from any network to a *variable* load impedance $\dot{Z}_R$ may be reduced to the simpler problem of power transfer in the circuit of Fig. 10–11. The impedance $\dot{Z}$ includes the impedances of the network and the internal impedance of the generator.

Three cases will be considered with $\dot{Z}$ as the fixed impedance and $\dot{Z}_R$ as the *variable*:

(1) Both $\dot{Z}$ and $\dot{Z}_R$ are pure resistances;

(2) Both $\dot{Z}$ and $\dot{Z}_R$ contain resistances and reactances;

(3) Both $\dot{Z}$ and $\dot{Z}_R$ contain resistances and reactances, but the power factor angle of $\dot{Z}_R$ is fixed.

Case 1: When $\dot{Z}=R$, and $\dot{Z}_R=R_R$, the current is

$$\dot{I}=\frac{\dot{E}}{R+R_R} \tag{10–46}$$

and the average power supplied to $\dot{Z}_R$ is

$$P=I^2R_R=\frac{E^2R_R}{(R+R_R)^2} \tag{10–47}$$

To find the condition for maximum power absorbed by $\dot{Z}_R$, differentiate equation (10–47) with respect to R_R and equate the result to zero. Thus, when E can be considered a constant,

$$\frac{\partial P}{\partial R_R}=\frac{E^2\left[(R+R_R)^2-2R_R(R+R_R)\right]}{(R+R_R)^4}=0$$

from which $$(R+R_R)^2-2R_R(R+R_R)=0$$

or $$R_R=R \text{ for maximum power transfer} \tag{10–48}$$

That is, if a load resistance, as $\dot{Z}_R=R_R$ of Fig. 10–11, is varied, maximum power is absorbed by the load when the load resistance is equal to the resistance looking back into the network at the junction points 3 and 4.

Example 10–4.—A variable resistive load R_R is connected across the terminals of a battery with constant internal emf E, Fig. 10–12. The fixed internal resistance of the battery is $R=3$ ohms and the voltage across the terminals at no load is 30 volts. Find the value of R_R for maximum power transfer. What is the efficiency of the system with this value of R_R?

Solution.—The value of R_R for maximum power transfer may be determined by assuming different values for R_R. When $R_R=1$, the current is

$$I=\frac{E}{R+R_R}=\frac{30}{3+1}=7.5 \text{ amp}$$

and the power delivered to R_R is

$$I^2R_R=7.5^2\times 1=56.25 \text{ watts}$$

The efficiency of the system is obtained by dividing the power delivered to R_R by the input to the entire system. Thus,

$$\text{Eff.} = \frac{I^2R_R}{EI} = \frac{56\ 25}{30 \times 7.5} = 0.25$$

The accompanying table gives the current, the power delivered to R_R, and the efficiency of the entire system for each of several values of R_R

R_R	$I=\frac{E}{R+R_R}$	I^2R_R	EI	$\text{Eff.}=\frac{I^2R_R}{EI}$
1	7.5	56.25	225	0.25
2	6.0	72.0	180	0.40
3	5.0	75.0	150	0.50
4	4.28	73.28	128.4	0.57
5	3.75	70.8	112.5	0.629

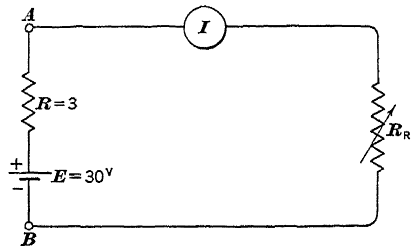


FIG. 10–12

It is obvious from the table that the condition for maximum power transfer is attained when $R_R = R = 3$. The efficiency of the system with this value of R_R is only 50 per cent. Furthermore, the voltage across the terminals A and B, Fig. 10–12, under this condition is only 50 per cent of the value at no load. Such variation in voltage would be objectionable in a lighting load where the light output depends on the voltage at the terminals of the lighting units.

Case 2: If both $\dot{Z}$ and $\dot{Z}_R$ contain resistances and reactances, then equation (10–47) becomes

$$P = \frac{E^2R_R}{(R+R_R)^2+(X+X_R)^2} \qquad (10\text{–}49)$$

If R_R is assumed to be constant, P is maximum when $X_R = -X$, for the denominator of the fraction is then a minimum while the numerator remains constant. When $X_R = -X$, equation (10–49) reduces to equation (10–47) which, when R_R is allowed to vary, is maximum when $R = R_R$. *Hence, the power in equation* (10–49) *is maximum when $\dot{Z}_R$ is the conjugate of $\dot{Z}$.* That is, when $\dot{Z}$ is capacitive, then $\dot{Z}_R$ must be inductive; and vice versa. Also, the resistance components of $\dot{Z}$ and $\dot{Z}_R$ must be equal. For maximum power transfer,

$$\left.\begin{aligned}\text{when } \dot{Z} &= R+jX\\ \text{then } \dot{Z}_R &= R-jX\end{aligned}\right\} \tag{10–50}$$

or

$$\left.\begin{aligned}\text{when } \dot{Z} &= R-jX\\ \text{then } \dot{Z}_R &= R+jX\end{aligned}\right\} \tag{10–51}$$

Case 3: When $\dot{Z}_R$ is varied while its power factor angle θ_R remains the same, the power equation (10–49) may be written as follows:

$$P=\frac{E^2Z_R\cos\theta_R}{(R+Z_R\cos\theta_R)^2+(X+Z_R\sin\theta_R)^2} \tag{10–52}$$

To determine the condition for maximum power transfer, differentiate equation (10–52) with respect to Z_R and equate the result to zero. Thus,

$$\frac{\partial P}{\partial Z_R}=\frac{[(R+Z_R\cos\theta_R)^2+(X+Z_R\sin\theta_R)^2]E^2\cos\theta_R}{[(R+Z_R\cos\theta_R)^2+(X+Z_R\sin\theta_R)^2]^2}-$$

$$\frac{E^2Z_R\cos\theta_R[2(R+Z_R\cos\theta_R)\cos\theta_R+2(X+Z_R\sin\theta_R)\sin\theta_R]}{[(R+Z_R\cos\theta_R)^2+(X+Z_R\sin\theta_R)^2]^2}=0$$

from which

$$(R+R_R)^2+(X+X_R)^2=2(RR_R+R_R^2+XX_R+X_R^2)$$

$$R_R^2+X_R^2=R^2+X^2$$

or

$$Z_R=Z \text{ for maximum power transfer} \tag{10–53}$$

Hence, when only the magnitude of $\dot{Z}_R$ can be varied, the absolute value of $\dot{Z}_R$ should be set equal to the absolute value of $\dot{Z}$ for maximum power transfer.

10–9. Superposition Theorem.—*In any network containing generators and linear bilateral impedances, the current flowing at any point due to the simultaneous action of a number of sources of emf distributed throughout the circuit is equal to the sum of the currents at this point which would exist if each source were considered separately, all other generators being replaced at the time by their internal impedances.*

In linear networks, electric currents are superposable. That is, a current flowing at a point may be considered as the algebraic sum of two or more currents flowing in the same direction or currents flowing in opposite directions. The principle of superposition, strictly speaking, does not hold true for non-linear circuits.

The principle of superposition is one of the most important theorems in network problems. Its use permits the solution of circuits without setting up a number of simultaneous equations involving complex numbers that depend on the physical nature of the circuit.

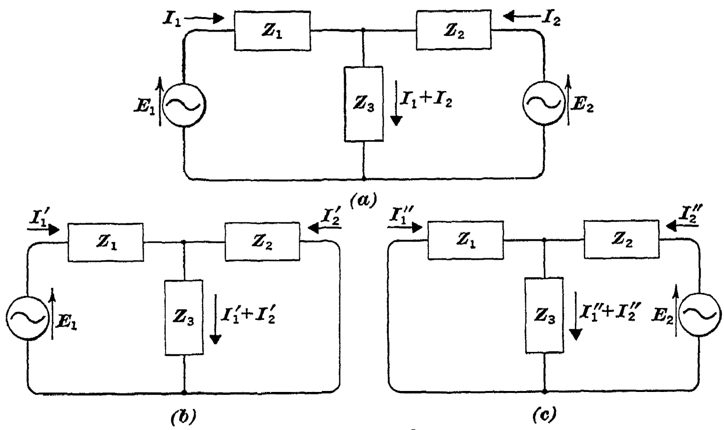


FIG. 10–13

When two voltages of different frequencies are applied to a *linear bilateral* network*, the principle of superposition permits a solution to be obtained for each frequency separately as if the other were absent. The actual result is found by adding the results for the two frequencies. It is obvious that this can be extended to the case of n voltages of n different frequencies.

An illustration of this theorem will now be given. Fig. 10–13(a) shows a network with two generators of the same frequency. The internal impedance of the generator producing emf $\dot{E}_1$ is assumed to be included in $\dot{Z}_1$, and that of the generator with emf $\dot{E}_2$ is included in $\dot{Z}_2$. The positive senses of current flow are indicated in Fig. 10–13(a).

With the generator producing $\dot{E}_2$ shorted out, currents will flow in the different parts of the network due to the generator

* See examples in Chapter 18 on Circuits With Non-Sinusoidal Voltages and Currents.

producing $\dot{E}_1$, as in Fig. 10–13(b). By application of Kirchhoff's Emf Law, the currents are:

$$\dot{I}_1' = \frac{\dot{E}_1}{\dot{Z}_1 + \dfrac{\dot{Z}_2\dot{Z}_3}{\dot{Z}_2+\dot{Z}_3}} = \frac{\dot{E}_1(\dot{Z}_2+\dot{Z}_3)}{\dot{Z}_1\dot{Z}_2+\dot{Z}_1\dot{Z}_3+\dot{Z}_2\dot{Z}_3} \tag{10–54}$$

$$\dot{I}_2' = \frac{-\dot{Z}_3}{\dot{Z}_2+\dot{Z}_3}\,\dot{I}_1' = \frac{-\dot{E}_1\dot{Z}_3}{\dot{Z}_1\dot{Z}_2+\dot{Z}_1\dot{Z}_3+\dot{Z}_2\dot{Z}_3} \tag{10–55}$$

and

$$\dot{I}_1' + \dot{I}_2' = \frac{\dot{E}_1\dot{Z}_2}{\dot{Z}_1\dot{Z}_2+\dot{Z}_1\dot{Z}_3+\dot{Z}_2\dot{Z}_3} \tag{10–56}$$

Similarly, with the generator producing $\dot{E}_1$ shorted out in Fig. 10–13(a), the currents flowing in the circuit shown in Fig. 10–13(c) are due to the generator producing $\dot{E}_2$. These currents are:

$$\dot{I}_2'' = \frac{\dot{E}_2}{\dot{Z}_2 + \dfrac{\dot{Z}_1\dot{Z}_3}{\dot{Z}_1+\dot{Z}_3}} = \frac{\dot{E}_2(\dot{Z}_1+\dot{Z}_3)}{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3} \tag{10–57}$$

$$\dot{I}_1'' = \frac{-\dot{Z}_3}{\dot{Z}_1+\dot{Z}_3}\,\dot{I}_2'' = \frac{-\dot{E}_2\dot{Z}_3}{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3} \tag{10–58}$$

and

$$\dot{I}_1'' + \dot{I}_2'' = \frac{\dot{E}_2\dot{Z}_1}{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3} \tag{10–59}$$

The actual currents flowing in the different parts of the circuit shown in Fig. 10–13(a) are:

$$\dot{I}_1 = \dot{I}_1' + \dot{I}_1'' = \frac{\dot{E}_1(\dot{Z}_2+\dot{Z}_3) - \dot{E}_2\dot{Z}_3}{\dot{Z}_1\dot{Z}_2+\dot{Z}_1\dot{Z}_3+\dot{Z}_2\dot{Z}_3} \tag{10–60}$$

$$\dot{I}_2 = \dot{I}_2' + \dot{I}_2'' = \frac{-\dot{E}_1\dot{Z}_3 + \dot{E}_2(\dot{Z}_1+\dot{Z}_3)}{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3} \tag{10–61}$$

$$\dot{I}_1 + \dot{I}_2 = \frac{\dot{E}_1\dot{Z}_2 + \dot{E}_2\dot{Z}_1}{\dot{Z}_1\dot{Z}_2+\dot{Z}_2\dot{Z}_3+\dot{Z}_1\dot{Z}_3} \tag{10–62}$$

Example 10–5.—Consider a circuit containing a pure resistance $R = 2$ ohms and two batteries having negligible internal resistance, as shown in Fig. 10–14(a). If the voltages of the batteries are $E_1 = 100$ volts and $E_2 = 50$ volts, find the current flowing in the resistance.

Solution.—With the battery producing E_2 shorted out, the current flowing in Fig 10–14(*b*) is

$$I_1 = \frac{E_1}{R} = \frac{100}{2} = 50 \text{ amp}$$

Since the positive sense of current flow is indicated in Fig. 10–14(*a*), the current I_1 is positive. With the battery producing E_1 shorted out in Fig 10–14(*a*), the current due to E_2 alone, as in Fig. 10–14(*c*), is

$$I_2 = \frac{-E_2}{R} = \frac{-50}{2} = -25 \text{ amp}$$

This current is negative because it is actually flowing from right to left.

The actual current flowing in the resistance is

$$I = I_1 + I_2 = 50 + (-25) = 25 \text{ amp}$$

Attention is called to the fact that the power dissipated in the resistor is

$$I^2R = 25^2 \times 2 = 1250 \text{ watts}$$

It is not

$$I_1^2R + I_2^2R = 50^2 \times 2 + (-25)^2 \times 2 = 6250 \text{ watts}$$

That is, the principle of superposition does not hold for power, because power is a quadratic function instead of a linear function of current.

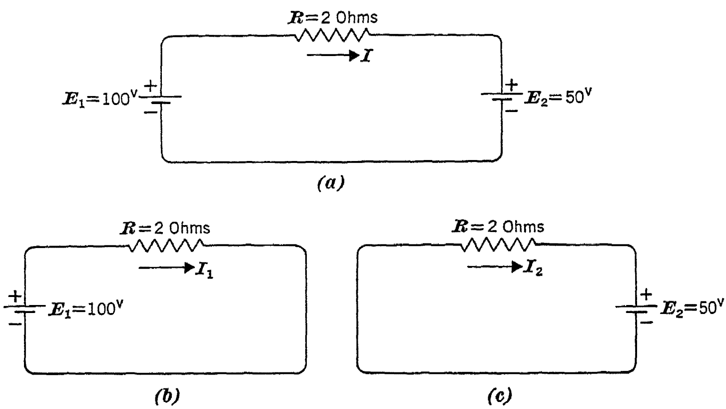


FIG. 10–14

10–10. Reciprocity Theorem.—The Reciprocity Theorem, sometimes known as the Reciprocal Theorem, may be stated in the following form: *In any linear bilateral passive network, the current $\dot{I}$ flowing in any branch as the result of an emf $\dot{E}$ developed by a source in a second branch is equal to the current that would be produced in the second branch if the source of emf $\dot{E}$ were transferred to the first branch, provided that either equal impedances or no impedances are interchanged in this transfer of the source of emf.*

This theorem may be proved by showing that it holds for a T-section, Fig. 10–15, since at a single frequency any linear bilateral network having two input terminals and two output terminals may be reduced to a simple T-section, as far as $\dot{I}_1$ and $\dot{I}_2$ are concerned. It can be seen from Fig. 10–15, which represents any linear bilateral network containing a single generator and an ammeter reading the magnitude of $\dot{I}$, that

$$\dot{I}_1 = \frac{\dot{E}}{\dot{Z}_1 + \dfrac{\dot{Z}_2\dot{Z}_3}{\dot{Z}_2 + \dot{Z}_3}}$$

and

$$\dot{I}_2 = \frac{\dot{Z}_3}{\dot{Z}_2 + \dot{Z}_3}\, \dot{I}_1 = \dot{I}$$

Combining and rearranging terms, we get:

$$\dot{I} = \frac{\dot{E}\dot{Z}_3}{\dot{Z}_1\dot{Z}_2 + \dot{Z}_2\dot{Z}_3 + \dot{Z}_1\dot{Z}_3} \qquad (10\text{–}63)$$

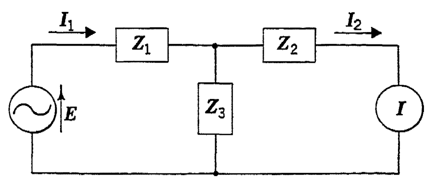


Fig. 10–15

If the positions of E and I in Fig. 10–15 are interchanged, there would only be a corresponding change in the positions of $\dot{Z}_1$ and $\dot{Z}_2$. It is obvious that the magnitude of $\dot{I}$ in equation (10–63) is not altered by interchanging $\dot{Z}_1$ and $\dot{Z}_2$, and the theorem is proved.

This theorem shows very clearly that any linear bilateral network transmits power with equal effectiveness in both directions, if the generator and the load being interchanged have the same impedance.

10–11. Compensation Theorem.—What is known as the Compensation Theorem may be stated as follows: *Any impedance drop in a network consisting of elements of any type may be simulated by a generator of zero internal impedance whose generated voltage at every instant is equal to the impedance drop being replaced.*

The truth of this theorem can be shown very easily with the aid of Kirchhoff's Laws. Fig. 10–16(a) shows a general network with a current I flowing in a branch 1–2 of impedance Z. The currents and the voltages in the different branches are completely deter-

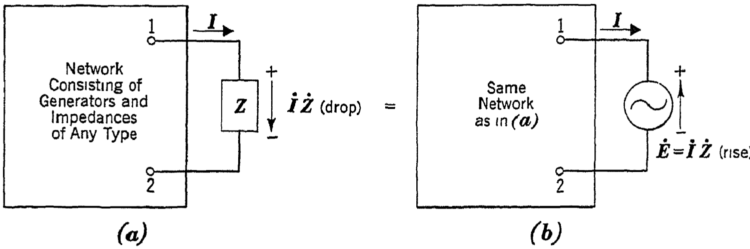


FIG. 10–16

mined by applying (a) Kirchhoff's Current Law at the different junction points and (b) Kirchhoff's Emf Law around the different meshes. These Kirchhoff's equations will be unaltered if the original circuit in Fig. 10–16(a) is replaced by that shown in Fig.

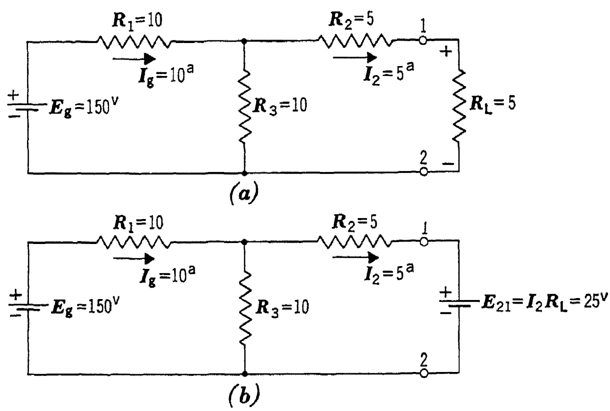


FIG. 10–17

10–16(b), where the generator voltage rise E must be inserted with its plus polarity at the plus polarity of the replaced voltage drop for every instant. That is, if the IZ drop has a plus polarity at 1 and a minus polarity at 2 (in the positive direction of I), then the plus polarity of E must be at 1. For example, if the I_2R_L drop in Fig. 10–17(a) is replaced by a battery of voltage rise

$E_{21} = I_2 R_L$ with the polarities shown in Fig. 10–17(*b*), the currents I_g and I_2 will be the same for Fig. 10–17(*a*) and Fig. 10–17(*b*).

This theorem holds for circuits with unilateral and non-linear elements, as well as for those with linear and bilateral elements. In the studies of carbon-grain telephone transmitters and reflection on long telephone lines, the Compensation Theorem is very useful. It must be kept in mind that the value of the generator voltage rise E must be changed when the current I is affected by a change in any part of the network and, also, that the positive polarity of E must be inserted at the positive polarity of the replaced voltage drop.

10–12. A Useful Network Theorem.—See Appendix E.

PROBLEMS

10–1. In a T-section like that shown in Fig. 10–1, there are three non-inductive resistances: $Z_1 = R_1 = 4$, $Z_2 = R_2 = 10$, and $Z_3 = R_3 = 5$. Find its equivalent π-section.

10–2. Repeat Problem 10–1 with the following impedances: $\dot{Z}_1 = 3 + j4$, $\dot{Z}_2 = 8 - j6$, and $\dot{Z}_3 = 10 + j0$.

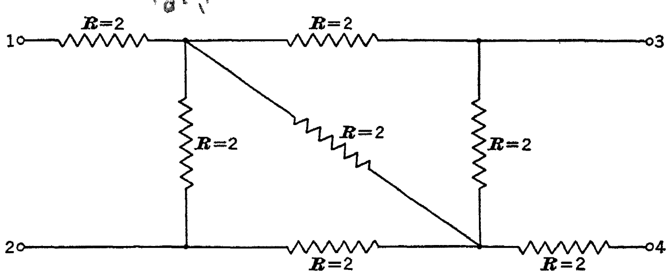


FIG. 10–18

10–3. A π-section like that shown in Fig. 10–2 is formed by the following three impedances: $\dot{Z}_A = 4 - j3$, $\dot{Z}_B = 2 + j7$, and $\dot{Z}_C = 10 + j12$. What is its equivalent T-section?

10–4. Construct a network containing six impedance elements, each having both resistance and reactance; and then reduce it to an equivalent T-section and an equivalent π-section.

10–5. Reduce the circuit of Fig. 10–18 to an equivalent T-section and an equivalent π-section.

10–6. A linear bilateral network having four terminals is to be reduced to an equivalent T-section and an equivalent π-section. Three impedance measurements are made, and the observed values are: $\dot{Z}_{O1} = 380 + j336$, $\dot{Z}_{S1} =$

$258+j162$, and $\dot{Z}_{O2}=330-j200$. Determine the impedance elements of the T-section and of the π-section.

10–7. A generator, having an internal impedance of $450+j75$ and a generated voltage of 100 volts, is connected to two terminals (input terminals) of the network in Problem 10–6. Determine, by Thévenin's Theorem, the equivalent generator which can replace the network and the actual generator as far as the current flowing in the load (which is connected to the remaining two terminals) is concerned.

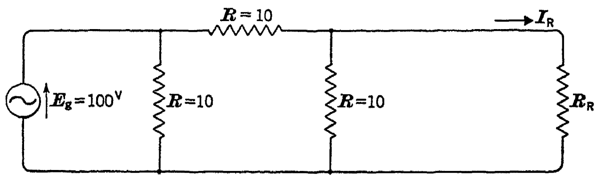


FIG. 10–19

10–8. By Thévenin's Theorem, replace the circuit of Fig. 10–19 by an equivalent generator as far as I_R is concerned.

10–9. Find, by Thévenin's Theorem, the equivalent generator which will replace the actual generator and circuit shown in Fig. 10–20 as far as the load current I_R is concerned.

10–10. By Norton's Theorem, replace the circuit of Fig. 10–19 by an equivalent constant-current generator as far as I_R is concerned.

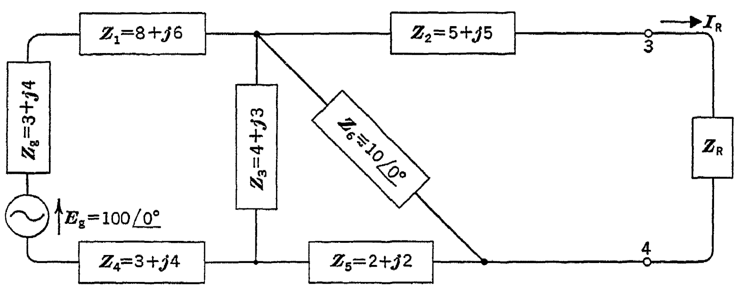


FIG. 10–20

10–11. A voltage source and an equivalent current source are shown in Fig. 10–21; and E_{34} is a voltage drop from 3 to 4. It is desirable to explain that:

(a) $E=I_{SC}R_{34}$ or $I_{SC}=\dfrac{E}{R_{34}}$

(b) $I_2=\dfrac{E-E_{34}}{R_{34}}=I_{SC}-\dfrac{E_{34}}{R_{34}}$ or $E_{34}=(I_{SC}-I_2)R_{34}=E-I_2R_{34}$

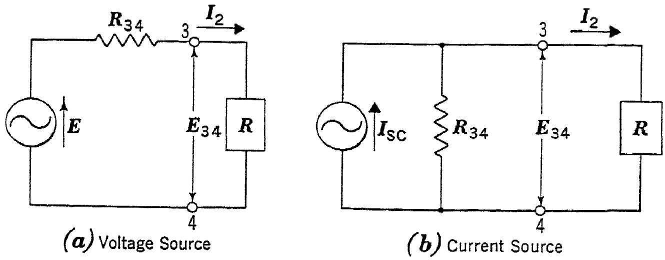


FIG. 10–21

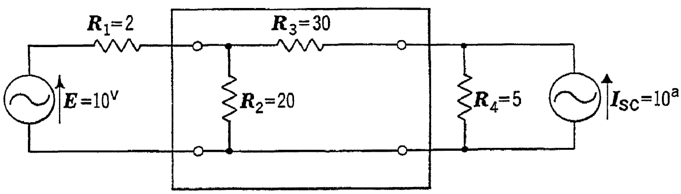


FIG. 10–22

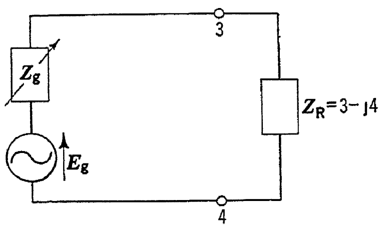


FIG. 10–23

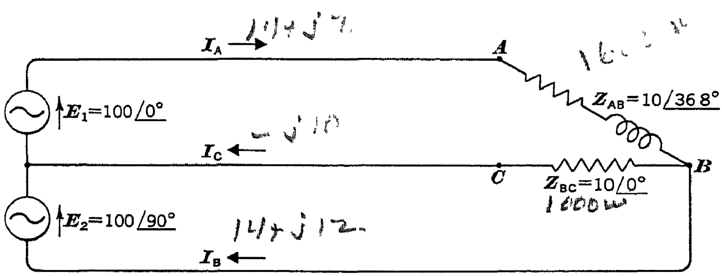


FIG. 10–24

10–12. In Fig. 10–22, replace the voltage generator by its equivalent current source, and also replace the current generator by its equivalent voltage source.

10–13. A variable resistive load is connected to a battery having a constant internal emf of 100 volts and an internal resistance of 2 ohms. Vary the load resistance from 0 to 4 ohms and determine the corresponding currents flowing in the circuit. Calculate the power, $P = I^2R_R$, transferred to the load for the different values of load resistance, and compare the value of load resistance at which maximum power occurs with the value of the internal resistance of the battery.

10–14. A variable resistive load is connected to a generator having a constant generated voltage of $\dot{E} = 100\underline{/0°}$ and an internal impedance of $\dot{Z}_g = 3+j4$. Determine the value of the resistive load for maximum power transfer.

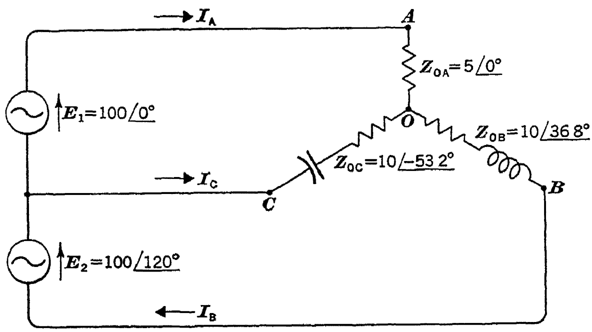


FIG. 10–25

10–15. In Fig. 10–23, the load $\dot{Z}_R = 3 - j4$ is fixed. Determine the generator impedance $\dot{Z}_G$ for maximum power transfer. (Note that $\dot{Z}_g$, and not $\dot{Z}_R$, is varied.)

10–16. Two generators having negligible internal impedance are connected to the T-section of Problem 10–1. Generator No. 1, for which $\dot{E}_1 = 100\underline{/0°}$, is connected to the terminals 1 and 3; and generator No. 2, for which $\dot{E}_2 = 50\underline{/0°}$, is connected to the terminals 2 and 3. The plus polarities are at 1 and 2. Determine, by the Superposition Theorem, the current flowing in the element Z_3.

10–17. Find, by the Superposition Theorem, the vector line currents $\dot{I}_A$, $\dot{I}_B$, and $\dot{I}_C$ in the circuit given in Fig. 10–24. The generators have negligible internal impedance. Also, find the power taken by the entire load.

10–18. Two generators having negligible internal impedance are connected to three single-phase loads, as shown in Fig. 10–25. Find the vector line currents by the use of the Superposition Theorem. What is the power taken by the Y-connected load?

10–19. Fig. 10–26 shows a pure resistance network with a generator of 10 volts and zero internal impedance. The currents are $I_1 = 2$ amp and $I_2 = 0.5$ amp. If an additional generator of 5 volts and zero internal impedance is connected in series with the load in such a direction as to oppose the load current, find the new value of I_1 by the use of the Reciprocity Theorem and the Superposition Theorem.

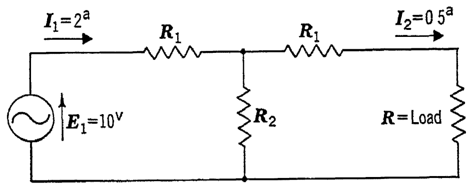


FIG. 10–26

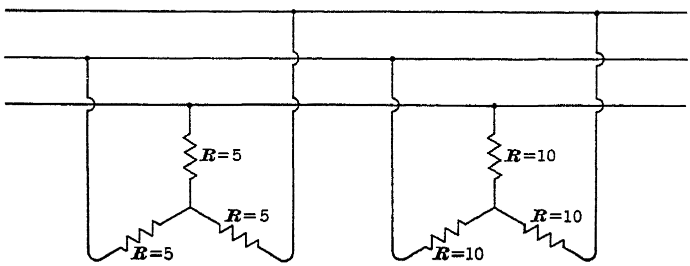


FIG. 10–27

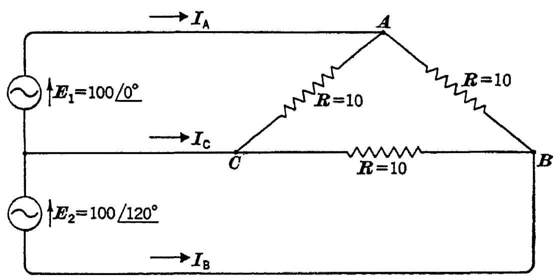


FIG. 10–28

10–20. Two Y-connected loads are connected in parallel, as shown in Fig. 10–27. Reduce the combination to a single Y-connected load which will take the same current and power as the original parallel combination. (Hint: Convert the Y-loads to Δ-loads.)

10–21. Two generators having negligible internal impedance are connected to a Δ-connected load, as shown in Fig. 10–28. Determine the three vector line currents.

10–22. Find the value of $\dot{Z}_R$ in Fig. 10–29 for maximum power transfer to Z_R.

10–23. What are the values of X_C and R_g in Fig. 10–30 for maximum power transfer to the load resistance $R = 6$ ohms?

10–24. The lattice network of pure resistances in Fig. 10–31(*a*) is to be replaced by an equivalent T-section, Fig. 10–31(*b*), as far as the input and output terminals are concerned. Find R_1, R_2, and R_3.

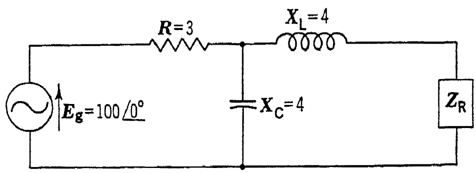


FIG. 10–29

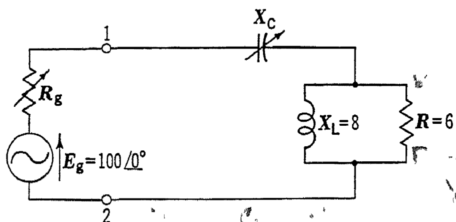


FIG. 10–30

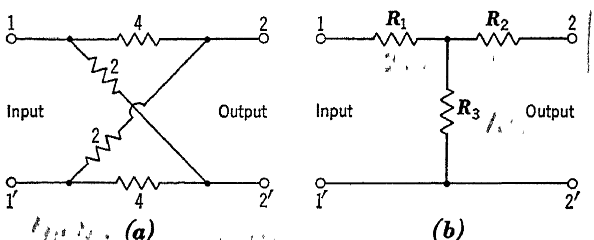


FIG. 10–31

10–25. A box contains a linear bilateral network which is composed of a generator, resistances, and reactances. Only the output terminals are available. On open circuit the terminal voltage is 100 volts; and on short circuit a current of 4 amp flows. When a capacitive reactance $X_C = 15$ ohms is connected across the output terminals, a current of 5 amp flows. Now this reactance of 15 ohms is removed and a load is connected across the output terminals. If both R and X components of the load can be varied, find the maximum power which can be delivered by the network to the load.

CHAPTER 11

MUTUAL INDUCTION AND COUPLED CIRCUITS

Up to the present point, only circuits having self-contained elements R, L, and C have been dealt with. That is, when a current in a circuit is varied, only the reactions that exist or arise in the circuit itself are considered. Thus, in the derivation of the equations given in the previous chapters, it is assumed that the circuit is either isolated or is so related in the neighborhood of other circuits that there can be no interaction between them. However, this is not always the case.

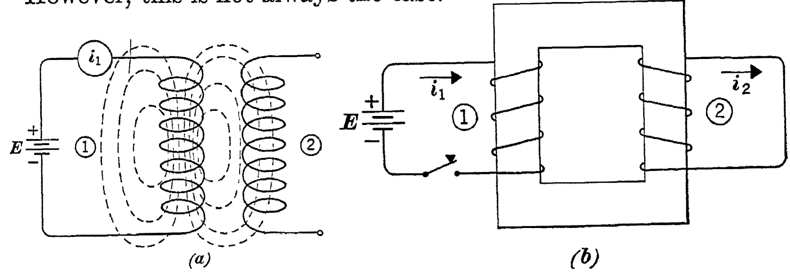


FIG. 11-1

11-1. Mutual Induction.—In general, when two or more circuits are in proximity, a part of the magnetic flux from one circuit may link with a second circuit as shown in Fig. 11-1(a). It can be seen from the figure that *not all* the flux Φ_1 produced by current i_1 in coil 1 can link coil 2. The difference between the flux linking one coil and that linking the other may be made smaller by bringing coil 2 closer to coil 1.

It is obvious that any change in the current i_1 is accompanied by a change in the flux linkages of each coil. As Faraday and Henry learned by actual experiments in 1831, the changes in the flux linkages of the coils induce in each coil an emf which is proportional to the time rate of change of flux linkages for the coil. If the flux is produced by current in the coil itself, the emf is spoken of as the emf of self-induction; but, if the flux is due to another circuit, the emf is spoken of as the emf of mutual induction. Thus, if i_1 is varying, the emf induced in coil 1 which opposes the applied emf is called the emf of self-induction and the emf appearing at the terminals of coil 2 is called the emf of mutual induction.

Next, consider the circuit shown in Fig. 11–1(*b*). When the switch S is closed, a current i_1 rushes into coil 1 in the direction indicated on the diagram. Strangely enough, a momentary current i_2 flows in coil 2. This momentary current is in the direction shown by the arrow, lasting but a very short time and dying out. If the switch S is now opened, another momentary current will flow in coil 2. But this time the direction of flow for i_2 is opposite to that when the switch is closed. These momentary currents set up in coil 2 whenever there is a change in i_1 may be called *induced currents.* The emfs which cause them to flow are the *induced emfs.* Circuits such as those shown in Fig. 11–1 are said to possess the property of *mutual inductance.* Thus, the rate of change of flux linkages of one circuit with respect to the change in current in an adjacent circuit gives us another fundamental circuit element.

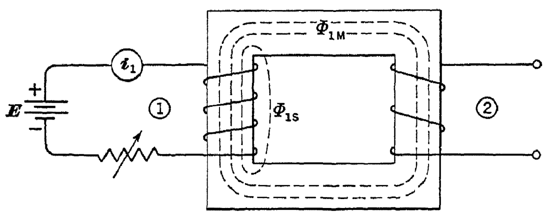


FIG. 11–2

The operation of much alternating current apparatus, such as the transformer and the induction motor, depends on mutual induction between windings. Also, the operation of certain radio and communication networks depends on the interaction between two or more circuits.

11–2. Mutual Flux and Leakage Flux.—If a source of emf is applied to coil 1 of an iron-core transformer, Fig. 11–2, a total flux Φ_1 will be produced by a current i_1 flowing in coil 1. The flux Φ_1 may be considered as consisting of two parts: (*a*) mutual flux and (*b*) leakage flux. That part of Φ_1 which is common to both coils is called mutual flux, and is designated as Φ_{1M}. That part of Φ_1 which links coil 1 and does not pass through coil 2 is called leakage flux or stray flux and is designated as Φ_{1S}. Thus,

$$\Phi_1 = \Phi_{1M} + \Phi_{1S} \tag{11–1}$$

The leakage flux would be zero if it were possible to have the corresponding turns of two coils occupy exactly the same positions.

Similarly, if a source of emf is applied to coil 2, with coil 1 left open, a total flux Φ_2 will be produced by a current i_2 flowing in coil 2. That part of Φ_2 which links coil 1 is indicated by Φ_{2M}; and that part which does not pass through coil 1 is given by Φ_{2S}. Obviously,

$$\Phi_2 = \Phi_{2M} + \Phi_{2S} \tag{11–2}$$

It should be recognized that the actual magnetic field produced by the current flowing in a coil is not so simple as that shown in Fig. 11–2; that is, it is not possible to have Φ_{1M} linking all the turns in coil 1 and all the turns in coil 2, and to have Φ_{1S} linking all the turns in coil 1. In the case of an air-core transformer, Fig. 11–1(a), it can be seen quite easily that part of Φ_{1M} links with only a fraction of the total number of turns in coil 2, and likewise Φ_{1S} links with only a fraction of the total turns in coil 1. Thus, it may be convenient to consider Φ_{1M} as a hypothetical or equivalent flux which, when linking all the turns in coil 1 and coil 2, will give the same total flux linkages as the true flux linkages of coil 1 and coil 2. Such a concept may also be applied to Φ_{1S}, Φ_{2M}, and Φ_{2S}.

The leakage flux is an important factor in determining the operating characteristics of iron-core and air-core transformers. In an iron-core transformer, the leakage flux or magnetic leakage usually is very small at no load. A clear picture of the magnetic field produced by the current in one winding is an aid to the understanding of the important and more complicated condition when there are currents in both windings.

11–3. Mutual Inductance.—In Chapter 3, the self-inductance of a coil was defined as the rate of change in the flux linkages of the coil with respect to its current. When the permeability of the surrounding medium is constant, the self-inductance of coil 1 is a constant. Thus,

$$L_1 = \frac{N_1\Phi_1}{i_1} \text{ henrys} \tag{11–3}$$

and the emf induced in it is given by

$$e_1 = -L_1 \frac{di_1}{dt} \text{ volts} \tag{11–4}$$

where Φ_1, e_1, and i_1 are in webers, volts, and amperes, respectively.

If the permeability of the surrounding medium varies with current strength, as in the case when the circuit contains iron, the self-inductance L_1 also varies with current strength.

In a similar manner, for a medium of constant permeability, mutual inductance may be defined as the flux linkages per ampere; or

$$M_{21} = \frac{N_2 \Phi_{1M}}{i_1} \text{ henrys} \qquad (11\text{–}5)$$

where Φ_{1M} is the mutual flux, in webers, linking the turns N_2 of coil 2.

The emf e_{21} induced in coil 2 due to a current varying in coil 1 can be expressed as

$$e_{21} = \pm M_{21} \frac{di_1}{dt} \qquad (11\text{–}6)$$

where M_{21} may be considered as the proportionality factor between the emf induced in coil 2 and the time rate of change of current in coil 1.

Like inductance L_1, M_{21} is assigned positive values. The sign $\pm$ on the right-hand side of the equation will be explained in Arts. 11–6 and 11–7. When the permeability of the surrounding medium varies with current strength, the mutual inductance also varies with current strength.

Likewise, if an emf is applied to coil 2 of Fig. 11–1(a), with coil 1 left open, the following equations are true for a medium of constant permeability:

$$M_{12} = \frac{N_1 \Phi_{2M}}{i_2} \text{ henrys} \qquad (11\text{–}7)$$

$$e_{12} = \pm M_{12} \frac{di_2}{dt} \qquad (11\text{–}8)$$

where M_{12} is the mutual inductance of coil 2 on coil 1 and e_{12} is the induced emf in coil 1 due to a current varying in coil 2. The mutual inductance M_{12} is assigned positive values.

If the induced emfs and currents and the time are given in volts, amperes, and seconds, respectively, then M_{21} and M_{12} are given in henrys. That is, when the current in coil 1 is changing at a rate of 1 ampere per second and an emf of 1 volt is induced in coil 2, the mutual inductance M_{21} is 1 henry. Other units, such as the millihenry (1mh $= 10^{-3}$ henry) and the microhenry (1μh $= 10^{-6}$ henry), are often used.

It can be shown from the energy point of view* that, when the permeability of the medium is constant,

$$M_{21} = M_{12} = M \tag{11-9}$$

The interpretation of equation (11–9) is that there is only one value of mutual inductance M between two coils in a medium of constant permeability. The statement is true without regard to the sizes, shapes, or positions of the two coils.

11–4. Factors Affecting Mutual Inductance.—The mutual inductance of two coils depends on the following factors:

(*a*) The physical dimensions of the two coils.

(*b*) The number of turns in each coil, N_1 and N_2.

(*c*) The distance between the two coils.

(*d*) The relative positions of the axes of the two coils (for example, if the axes of the coils are perpendicular to each other, the mutual inductance is practically zero).

(*e*) The permeability of the medium.

For a very special case where two long coils are wound on the same core in such a manner that all the flux set up by one can be considered to cut the other, the mutual inductance of the two coils in the rationalized MKS system can be expressed as follows:

$$M = \frac{N_1 N_2 A \mu}{l} \text{ henrys} \tag{11-10}$$

where M = mutual inductance of coils, in henrys;
N_1 = number of turns in coil 1;
N_2 = number of turns in coil 2;
A = area of core, in square meters;
μ = permeability of medium, in henry/meter (for air, $\mu_0 = 4\pi \times 10^{-7}$ henry/meter);
l = length of core, in meters.

See equation (3–14) for the analogous equation for self inductance.

* See "Principles of Alternating Currents," Second Edition, by Ralph R. Lawrence, p. 187 and 188. Also refer to "Electric Circuit Analysis," by M. G. Malti, p. 139 and 140; "Fundamentals of Electrical Engineering," by M. B. Reed, p. 227–229; and "Alternating Current Machinery," by J. G. Tarboux, p. 135–138.

11–5. Voltage Equations of Circuits Having *R*, *L*, and *M*. Fig. 11–3 shows a circuit containing a transformer with a core of material of constant permeability. With an alternating emf e_1 applied to circuit 1, currents will flow in the two circuits. Any change in the current of one circuit must be accompanied by a change of current in the other, since there is mutual induction between the two.

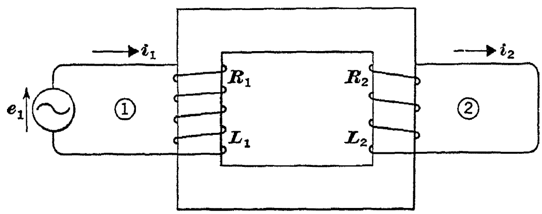


FIG. 11–3

As i_1 introduces an emf in circuit 2 and i_2 reacts back on circuit 1, it is necessary to set up two simultaneous Kirchhoff's emf equations in order to determine the values of the currents. If the circuits are traced in the direction of the current arrows, the differential equations for the circuit in Fig. 11–3 are:

$$e_1 = R_1 i_1 + L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \qquad \textbf{(11–11)}$$

$$0 = R_2 i_2 + L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} \qquad \textbf{(11–12)}$$

where L_1 and L_2 are the self-inductances of the coils and M is the mutual inductance. The negative sign is used before the terms containing M because of the manner in which the coils are wound on the core and the choice of positive sense of current flow. This will be considered further in the next two articles.

If circuit 2 were not short-circuited but had an emf e_2 across its terminals, equation (11–12) would become

$$e_2 = R_2 i_2 + L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} \qquad \textbf{(11–13)}$$

Equations (11–11) and (11–13) hold, irrespective of the wave shape of the emfs and the currents. When L and M are not constants, the solution of the foregoing equations is very difficult.

For the case when the voltages and currents are sinusoidal and the permeability of the medium is constant, equations (11–11) and (11–12) become equations (11–14) and (11–15). Let the applied emfs and the resulting currents in equations (11–11) and (11–12) be:

$$e_1 = E_m \sin \omega t \qquad e_2 = 0$$
$$i_1 = I_{m1} \sin (\omega t + \beta) \qquad i_2 = I_{m2} \sin (\omega t + \gamma)$$

where β and γ are phase angles.

Substitution of these values in equations (11–11) and (11–12) gives:

$$E_m \sin \omega t = R_1 I_{m1} \sin (\omega t + \beta) + \omega L_1 I_{m1} \cos (\omega t + \beta) - \omega M I_{m2} \cos (\omega t + \gamma) \quad \textbf{(11–14)}$$

$$0 = R_2 I_{m2} \sin (\omega t + \gamma) + \omega L_2 I_{m2} \cos (\omega t + \gamma) - \omega M I_{m1} \cos (\omega t + \beta) \quad \textbf{(11–15)}$$

If the circuit elements are independent of current strength, the steady state expressions in complex form for equations (11–14) and (11–15) are:

$$\dot{E}_1 = R_1 \dot{I}_1 + j\omega L_1 \dot{I}_1 - j\omega M \dot{I}_2 = (R_1 + j\omega L_1)\dot{I}_1 - j\omega M \dot{I}_2 \quad \textbf{(11–16)}$$

$$0 = R_2 \dot{I}_2 + j\omega L_2 \dot{I}_2 - j\omega M \dot{I}_1 = (R_2 + j\omega L_2)\dot{I}_2 - j\omega M \dot{I}_1 \quad \textbf{(11–17)}$$

or

$$\dot{E}_1 = \dot{Z}_1 \dot{I}_1 - \dot{Z}_m \dot{I}_2 \quad \textbf{(11–16a)}$$

$$0 = -\dot{Z}_m \dot{I}_1 + \dot{Z}_2 \dot{I}_2 \quad \textbf{(11–17a)}$$

where

$$\dot{Z}_1 = R_1 + j\omega L_1$$
$$\dot{Z}_2 = R_2 + j\omega L_2$$
$$\dot{Z}_m = +j\omega M$$

11–6. Sign of Terms Containing M.—It must be understood that the positive sense of current flow in a circuit, such as that shown in Fig. 11–3, may be chosen quite arbitrarily. The arrows indicate only the positive sense and not the direction of current flow at all times. For two coils wound on a core in a given manner and with a definite choice of positive senses of current flow, the following rules should be kept in mind when writing Kirchhoff's emf equation for any closed loop:

(*a*) The sign of the M-term will be the *same* as that of the L-term if the emf of mutual induction acts in the same direction as the emf of self-induction. Another way of looking at this is that, if in coil 1 the mmf (or flux) due to i_2

flowing in coil 2 aids the mmf (or flux) due to i_1 flowing in coil 1, then the sign of the M-term is the *same* as that of the L-term.

(*b*) The sign of the M-term will be *opposite* to that of the L-term if the emfs or mmfs in (*a*) oppose each other.

When these rules are applied to the conditions shown in Fig. 11–3 for coil 1, it can be seen that:

(1) the mmf (or flux) due to i_1 flowing in coil 1 is downward through coil 1;

(2) the mmf (or flux) due to i_2 flowing in coil 2 is downward through coil 2 but upward through coil 1.

Hence, the two mmfs (or fluxes) oppose each other in coil 1, and the sign of $M\frac{di_2}{dt}$ will be opposite to that of $L_1\frac{di_1}{dt}$. Since the sign of $L_1\frac{di_1}{dt}$ is positive, then that of $M\frac{di_2}{dt}$ will be negative, as indicated in equation (11–11).

In coil 2, the two mmfs (or fluxes) also oppose each other, and the sign of $M\frac{di_1}{dt}$ is negative because $L_2\frac{di_2}{dt}$ is positive, as indicated in equation (11–12).

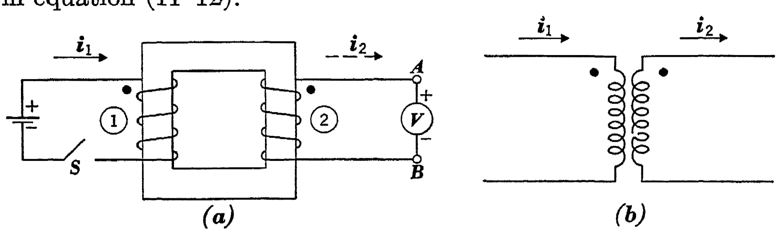


FIG. 11–4

It can be shown that changing the positive sense of either I_1 or I_2 (but not both at the same time) will alter the signs of the terms containing M. Changing the winding sense of either coil 1 or coil 2 will also change the signs of the terms containing M.

11–7. Coil Polarities.—In practice, coil terminals are marked by some symbol such as the dots in Fig. 11–4. These dots indicate the relative polarities of the two coils. That is, if at some instant the dot terminal of coil 1 is positive, then the dot terminal of coil 2 will also be positive at the same instant. It is important to know the polarities of coils, especially when it is necessary to connect them in series or in parallel.

The positions of the dots to be placed on the coil terminals may be determined by test in the following manner: Fig. 11–4(a) shows a circuit with the (+) terminal of the battery connected to the top or dot terminal of coil 1. With the dot terminal of coil 1 arbitrarily chosen as (+), then it remains to locate the (+) or dot terminal of coil 2 for the same instant. When the switch S is closed suddenly, an instantaneous current i_1 flows in the direction of the arrow. If a d-c voltmeter across A and B should deflect up scale, then the dot should be associated with the coil terminal connected to the (+) terminal of the voltmeter (in this case, terminal A). If the voltmeter should deflect down scale, or backward, then the dot should be associated with the coil terminal connected to the (−) terminal of the d-c voltmeter (in this case, terminal B). For the case given in Fig. 11–4(a), the voltmeter will read up scale and the dot should be associated with the coil terminal A, as shown. Thus, at some instant when the dot terminal of coil 1 is positive, then the dot terminal of coil 2 is positive; and, if A and B were connected together, a current i_2 would flow in the direction of the dotted arrow in Fig. 11–4(a).

It should be noted that the winding senses and the current senses in Fig. 11–4(a) are the same as those in Fig. 11–3. The sign of $M\frac{di_2}{dt}$ is opposite to that of $L_1\frac{di_1}{dt}$ and the sign of $M\frac{di_1}{dt}$ is opposite to that of $L_2\frac{di_2}{dt}$, as indicated in equations (11–11) and (11–12). Hence, the following rule may be used in determining the sign of the terms containing M:

Dot Rule: If both current senses are directed toward the dots (or both are directed away from the dots), then the sign of the term containing M will be the same as that of the L-term.

That is, when the sign of the L-term is positive, then the sign of the M-term will also be positive if both current senses are directed toward the dots (or both are directed away from the dots). If such a rule is to be followed, then Fig. 11–4(a) may be represented by its equivalent in Fig. 11–4(b).

Example 11–1.—For the circuit shown in Fig. 11–5(a), draw its equivalent circuit with the same current senses similar to that shown in Fig. 11–4(a). Also, find the sign of the terms containing M by the dot rule.

Solution.—The equivalent circuit of Fig. 11–5(a) can be represented by either Fig. 11–5(b) or Fig. 11–5(c). For the given locations of the dots, the winding senses of the coils are determined in the following manner. Arbitrarily choose a winding sense for coil L_1, as that shown in Fig. 11–5(b). Then, for the given location of the dot for coil L_2, the winding sense of coil L_2 in Fig. 11–5(b) is found by the coil polarity test.

When the dot rule is applied, it is advisable to locate the current arrows near the dots, as shown in Fig. 11–5(b). The current arrows must be placed external to the primary and secondary coils. When each loop is traced in the direction of the current arrow, the sign of the term $L_1 \frac{di_1}{dt}$ or that of the term $L_2 \frac{di_2}{dt}$ is positive. Then with both current senses toward the dots in Fig. 11–5(a), the sign of each term containing M is positive; that is, it is the same as that of each term containing L.

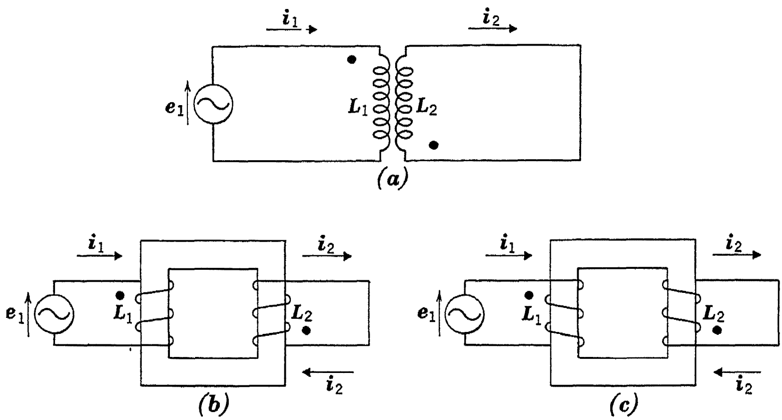


FIG. 11–5

The student should check the signs of the M-terms in Example 11–1 by the mmf (or flux) method discussed in the preceding article.

11–8. Coefficient of Coupling of Two Coils.—The design of transformers for use in power and radio networks differs quite widely in the two cases. In the case of a commercial power transformer with an iron core, good voltage regulation requires that as large a proportion as possible of the flux which links with one coil should link also with the other coil. This introduces a factor k called the *coefficient of coupling*. It is a measure of the magnetic proximity of the two coils. When all of the flux links both coils, the coefficient of coupling is equal to unity. This is the maximum value k can have. If the coils are placed a small distance apart, k

is less than unity. As the distance is increased indefinitely, the value of k becomes smaller, approaching zero as a limit. The value of k depends largely on the permeability of the medium.

It can be shown that, if L_1 and L_2 are the self-inductances of two coils and M is the mutual inductance between the two coils coupled through a magnetic flux, the coefficient of coupling of the two coils is

$$k = \frac{M}{\sqrt{L_1 L_2}} \tag{11-18}$$

For the same frequency in the two coils,

$$k = \frac{\omega M}{\omega\sqrt{L_1 L_2}} = \frac{X_m}{\sqrt{X_1 X_2}} \tag{11-19}$$

where $X_m = \omega M$, $X_1 = \omega L_1$, and $X_2 = \omega L_2$.

Two coils may be said to be: (*a*) close-coupled or (*b*) loose-coupled. In power work, the commercial transformers with iron cores of high permeability may have values of k as high as 0.98. In radio work, two circuits are said to be loosely coupled when the coefficient of coupling is less than 0.5. It is quite often true that the close coupling used in power transformers may not be desirable in transformers for use at radio frequencies. The degree of coupling that is desirable depends on the purpose for which the circuits are to be used.

11–9. Types of Coupled Circuits.—When electric circuits are located in the vicinity of each other or are so interconnected as to allow a transfer of electrical energy from one to the other, they are called coupled circuits. Circuits may be connected or coupled together in a number of ways, as shown in Fig. 11–6. The characteristic of coupled circuits is that the variation of impedances in one circuit affects the current flowing in the other. Thus, coupled circuits may be used to change the impedance of any portion of a network. They may be used to suppress a current of one frequency while responding to a current of another frequency.

Coupled circuits are quite often grouped in the following classification:

(1) *Direct Coupling*

(*a*) *Resistance Coupling:* Circuit 1 and circuit 2 are connected through a common resistance R_m, Fig. 11–6(*a*).

(*b*) *Inductance Coupling:* Circuit 1 and circuit 2 are connected through a common inductance L_m, Fig. 11–6(*b*); when the resistance of the common coil must be considered, then the common element is an impedance.

(*c*) *Capacitance Coupling:* Circuit 1 and circuit 2 are connected through a common condenser C_m, Fig. 11–6(*c*).

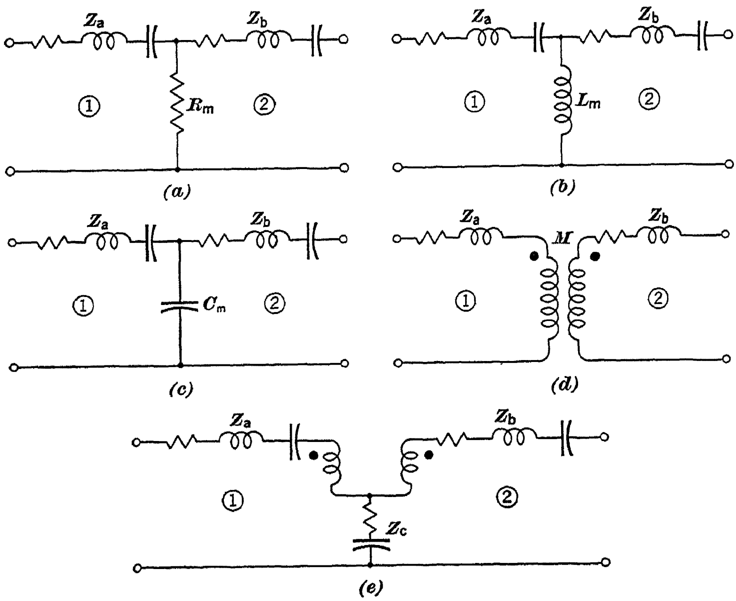


FIG. 11–6

(2) *Inductive Coupling:* The coupling is through a common magnetic flux linking the two circuits, as shown in Fig. 11–6(*d*). Such a coupling is often referred to as *magnetic, indirect, transformer*, or *inductive coupling*. (There may also be coupling through electrostatic flux connecting the two circuits.)

(3) *Combination Coupling:* Here the coupling is a combination of direct and indirect coupling, as in Fig. 11–6(*e*).

11–10. Voltage Equations of Circuits Having *R*, *L*, *C*, and *M*. When, in Fig. 11–3, circuit 1 contains C_1 in series with R_1 and the self-inductance L_1 and circuit 2 contains C_2 in series with R_2 and the self-inductance L_2, equations (11–11) and (11–12) become:

$$e_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int i_1\,dt - M\frac{di_2}{dt} \qquad (11\text{–}20)$$

$$0 = R_2 i_2 + L_2\frac{di_2}{dt} + \frac{1}{C_2}\int i_2\,dt - M\frac{di_1}{dt} \qquad (11\text{–}21)$$

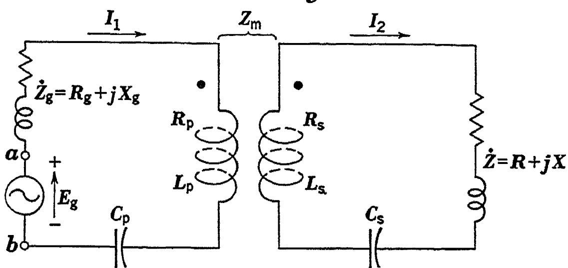


FIG. 11–7

If the circuit elements are linear and the voltages and currents are sinusoidal, the steady state complex forms* are:

$$\dot{E}_1 = R_1\dot{I}_1 + j\left(\omega L_1 - \frac{1}{\omega C_1}\right)\dot{I}_1 - j\omega M\dot{I}_2 = \dot{Z}_1\dot{I}_1 - \dot{Z}_m\dot{I}_2 \quad (11\text{–}22)$$

$$0 = R_2\dot{I}_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\dot{I}_2 - j\omega M\dot{I}_1 = \dot{Z}_2\dot{I}_2 - \dot{Z}_m\dot{I}_1 \quad (11\text{–}23)$$

where $\dot{Z}_1 = R_1 + j\left(\omega L_1 - \frac{1}{\omega C_1}\right)$ = the self-impedance of circuit 1, with circuit 2 open;

$\dot{Z}_2 = R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)$ = the self-impedance of circuit 2, with circuit 1 open;

$\dot{Z}_m = +j\omega M$ is the mutual impedance (assumed to be a pure reactance).

11–11. Steady-State Solution of Coupled Circuits.—In Fig. 11–7 are shown two circuits coupled by a mutual impedance $\dot{Z}_m = j\omega M$ (assumed to be a pure reactance). A generator is connected in one of the circuits, and a load $\dot{Z}$ is connected in the other. The coil or circuit receiving energy from a source is quite commonly referred to as the *primary*, and the coil or circuit delivering energy is called the *secondary*. The steady-state solution of the circuit shown in Fig. 11–7 will now be considered.

* Some authors prefer the use of double subscripts, such as $\dot{Z}_{11}$ for $\dot{Z}_1$, $\dot{Z}_{22}$ for $\dot{Z}_2$, and $\dot{Z}_{12} = \dot{Z}_{21}$ for $\dot{Z}_m$.

When the dots are used to indicate the relative polarities of the coils, it is not necessary to show also the winding senses on the same diagram. In order that the student may associate the dot notation with the winding senses, however, it may be well to have both on some of the diagrams. For the manner in which the coils are wound and the choice of positive sense of current flow in Fig. 11–7, the sign of the M-term is negative when the L-term or $j\omega L$ is positive as determined by the mmf or flux method. The dot rule checks this, as one current sense is toward the dot and the other is away from the dot. Then, by application of Kirchhoff's Emf Law, we obtain the steady-state relations:

$$\dot{E}_g=\left[R_g+jX_g+R_p+j\left(\omega L_p-\frac{1}{\omega C_p}\right)\right]\dot{I}_1 - j\omega M\dot{I}_2=\dot{Z}_1\dot{I}_1-\dot{Z}_m\dot{I}_2 \quad (11\text{–}24)$$

$$0=-j\omega M\dot{I}_1+\left[R+jX+R_s+j\left(\omega L_s-\frac{1}{\omega C_s}\right)\right]\dot{I}_2 = -\dot{Z}_m\dot{I}_1+\dot{Z}_2\dot{I}_2 \quad (11\text{–}25)$$

where $\dot{Z}_1$ = all impedances in circuit 1 including that of the generator, with secondary open-circuited;

$\dot{Z}_2$ = all impedances in circuit 2 including that of the load, with primary open-circuited;

$\dot{Z}_m=+j\omega M.$

The current $\dot{I}_1$ flowing in circuit 1 induces an emf $\dot{Z}_m\dot{I}_1$ in circuit 2. The current $\dot{I}_2$ flowing in circuit 2 is due to the induced emf $\dot{Z}_m\dot{I}_1$. With $\dot{I}_2$ flowing, an emf $\dot{Z}_m\dot{I}_2$ is induced in circuit 1.

The solutions for $\dot{I}_1$ and $\dot{I}_2$ can be found by solving equations (11–24) and (11–25) simultaneously. This may be done by the use of determinants, or otherwise. Thus,

$$\dot{I}_1=\frac{\begin{vmatrix}\dot{E}_g & -\dot{Z}_m\\ 0 & \dot{Z}_2\end{vmatrix}}{\begin{vmatrix}\dot{Z}_1 & -\dot{Z}_m\\ -\dot{Z}_m & \dot{Z}_2\end{vmatrix}}=\frac{\dot{E}_g\dot{Z}_2}{\dot{Z}_1\dot{Z}_2-\dot{Z}_m^2} \quad (11\text{–}26)$$

$$\dot{I}_2=\frac{\begin{vmatrix}\dot{Z}_1 & \dot{E}_g\\ -\dot{Z}_m & 0\end{vmatrix}}{\begin{vmatrix}\dot{Z}_1 & -\dot{Z}_m\\ -\dot{Z}_m & \dot{Z}_2\end{vmatrix}}=\frac{\dot{E}_g\dot{Z}_m}{\dot{Z}_1\dot{Z}_2-\dot{Z}_m^2} \quad (11\text{–}27)$$

When $\dot{I}_1$ is known, it is often convenient to find $\dot{I}_2$ from a relationship based on equation (11–25). From this equation,

$$\dot{Z}_m\dot{I}_1 = \dot{Z}_2\dot{I}_2 \tag{11–28}$$

and

$$\dot{I}_2 = \frac{\dot{Z}_m}{\dot{Z}_2}\dot{I}_1 \tag{11–29}$$

Equation (11–28) states that the voltage $\dot{Z}_m\dot{I}_1$ induced in the secondary (with no external voltage applied in the secondary circuit) is consumed in the impedance drop $\dot{Z}_2\dot{I}_2$. The voltage Z_mI_1 may be measured at the terminals of the secondary when it is open-circuited; and, I_1 being known, Z_m can be determined.

If, in Fig. 11–7, either the winding sense of one of the coils or the positive sense of the current $\dot{I}_2$ should be reversed, then equations (11–24) and (11–25) would become:

$$\dot{E}_g = \dot{Z}_1\dot{I}_1 + \dot{Z}_m\dot{I}_2 \tag{11–30}$$

$$0 = \dot{Z}_m\dot{I}_1 + \dot{Z}_2\dot{I}_2 \tag{11–31}$$

where $\dot{Z}_m = +j\omega M$.

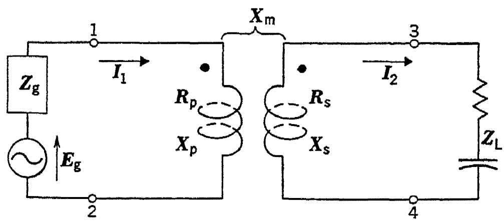


FIG. 11–8

When working coupled circuit problems, it is suggested to write first the Kirchhoff's emf equations. The currents can then be found by solving the equations simultaneously, without trying to remember the equations for the currents, such as equations (11–26) to (11–29). If it is desirable to use equations (11–27) to (11–29), the voltage equations must be of the form given by equations (11–24) and (11–25), that is, with minus signs for the Z_m-terms. It can be shown that $\dot{I}_1$ is not affected by the sign of the Z_m-term. The sign of the Z_m-term affects the sign of $\dot{I}_2$, but not its magnitude.

Example 11–2.—Fig. 11–8 shows a load connected to a generator through an air-core transformer. The voltage of the generator is 1000 volts, and the internal impedance of the generator is $\dot{Z}_g = 300 + j300$. The load is $\dot{Z}_L = 500$

$-j300$. If X_p is the reactance due to the self-inductance L_p and X_s is the reactance due to the self-inductance L_s, then the impedances of the transformer are:

$$\dot{Z}_p = R_p + jX_p = 200 + j1000$$
$$\dot{Z}_s = R_s + jX_s = 50 + j500$$
$$\dot{Z}_m = jX_m = j50$$

(*a*) What are the vector currents $\dot{I}_1$ and $\dot{I}_2$? (*b*) What is the voltage induced in circuit 2 due to the current $\dot{I}_1$ flowing in circuit 1?

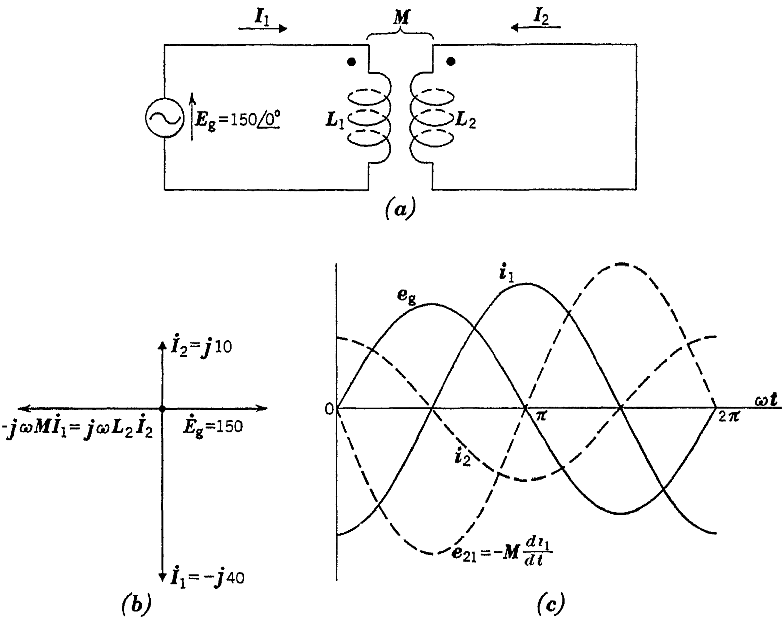


FIG. 11–9

Solution.—(*a*) Let the generator voltage be the reference vector. Then

$$\dot{E}_g = 1000\underline{/0°}$$

The total impedance in circuit 1, including that of the generator, is

$$\dot{Z}_1 = \dot{Z}_g + \dot{Z}_p = 500 + j1300 = 1393\underline{/69°}$$

The total impedance in circuit 2, including that of the load, is

$$\dot{Z}_2 = \dot{Z}_s + \dot{Z}_L = 550 + j200 = 585\underline{/20°}$$

For the winding and current senses shown in Fig. 11–8,

$$\dot{E}_g = \dot{Z}_1\dot{I}_1 - \dot{Z}_m\dot{I}_2 = (1393\underline{/69°})\dot{I}_1 - (j50)\dot{I}_2$$
$$0 = -\dot{Z}_m\dot{I}_1 + \dot{Z}_2\dot{I}_2 = -(j50)\dot{I}_1 + (585\underline{/20°})\dot{I}_2$$

The currents $\dot{I}_1$ and $\dot{I}_2$ may be determined by solving the last two equations simultaneously. However, these equations are of the form similar to equations (11–24) and (11–25), and equations (11–26) and (11–29) will hold. Thus, the primary current I_1 can be found from equation (11–26), or

$$\dot{I}_1 = \frac{1000\underline{/0°} \times 585\underline{/20°}}{1393\underline{/69°} \times 585\underline{/20°} - (j50)^2} = 0.718\underline{/-68.8°}$$

The secondary current $\dot{I}_2$ can be determined from equation (11–27). But, since $\dot{I}_1$ is known, it is more convenient to use equation (11–29) to find $\dot{I}_2$. Thus,

$$\dot{I}_2 = \frac{50\underline{/90°}}{585\underline{/20°}} \times 0.718\underline{/-68.8°} = 0.0614\underline{/1.2°}$$

(*b*) The voltage induced in circuit 2 due to the current $\dot{I}_1$ flowing in circuit 1 is

$$\dot{Z}_m \dot{I}_1 = 50\underline{/90°} \times 0.718\underline{/-68.8°} = 35.9\underline{/21.2°}$$

Example 11–3.—An air-core transformer, Fig. 11–9(*a*), has negligible resistances and the following self-inductances: $L_1 = 0.01$ henry and $L_2 = 0.04$ henry. The coefficient of coupling is 0.5, the generator voltage is 150 volts, and $\omega = 500$ radians per second.

(*a*) Write Kirchhoff's emf equations.
(*b*) Find the vector currents $\dot{I}_1$ and $\dot{I}_2$.
(*c*) Determine the voltage induced in circuit 2 due to the current flowing in circuit 1; $\dot{E}_{21} = -j\omega M \dot{I}_1$.
(*d*) Draw the vector diagram.
(*e*) Sketch the waves of e_g, i_1, i_2, and the induced emf in circuit 2, or $e_{21} = -M \dfrac{di_1}{dt}$.

Solution.—(*a*) If the circuits are traced in the direction of the current arrows, Kirchhoff's emf equations are:

$$e_g = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$0 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

In the complex form

$$\dot{E}_g = j\omega L_1 \dot{I}_1 + j\omega M \dot{I}_2 = \dot{Z}_1 \dot{I}_1 + \dot{Z}_m \dot{I}_2 \quad \text{[See equation (11–30)]}$$
$$0 = j\omega M \dot{I}_1 + j\omega L_2 \dot{I}_2 = \dot{Z}_m \dot{I}_1 + \dot{Z}_2 \dot{I}_2 \quad \text{[See equation (11–31)]}$$

In this case,

$$M = k\sqrt{L_1 L_2} = 0.5\sqrt{0.01 \times 0.04} = 0.01 \text{ henry}$$
$$\dot{Z}_1 = j\omega L_1 = j500 \times 0.01 = j5$$
$$\dot{Z}_2 = j\omega L_2 = j500 \times 0.04 = j20$$
$$\dot{Z}_m = j\omega M = j500 \times 0.01 = j5$$

Then

$$150\underline{/0°} = j5\dot{I}_1 + j5\dot{I}_2$$
$$0 = j5\dot{I}_1 + j20\dot{I}_2$$

(*b*) The vector currents can be found by solving the emf equations simultaneously. Thus,

$$\dot{I}_1 = \frac{\dot{E}_g \dot{Z}_2}{\dot{Z}_1 \dot{Z}_2 - \dot{Z}_m^2} = \frac{150\underline{/0^\circ} \times (j20)}{(j5)(j20) - (j5)^2} = -j40$$

$$\dot{I}_2 = \frac{-\dot{E}_g \dot{Z}_m}{\dot{Z}_1 \dot{Z}_2 - \dot{Z}_m^2} = \frac{-150\underline{/0^\circ} \times (j5)}{(j5)(j20) - (j5)^2} = j10$$

or

$$\dot{I}_2 = \frac{-\dot{Z}_m}{\dot{Z}_2} \dot{I}_1 = \frac{-j5}{j20} \times (-j40) = j10$$

(*c*) The voltage induced in circuit 2, which is effectively in series with L_2 in circuit 2, is

$$\dot{E}_{21} = -j\omega M \dot{I}_1 = -\dot{Z}_m \dot{I}_1 = \dot{Z}_2 \dot{I}_2 = j\omega L_2 \dot{I}_2$$
$$-j\omega M \dot{I}_1 = -(j5)(-j40) = 200\underline{/180^\circ}$$

(*d*) The vector diagram is shown in Fig. 11–9(*b*)

(*e*) The waves of e_g, i_1, i_2, and the induced emf $-M\frac{di_1}{dt}$ in circuit 2 are drawn in Fig. 9(*c*) in the following order:

(1) Start with e_g as reference.
(2) i_1 lags e_g by 90° because the circuit is purely inductive.
(3) The induced emf $-M\frac{di_1}{dt}$ lags i_1 by 90°.
(4) Circuit 2 being purely inductive, i_2 would lag behind $L_2\frac{di_2}{dt}$ $\left(\text{in this case, } L_2\frac{di_2}{dt} = e_{21} = -M\frac{di_1}{dt}\right)$ by 90°, or $\dot{I}_2$ lags behind $j\omega L_2 \dot{I}_2$ by 90°.

11–12. Driving-Point and Transfer Impedances.—The driving-point impedance, sometimes called the input impedance at two specified terminals, is the ratio of the input voltage to the input current. For the input terminals *a* and *b* in Fig. 11–7, it is the ratio $\frac{\dot{E}_g}{\dot{I}_1}$. From equation (11–26), the input or driving-point impedance at *a* and *b*, designated by the symbol $\dot{Z}_1'$, is

$$\dot{Z}_1' = \frac{\dot{E}_g}{\dot{I}_1} = \dot{Z}_1 - \frac{\dot{Z}_m^2}{\dot{Z}_2} \tag{11–32}$$

Without the effect of circuit 2, that is, with circuit 2 open-circuited, the impedance of circuit 1 is $\dot{Z}_1$. As a result of the coupling to circuit 2, the equivalent impedance of circuit 1 is the impedance $\dot{Z}_1$ of circuit 1 modified by the amount $\frac{-\dot{Z}_m^2}{\dot{Z}_2}$. A knowledge of the driving-point impedance $\dot{Z}_1'$ is often useful, for it is an equivalent impedance of the form $R_1' + jX_1'$ at the input terminals (see Art. 11–15, Action of the Coupled Impedance). It gives indication of the magnitude and phase angle of the input current with respect to a given input voltage.

The transfer impedance is defined as the ratio of a voltage impressed in one part of a network to a current in another part of the network. For the circuit shown in Fig. 11–7, the ratio $\frac{\dot{E}_g}{\dot{I}_2}=\dot{Z}'_{12}$ is called the transfer impedance. The transfer impedance is so called and designated by the symbol $\dot{Z}'_{12}$ to distinguish it from the input impedance $\dot{Z}'_1$. For the set of equations (11–24) and (11–25), equation (11–27) gives

$$\dot{Z}'_{12}=\frac{\dot{E}_g}{\dot{I}_2}=\frac{\dot{Z}_1\dot{Z}_2-\dot{Z}_m^2}{\dot{Z}_m} \tag{11–33}$$

The transfer impedance has the dimensions of impedance because it represents the ratio of a sinusoidal voltage to a sinusoidal current. The angle of a transfer impedance may have any value, that is, may be in any quadrant, and it is not significant as a measure of power. If the transfer impedance $\dot{Z}'_{12}$ for the network in Fig. 11–7 is known (its value depends on the load impedance), the load current can be found for a given applied voltage. Thus,

$$\dot{I}_2=\frac{\dot{E}_g}{\dot{Z}'_{12}} \tag{11–33a}$$

Example 11–4.—Consider again the circuit in Example 11–3. (*a*) Find the driving-point impedance $\dot{Z}'_1$ and the transfer impedance $\dot{Z}'_{12}$. (*b*) Determine $\dot{I}_1$ and $\dot{I}_2$ by using $\dot{E}_g$, $\dot{Z}'_1$, and $\dot{Z}'_{12}$.

Solution.—(*a*) From equation (11–32), the driving-point impedance is

$$\dot{Z}'_1=j5-\frac{(j5)^2}{j20}=j3.75$$

Since the voltage relations are of the form of equations (11–30) and (11–31), the transfer impedance is

$$\dot{Z}'_{12}=\frac{\dot{E}_g}{\dot{I}_2}=\frac{\dot{Z}_1\dot{Z}_2-\dot{Z}_m^2}{-\dot{Z}_m}=\frac{(j5)(j20)-(j5)^2}{-j5}=-j15$$

(*b*) From equation (11–32),

$$\dot{I}_1=\frac{\dot{E}_g}{\dot{Z}'_1}=\frac{150\underline{/0^\circ}}{j3.75}=-j40$$

From equation (11–33a),

$$\dot{I}_2=\frac{\dot{E}_g}{\dot{Z}'_{12}}=\frac{150\underline{/0^\circ}}{-j15}=j10$$

11–13. Equivalent T-Section of a Transformer.—In the solution of transformer coupled circuits, it is sometimes convenient to replace the transformer alone by an equivalent T-section, as far as the solutions for $\dot{I}_1$ and $\dot{I}_2$ are concerned. Students are more familiar with a T-section than with two coils coupled to each other. This may be considered as a reason for the use of equivalent T-sections for transformers. The concept of equivalent T-sections will be used quite extensively in the next chapter.

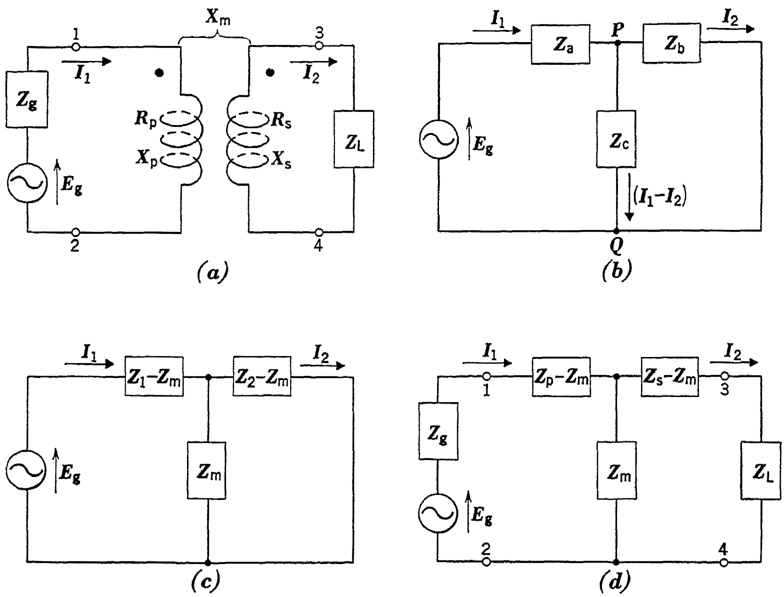


FIG. 11–10

Let it be desirable to replace the transformer alone in Fig. 11–10(*a*) by an equivalent T-section. This may be done in the following way. The self-impedance of circuit 1 in Fig. 11–10(*b*) is determined with circuit 2 open. Thus,

$$\dot{Z}_1 = \dot{Z}_a + \dot{Z}_c$$

or

$$\dot{Z}_a = \dot{Z}_1 - \dot{Z}_c$$

The self-impedance of circuit 2 is obtained with circuit 1 open. Then,

$$\dot{Z}_2 = \dot{Z}_b + \dot{Z}_c$$

or

$$\dot{Z}_b = \dot{Z}_2 - \dot{Z}_c$$

Between P and Q in Fig. 11–10(b),

$$\dot{Z}_c(\dot{I}_1-\dot{I}_2)=\dot{Z}_b\dot{I}_2=(\dot{Z}_2-\dot{Z}_c)\dot{I}_2$$

which reduces to

$$\dot{Z}_c\dot{I}_1=\dot{Z}_2\dot{I}_2$$

The loop equation for circuit 2 in Fig. 11–10(a) is

$$0=-\dot{Z}_m\dot{I}_1+\dot{Z}_2\dot{I}_2$$

or

$$\dot{Z}_m\dot{I}_1=\dot{Z}_2\dot{I}_2$$

Hence, by comparison, $\dot{Z}_c=\dot{Z}_m$ (not $\dot{Z}_c=-\dot{Z}_m$) for the winding and current senses in Fig. 11–10(a). Also,

$$\dot{Z}_a=\dot{Z}_1-\dot{Z}_m$$
$$\dot{Z}_b=\dot{Z}_2-\dot{Z}_m$$

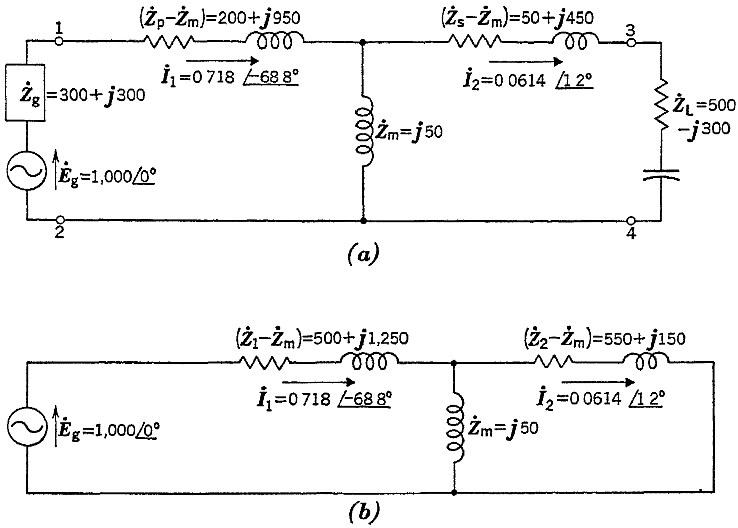


FIG. 11-11

Thus, a transformer coupled circuit with the winding and current senses as indicated in Fig. 11–10(a) may be represented by the T-section in Fig. 11–10(c), as far as $\dot{I}_1$ and $\dot{I}_2$ are concerned. Since $\dot{Z}_1=\dot{Z}_p+\dot{Z}_g$ and $\dot{Z}_2=\dot{Z}_s+\dot{Z}_L$, the equivalent T-section for the transformer alone is shown in Fig. 11–10(d). For other winding and current senses, see Problems 11–29, 11–30, and 11–31 for the corresponding equivalent T-sections.

Example 11–5.—What is the equivalent T-section for the circuit of Example 11–2 in Fig. 11–8?

Solution.—With the winding sense and current sense as shown in Fig. 11–8,

$$\dot{Z}_m = j50$$
$$\dot{Z}_p - \dot{Z}_m = (200 + j1000) - j50 = 200 + j950$$
$$\dot{Z}_s - \dot{Z}_m = (50 + j500) - j50 = 50 + j450$$

The equivalent T-section of the circuit may be represented as in Fig. 11–11(*a*) or Fig. 11–11(*b*).

11–14. The Ideal Transformer.—The ideal transformer is a device which is useful in the analysis of certain power and communication circuits. It is defined as a mechanism having the following properties:

(1) There are no resistances in the windings.

(2) There are no core losses (hysteresis and eddy-current losses) in the magnetic circuit.

(3) The coefficient of coupling is equal to unity (no leakage).

(4) The permeability of the core is so high (or reluctance is practically zero) that, theoretically, no mmf is necessary to produce the required flux. (Some authors prefer to use the property of infinite primary self-inductance L_1 and an infinite secondary self-inductance L_2 but a finite value for the ratio $\frac{L_1}{L_2}$.)

From these characteristics, it is apparent that an ideal transformer can never be realized physically. The modern iron-core transformer has so nearly approached perfection that, with a fair degree of approximation, it may be considered a perfect transforming device. It is now desirable to consider how the voltages, currents, and load impedances transform from one circuit to the other in a perfect transformer.

(*a*) *Voltage Ratio:* If the coefficient of coupling is unity, the mutual flux passing through all the turns of the two coils is the same and it must induce the same emf per turn in each winding. For N_1 turns in coil 1 and N_2 turns in coil 2, the total induced emf in each coil must then be proportional to the number of turns in that coil; that is, $e_1 = -N_1\frac{d\Phi_M}{dt}$ and $e_2 = -N_2\frac{d\Phi_M}{dt}$, and

$$\frac{E_1}{E_2} = \frac{N_1}{N_2} = a \qquad \textbf{(11–34)}$$

where E_1 and E_2 are the effective values of the induced emfs. The factor $a = \frac{N_1}{N_2}$ is sometimes referred to as the *ratio of transformation* or the *turns ratio.*

Equation (11–34) states that the induced emfs transform in direct proportion to the numbers of turns. In an ideal transformer, the terminal voltages will be equal to the induced emfs, and

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} = a \tag{11–35}$$

For an ordinary power transformer, the terminal voltages differ slightly from the induced emfs. When the difference is a very small percentage, it may be assumed for most practical purposes that the primary and secondary terminal voltages are approximately proportional to the corresponding numbers of turns.

(*b*) *Current Ratio:* According to properties (1), (2), and (4), an ideal transformer has no losses and it takes no current from the line when the secondary is open-circuited. That is, it requires no component of primary current to take care of hysteresis and eddy-current losses and requires no magnetizing component of current in the primary to set up the necessary flux. In the case of a commercial transformer, the total primary current contains both magnetizing and hysteresis and eddy-current components in addition to the load component which comes from the secondary. Under ideal conditions it is reasonable to accept that, when a load is connected to the secondary, the volt-ampere input to the primary will be equal to the volt-ampere output of the secondary. Thus,

$$E_1I_1 = E_2I_2 \tag{11–36}$$

from which

$$\frac{I_1}{I_2} = \frac{E_2}{E_1} = \frac{N_2}{N_1} = \frac{1}{a} \tag{11–37}$$

and

$$N_1I_1 = N_2I_2 \tag{11–38}$$

It can be seen from equation (11–37) that the currents transform in inverse proportion to the turns ratio. Equation (11–38) indicates that the ampere-turns of the primary winding are equal to the ampere-turns of the secondary winding (sometimes this is referred to as the balancing of ampere-turns).

(c) *Impedance Ratio:* It has been shown that a transformer is a device which transforms alternating voltage and alternating current. An ideal transformer may be looked upon as a mechanism for transforming an impedance, that is, for making a given impedance, as R_L, appear larger or smaller, as R'_L at the terminals 1 and 1′ of Fig. 11–12, for matching purposes. This will now be considered.

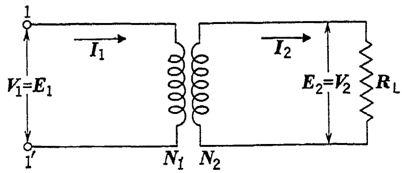


FIG. 11–12

Fig. 11–12 shows an ideal transformer with a load resistance R_L connected to its secondary. It is obvious from the circuit that $\frac{V_2}{I_2} = R_L$. If R'_L is to appear at the terminals 1 and 1′, then $\frac{V_1}{I_1} = R'_L$. It will be shown that $R'_L = a^2R_L$. From the combination of equations (11–35) and (11–37),

$$\frac{V_1/I_1}{V_2/I_2} = \left(\frac{N_1}{N_2}\right)^2$$

or

$$\frac{V_1}{I_1} = \left(\frac{N_1}{N_2}\right)^2 \frac{V_2}{I_2} = \left(\frac{N_1}{N_2}\right)^2 R_L = a^2R_L \tag{11–39}$$

Thus, as far as $\dot{I}_1$ is concerned, the load resistance R_L plus the ideal transformer in Fig. 11–12 may be viewed as equivalent to a resistance connected across the input terminals 1 and 1′ and having the value

$$R'_L = a^2R_L \tag{11–40}$$

That is, when viewed from the input terminals, the load resistance R_L in the secondary can be replaced by a resistance $R'_L = a^2R_L$ in the primary. The relation between R'_L and R_L may also be written as follows:

$$\frac{R'_L}{R_L} = a^2 = \left(\frac{N_1}{N_2}\right)^2 \tag{11–41}$$

which states that the resistance transforms as the square of the turns ratio.

This result can be extended to an ideal transformer with a load impedance $\dot{Z}_L$ across its secondary. Thus,

$$\dot{Z}'_L = a^2\dot{Z}_L \tag{11–42}$$

It may be seen from equations (11–40) and (11–42) that an ideal transformer will modify the magnitude of the load impedance, as viewed from the input terminals, without changing its angle.

Example 11–6.—An ideal transformer has a turns ratio $\frac{N_1}{N_2} = 2$. (*a*) If an emf of 100 volts is first impressed across the terminals of the primary of N_1 turns, what is the voltage across the terminals of the open-circuited secondary? (*b*) If a load resistance $R_L = 10$ ohms is next connected across the secondary, what will be the primary and secondary currents? (*c*) What is R'_L, the equivalent of resistance R_L in the primary circuit?

Solution.—(*a*) By equation (11–35),

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} = 2$$

from which

$$V_2 = \frac{V_1}{2} = \frac{100}{2} = 50 \text{ volts}$$

(*b*) The currents are:

$$I_2 = \frac{V_2}{R_L} = \frac{50}{10} = 5 \text{ amp}$$

$$I_1 = \frac{N_2}{N_1} I_2 = \frac{1}{2} \times 5 = 2.5 \text{ amp}$$

(*c*) From equation (11–41),

$$R'_L = \left(\frac{N_1}{N_2}\right)^2 R_L = 2^2 \times 10 = 40 \text{ ohms}$$

As a check,

$$I_1 = \frac{V_1}{R'_L} = \frac{100}{40} = 2.5 \text{ amp}$$

11–15. Action of the Coupled Impedance.—Important properties of coupled circuits can be determined by examining the nature of the coupled impedance $\frac{\dot{Z}_m^2}{\dot{Z}_2}$. This may be done effectively by first showing that, for a given frequency and as far as the generator current is concerned, the coupled circuit shown in Fig. 11–7 can be reduced to a simple series circuit containing a resistive element and a reactive element. Then the effect of the coupling on the resistance and reactance in the primary may be found by examining the expression for the impedance of this series circuit. Let $\dot{Z}_1$,

$\dot{Z}_2$, and $\dot{Z}_m$ be defined as in Art. 11–11 in connection with Fig. 11–7 and have the following forms:

$$\dot{Z}_1 = R_1 + jX_1; \quad \dot{Z}_2 = R_2 + jX_2; \quad \dot{Z}_m = j\omega M$$

By substituting these values in equation (11–32) and rearranging terms, we get:

$$Z_1' = \left[R_1 + \frac{(\omega M)^2}{R_2^2 + X_2^2} R_2\right] + j\left[X_1 - \frac{(\omega M)^2}{R_2^2 + X_2^2} X_2\right] = R_1' + jX_1' \quad \textbf{(11–43)}$$

where
$$R_1' = R_1 + \frac{(\omega M)^2}{R_2^2 + X_2^2} R_2 = \text{Resistive element}$$

$$X_1' = X_1 - \frac{(\omega M)^2}{R_2^2 + X_2^2} X_2 = \text{Reactive element}$$

Thus, for a single frequency and as far as $\dot{I}_1$ is concerned, the coupled circuit in Fig. 11–7 can be replaced by a series circuit containing R_1' and X_1' connected across a and b.

Equation (11–43) gives a convenient form for studying the effect of the coupled impedance on the primary circuit. It is evident that, as a result of the coupling, the primary resistance R_1 is increased by the amount $\frac{(\omega M)^2}{R_2^2 + X_2^2} R_2$, and the primary reactance X_1 is modified by the amount $\frac{-(\omega M)^2}{R_2^2 + X_2^2} X_2$. The reflected resistance and reactance, in association with I_1, will consume the same energy and reactive volt-amperes as are transferred to the secondary. If conditions are such that both X_1 and X_2 are inductive, then $\frac{-(\omega M)^2}{R_2^2 + X_2^2} X_2$ neutralizes part of X_1. When the mutual inductance M is small and the secondary impedance Z_2 is large, the coupled impedance $\frac{\dot{Z}_m^2}{\dot{Z}_2}$ is small and the secondary has little effect on the primary. When M is not too small and Z_2 is low, as in the case of a radio coupled circuit with tuned secondary, the coupled impedance will be an important factor in determining the primary current. This in turn will affect the voltage $\omega M I_1$ induced in the secondary and the secondary current $I_2 = \frac{\omega M I_1}{Z_2}$.

11–16. Coupled Impedance Under "Ideal" Conditions.—To further illustrate the nature of the coupled impedance $\frac{\dot{Z}_m^2}{\dot{Z}_2}$, consider the circuit shown in Fig. 11–13 as an example. It should be noted here that R_p and X_p are the resistance and reactance (due to self-inductance L_p) of the primary coil *only*; R_s and X_s are the resistance and reactance (due to self-inductance L_s) of the secondary coil *only*; $\dot{Z}=R+jX$ is the load impedance; and $\dot{Z}_g=R_g+jX_g$ is the internal impedance of the generator. Thus, for this case,

$$\dot{Z}_1 = R_1+jX_1 = (R_g+R_p)+j(X_g+X_p)$$
$$\dot{Z}_2 = R_2+jX_2 = (R+R_s)+j(X+X_s)$$
$$\dot{Z}_m = +j\omega M = +jX_m$$

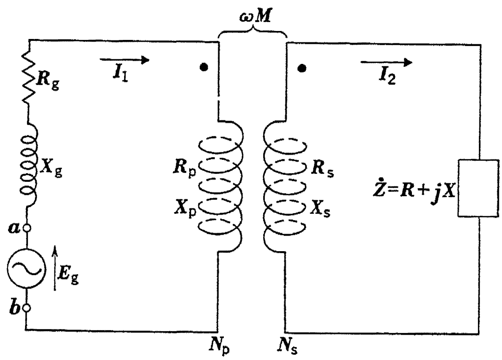


FIG. 11–13

When these values of $\dot{Z}_1$, $\dot{Z}_2$, and $\dot{Z}_m$ are substituted in equation (11–32) and terms are rearranged, the driving-point impedance at the terminals a and b is

$$\dot{Z}_1' = \left[(R_g+R_p)+\frac{(\omega M)^2(R+R_s)}{(R+R_s)^2+(X+X_s)^2}\right]+ j\left[(X_g+X_p)-\frac{(\omega M)^2(X+X_s)}{(R+R_s)^2+(X+X_s)^2}\right] \quad \textbf{(11–43a)}$$

It is interesting to study the last equation under "ideal" conditions which convert the actual transformer approximately into an ideal transformer. That is, make the following assumptions:

(1) X_p and X_s are very large in comparison with all other quantities; that is, L_p and L_s are large.
(2) R_p and R_s are very small in comparison with all other quantities.
(3) The coefficient of coupling k is practically unity so that, as an approximation from equation (11–19),

$$(\omega M)^2 = X_p X_s$$

Then,

$$\dot{Z}_1' = \left[R_g + \frac{X_p X_s R}{(X+X_s)^2}\right] + j\left[X_g + X_p - \frac{X_p X_s}{X+X_s}\right] \quad \textbf{(11–44)}$$

or

$$\dot{Z}_1' = \left[R_g + \frac{X_p X_s R}{(X+X_s)^2}\right] + j\left[X_g + \frac{X_p X + X_p X_s - X_p X_s}{X+X_s}\right]$$

Now, neglecting X in comparison with X_s, we obtain:

$$\dot{Z}_1' = \left[R_g + \frac{X_p}{X_s} R\right] + j\left[X_g + \frac{X_p}{X_s} X\right] \quad \textbf{(11–44a)}$$

This last equation will have more meaning when it is shown that $\frac{X_p}{X_s}$ is equal to $\left(\frac{N_p}{N_s}\right)^2 = a^2$ and $\dot{Z}_1'$ is given as the vector sum of $\dot{Z}_g$ and $a^2\dot{Z}$, as indicated in equation (11–44c). This would check equation (11–42), which states that the combination of an ideal transformer and a load $\dot{Z}$ is equivalent to $a^2\dot{Z}$ in the primary circuit.

The flux is

$$\Phi = \frac{\text{Magnetomotive force}}{\text{Reluctance}} = \frac{Ni}{\text{Reluctance}}$$

and the inductance is

$$L = \frac{N\Phi}{i} = \frac{N^2}{\text{Reluctance}}$$

where the reluctance relates to the flux path. If the reluctances of the flux paths can be assumed to be the same, the ratio of the coil inductances will be proportional to the ratio of the squares of the numbers of turns. Thus,

$$\frac{X_p}{X_s} = \frac{L_p}{L_s} = \frac{N_p^2}{N_s^2} = a^2$$

and equation (11–44a) may be written as follows:

$$\dot{Z}_1' = [R_g + a^2 R] + j[X_g + a^2 X] \quad \textbf{(11–44b)}$$

or

$$\dot{Z}' = (R_g + jX_g) + a^2(R + jX) = \dot{Z}_g + a^2\dot{Z} \quad \textbf{(11–44c)}$$

It should be noted that X_p and X_s do not appear in equation (11–44b). This means that, because of the coupling in an ideal transformer, the effect of X_s exactly neutralizes that of X_p. Some indication of this effect may be noted in the value of the term $\left(X_p - \frac{X_p X_s}{X+X_s}\right) = X_p - \frac{X_p}{1+\frac{X}{X_s}}$ of equation (11–44) when $X=0$.

PROBLEMS

11–1. In Fig. 11–1(*a*), coil 1 has 200 turns and coil 2 has 2000 turns. When the current in circuit 1 is $i_1 = 2$ amp, the total equivalent flux linking coil 1 is $\Phi_1 = 50{,}000$ maxwells and the total equivalent flux linking coil 2 is $\Phi_{1M} = 45{,}000$ maxwells. (*a*) What is the self-inductance L_1 of coil 1? (*b*) What is the mutual inductance M_{21} of the two coils? (*c*) If the current I_1 is reduced uniformly from 2 amp to zero in 0.001 second, what is the emf e_{21} induced in coil 2?

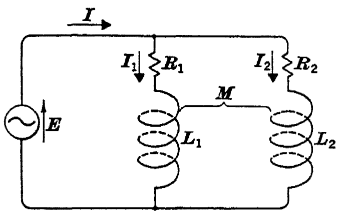


Fig 11–14

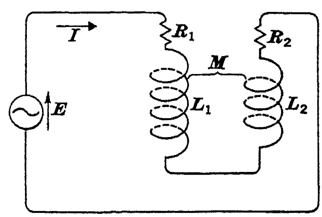


Fig. 11–15

11–2. Show from the energy point of view that the coefficients of mutual induction, M_{12} and M_{21}, of two coils in a medium of constant permeability are equal.

11–3. Write Kirchhoff's emf equations for the circuit given in Fig. 11–14, where the axes of the two coils do not coincide.

11–4. Indicate on Fig. 11–14 dots for relative coil polarities so that the correct emf equations can be determined

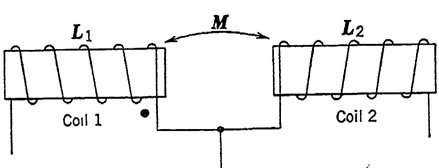


Fig. 11–16

11–5. If two coils connected in series are so arranged as to have mutual inductance as shown in Fig. 11–15, show that the resultant inductance is either $L = L_1 + L_2 + 2M$ or $L = L_1 + L_2 - 2M$. The axes of the coils do not coincide.

11–6. Write Kirchhoff's emf equation for the circuit of Fig. 11–15, where the axes of the coils do not coincide.

11–7. Determine the dot terminal for coil 2 in Fig. 11–16.

11–8. Write Kirchhoff's emf equations for the circuit shown in Fig 11–17

11–9. Write Kirchhoff's emf equations for the circuit shown in Fig. 11–18.

11–10. Show that $k=\dfrac{M}{\sqrt{L_1L_2}}$. State the assumption or assumptions made in the derivation.

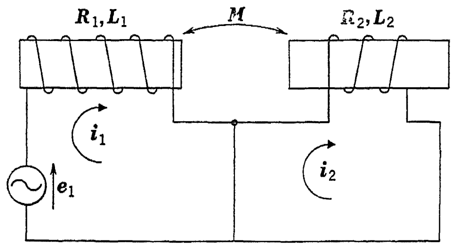


FIG. 11–17

11–11. An air-core transformer has the following constants:

$$L_1=0.005 \text{ henry}; \quad L_2=0.01 \text{ henry}; \quad M=0.002 \text{ henry}$$

What is the coefficient of coupling?

11–12. An air-core transformer has the following primary and secondary inductances:

$$L_1=0.01 \text{ henry} \qquad L_2=0.04 \text{ henry}$$

What are the values of M when $k=0$ 98, 0 9, 0.6, and 0.4?

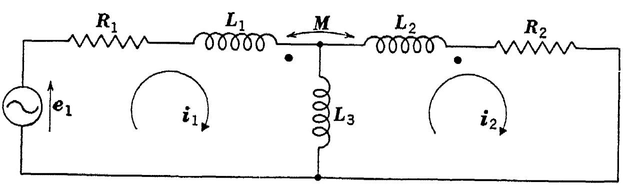


FIG. 11–18

11–13. Two coils, with $N_1=100$ turns and $N_2=1000$ turns, are coupled together as shown in Fig. 11–1 (a). With coil 2 short-circuited and a current of 1 amp flowing in coil 1, the total equivalent flux Φ_1 is 1000 maxwells, of which $\Phi_{1M}=500$ maxwells link with N_2. Under a similar condition with coil 1 short-circuited, a current of 1 amp in coil 2 establishes a total equivalent flux $\Phi_2=$ 10,000 maxwells, of which $\Phi_{2M}=5000$ maxwells link with N_1. What is the coefficient of coupling?

11–14. When two coils are interlinked magnetically, the following relations hold:

$$\Phi_1 = \frac{L_1 i_1}{N_1} \quad ; \quad \Phi_2 = \frac{L_2 i_2}{N_2}$$

$$\Phi_{1M} = \frac{M_{21} i_1}{N_2} = \frac{M i_1}{N_2}; \quad \Phi_{2M} = \frac{M_{12} i_2}{N_1} = \frac{M i_2}{N_1}$$

$$k_1 = \frac{\Phi_{1M}}{\Phi_1} \quad ; \quad k_2 = \frac{\Phi_{2M}}{\Phi_2}$$

Show that the coefficient of coupling $k = \sqrt{k_1 k_2}$.

11–15. In the circuit in Fig. 11–14,

$R_1 = 80$ ohms	$L_1 = 0.08$ henry	$k = 0.6$
$R_2 = 20$ ohms	$L_2 = 0.02$ henry	$\dot{E} = 100\underline{/0°}$ volts
		$\omega = 1000$ radians/sec

Find $\dot{I}_1$, $\dot{I}_2$, $\dot{I}$, and the total power delivered by the generator for: (*a*) the windings as shown; (*b*) the connections to coil 2 reversed.

Draw the vector diagram for each case, showing the generator voltage and all three current vectors.

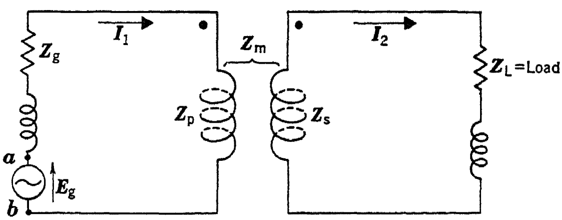


FIG. 11–19

11–16. The generator of the circuit shown in Fig. 11–19 has an internal impedance of $\dot{Z}_g = 300 + j400$. The circuit impedances are:

$\dot{Z}_p = 50 + j750$	$\dot{Z}_L = 600 + j800$
$\dot{Z}_s = 30 + j500$	$\dot{Z}_m = j400$

Find the currents $\dot{I}_1$ and $\dot{I}_2$, assuming that $\dot{E}_g = 1\underline{/0°}$.

11–17. Repeat Problem 11–16 with the positive sense of $\dot{I}_2$ reversed.

11–18. An air-core transformer has the following characteristics:

$R_p = 5$ ohms	$R_s = 3$ ohms	$X_m = 0.6$ ohm
$X_p = 5$ ohms	$X_s = 3$ ohms	

The positive senses of the currents and the coil windings are as shown in Fig. 11–19. If the primary is connected to a 60-cycle, 110-volt generator with negligible internal impedance, what will be the steady state primary and secondary currents when the secondary is short-circuited? Use the generator voltage as reference.

11–19. Determine the driving-point impedance at the terminals *a* and *b* of Fig. 11–19 and the transfer impedance of the air-core transformer in Problem 11–18 when the secondary is short-circuited. How are the primary resistance and the primary reactance affected?

11–20. Derive an expression for the input impedance of the circuit in terms of R_1, R_2, L_1, L_2, and M in Problem 11–4.

11–21. How much capacitance must be inserted in series with the primary of Problem 11–18 to bring about the unity power factor condition at the primary terminals?

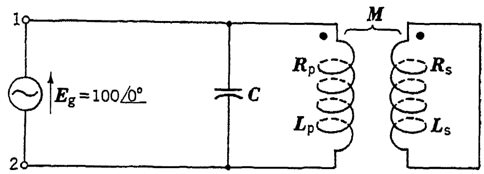


FIG. 11–20

11–22. How much capacitance or inductance must be inserted in series with the secondary of Problem 11–18 to bring about the unity power factor condition at the primary terminals when R_s is made equal to zero?

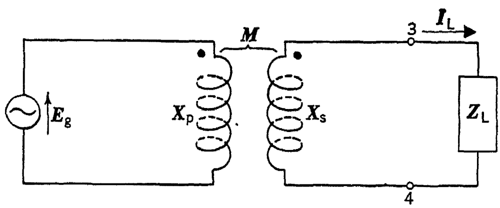


FIG 11–21

11–23. The circuit elements shown in Fig. 11–20 have the following values:

$R_p = 10$ ohms $R_s = 0$
$L_p = 0.4$ henry $L_s = 0.1$ henry
$M = 0.1$ henry $\omega = 500$ radians/sec

What is the value of C in order that the entire circuit will act like a non-inductive resistance at the generator terminals 1 and 2?

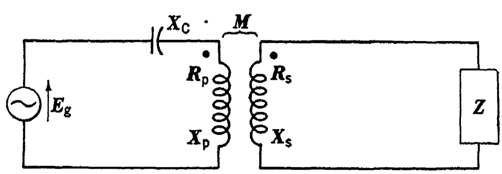


FIG. 11–22

11–24. The constants for the network shown in Fig. 11–21 are:

$X_p = 100$ ohms $k = 0.5$
$X_s = 25$ ohms $\dot{E}_g = 100\underline{/0°}$ volts

(*a*) As far as $\dot{I}_L$ is concerned, find by Thévenin's Theorem the equivalent generator which will replace the network to the left of terminals 3 and 4. (*b*) Repeat part (*a*), using Norton's Theorem.

11–25. The circuit shown in Fig. 11–22 has the following characteristics:

$R_p = 10$ ohms $\quad R_s = 2$ ohms
$X_p = 40$ ohms $\quad X_s = 10$ ohms
$X_C = 100$ ohms $\quad k = 0.5$

Determine the load impedance $\dot{Z}$ such that maximum power will be delivered to the load. What is the maximum power delivered to $\dot{Z}$ if $\dot{E}_g = 10\angle 0°$?

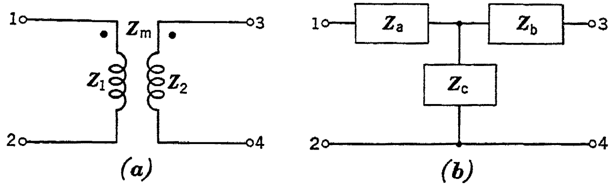


FIG. 11–23

11–26. A coil A with a resistance of 3 ohms and a coil B with a resistance of 4 ohms are so placed with respect to each other that they have mutual inductance. When coil A is connected to a 60-cycle, 110-volt source, the open-circuit voltage measured at the terminals of B is 30 volts. Under these conditions the current in coil A is 7.5 amp. When coil B is connected to the 60-cycle, 110-volt source, the open-circuit potential of coil A is 60 volts. Determine the self-inductances of coils A and B.

11–27. Find the equivalent T-section for Problem 11–18. Indicate values of the vector currents.

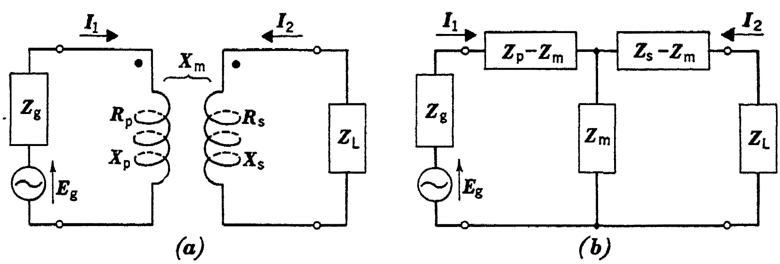


FIG 11–24

11–28. The inductively-coupled circuit in Fig. 11–23(*a*) is to be represented by the equivalent T-section in Fig. 11–23(*b*). Find $\dot{Z}_a$, $\dot{Z}_b$, and $\dot{Z}_c$ in terms of the primary self-impedance $\dot{Z}_1$, the secondary self-impedance $\dot{Z}_2$, and the mutual impedance $\dot{Z}_m$ by open-circuit and short-circuit tests, as indicated in Art. 10–5.

11–29. It has been shown that, with the winding and current senses indicated in Fig. 11–10(*a*), the equivalent T-section is shown in Fig. 11–10(*d*). If the positive sense of I_2 is reversed, as given in Fig. 11–24(*a*), show by writing loop equations that the equivalent T-section may be obtained by reversing the positive sense of $\dot{I}_2$ in the T-section of Fig. 11–10(*d*), as shown in Fig. 11–24(*b*).

11–30. If the positive sense of $\dot{I}_1$ is reversed from that given in Fig. 11–10(*a*), as indicated in Fig. 11–25(*a*), show by writing loop equations that the equivalent T-section may be obtained by reversing the positive sense of $\dot{I}_1$ from that indicated in Fig. 11–10(*d*), as given in Fig. 11–25(*b*).

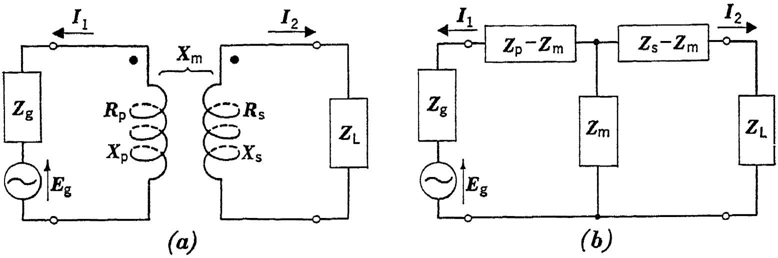


FIG. 11–25

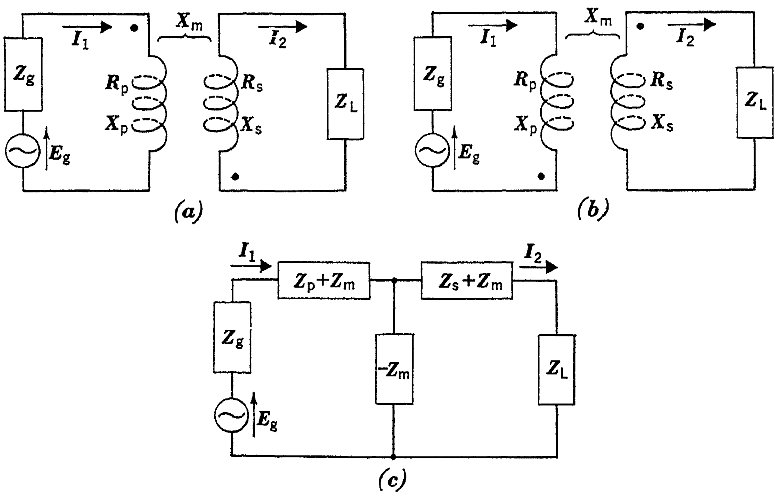


FIG. 11–26

11–31. If the locations of the coil dots are changed from those shown in Fig. 11–10(*a*), as indicated in Fig. 11–26(*a*) or Fig. 11–26(*b*), show by writing loop equations that the equivalent T-section may be obtained by changing $+\dot{Z}_m$ in Fig. 11–10(*d*) to $-\dot{Z}_m$, as given in Fig. 11–26(*c*).

11–32. Fig. 11–27 shows a circuit with an ideal transformer having a turns ratio $\frac{N_1}{N_2} = 2$. (*a*) Find the values of R', X_L', and X_C' such that, when they are connected in series across the terminals 1 and 2, the same current $\dot{I}_1$ will flow as in Fig. 11–27. (*b*) If the frequency is 60 cps, find the equivalent L' and C'.

11–33. Refer to Fig. 11–27. What is the required voltage E in order that the generator may deliver 300 watts?

11–34. If the generator in Fig. 11–27 has an internal impedance of $\dot{Z}_g = 300 + j400$, what turns ratio of the ideal transformer will give maximum power to the load?

11–35. An air-core transformer has the following constants:

$R_1 = 2$ ohms	$R_2 = 10$ ohms	$M = 0.003$ henry
$L_1 = 0.002$ henry	$L_2 = 0.01$ henry	

When the secondary is short-circuited, determine: (*a*) the potential that must be impressed on the primary to produce a secondary current of 10 amp at 500 cycles; (*b*) the magnitude of the primary current; (*c*) the primary power factor.

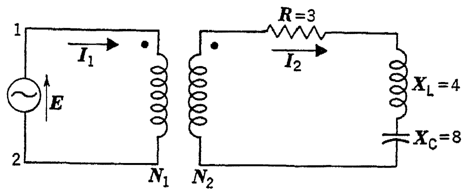


FIG. 11–27

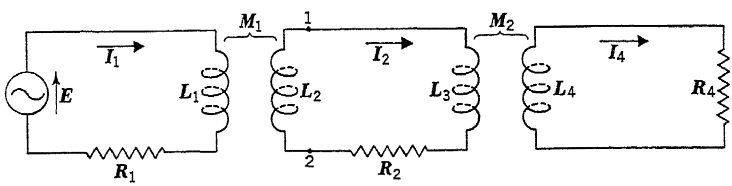


FIG. 11–28

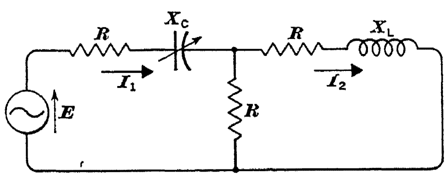


FIG. 11–29

11–36. In the circuit shown in Fig. 11–28,

$L_1 = 150\ \mu h$	$M_1 = 50\ \mu h$	$R_1 = 40$ ohms
$L_2 = 200\ \mu h$	$M_2 = 100\ \mu h$	$R_2 = 60$ ohms
$L_3 = 100\ \mu h$	$\dot{E} = 12\underline{/0°}$ volts	$R_4 = 80$ ohms
$L_4 = 150\ \mu h$	$\omega = 2 \times 10^6$ radians/sec	

Find I_4. (Hint: Either replace the portion of the circuit to the left of terminals 1 and 2 by an equivalent generator, using Thévenin's Theorem, or write three simultaneous equations, one for each mesh, and solve for I_4 by the use of determinants.)

11–37. A circuit with pure resistance coupling is shown in Fig. 11–29. Determine the value of X_C when I_2 is maximum.

11–38. Two loads are connected to the secondaries of a three-winding transformer as shown in Fig. 11–30. It is assumed that all of the flux set up by any one coil links completely with all three coils. Determine I_1, I_2, and I_3 for $f = 60$ cps.

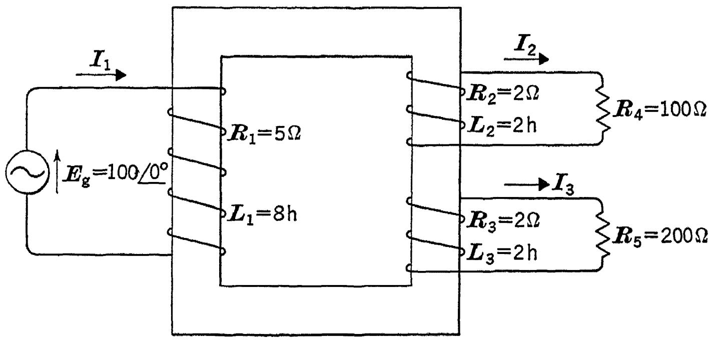


FIG. 11–30

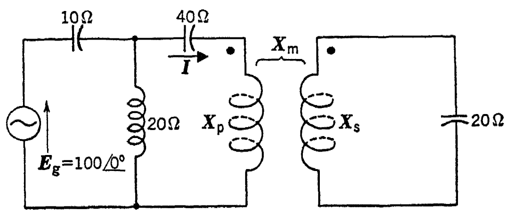


FIG. 11–31

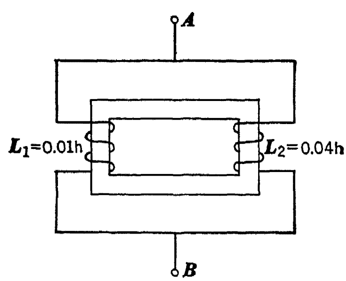


FIG. 11–32

11–39. If, in Fig. 11–31, $X_p = 40$ ohms, $X_s = 10$ ohms, and $k = 0.5$, find the current in the 40-ohm condenser.

11–40. If, in Fig. 11–32, the coefficient of coupling is $k = 0.5$, determine the value of the total inductance between A and B.

11–41. Find the maximum power in watts delivered to R_2 when X_C is the only variable in Fig. 11–33.

11–42. For the circuit shown in Fig. 11–34, carry out the following directions:

(*a*) Place the proper polarity marks on the coils with inductances L_p and L_s.
(*b*) Write the differential equations for determining the two loop currents i_1 and i_2.
(*c*) Write the corresponding steady state complex equations, assuming that all voltages and currents are sinusoidal.

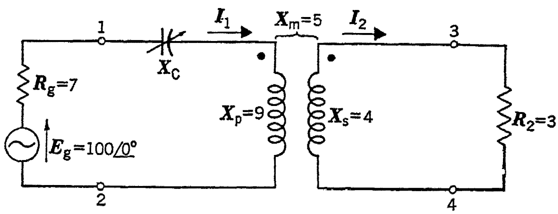


FIG. 11–33

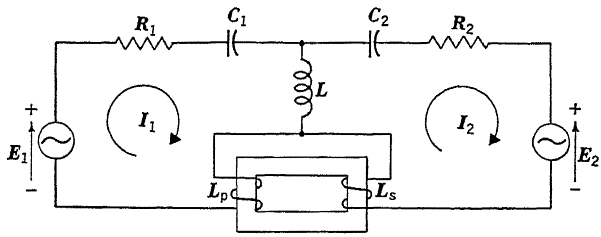


FIG. 11–34

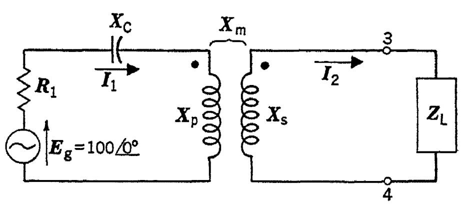


FIG. 11–35

11–43. Fig. 11–35 shows a coupled circuit with the following values:

$R_1 = 10$ ohms	$X_m = 25$ ohms
$X_C = 10$ ohms	$X_s = 100$ ohms
$X_p = 10$ ohms	$\dot{E}_g = 100\underline{/0°}$

(*a*) Replace the network to the left of terminals 3 and 4 by a Norton's generator.

(b) Determine the resistance and reactance of the load impedance $\dot{Z}_L$ for maximum power transfer to Z_L.

(c) What is the maximum power transferred to Z_L with the value determined in part (b)?

11–44. Fig. 11–36 shows a coupled circuit with the following values:

$L_1 = 100\ \mu h$ $\quad M = 50\ \mu h$
$L_2 = 100\ \mu h$ $\quad C = 100\ \mu\mu f$

(a) Determine the driving-point impedance at the terminals 1 and 2

(b) Find the frequency or frequencies (other than zero or infinity) at which the magnitude of I_1 is minimum.

(c) Find the frequency or frequencies (other than zero or infinity) at which the magnitude of I_1 is maximum.

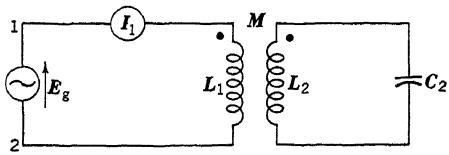


FIG. 11–36

11–45. Fig. 11–37 shows two coils in parallel. The mutual inductance is M. Assume that wattmeters W_1 and W_2 are to be connected in the usual way to measure the power in each branch as if there were no mutual effect. Also assume that the current vectors are as shown in Fig. 11–38.

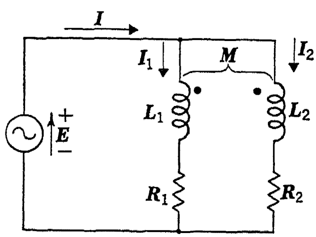


FIG. 11–37

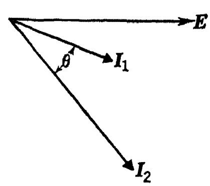


FIG. 11–38

Check the correctness of both of the following sets of answers:

(a) $W_1 = I_1^2 R_1 + I_1 I_2 X_M \sin\theta$

$W_2 = I_2^2 R_2 - I_1 I_2 X_M \sin\theta$

(b) $$W_1 = \frac{I_1^2[R_1 R_2^2 + R_2 X_M^2 + R_1 X_2^2 - X_M X_1 R_2 - X_M X_2 R_1]}{R_2^2 + (X_2 - X_M)^2}$$

$$W_2 = \frac{I_2^2[R_2 R_1^2 + R_1 X_M^2 + R_2 X_1^2 - X_M X_2 R_1 - X_M X_1 R_2]}{R_1^2 + (X_1 - X_M)^2}$$

CHAPTER 12

T-SECTIONS OF MODIFIED OR EQUIVALENT COUPLED CIRCUITS

Sometimes, it is of great assistance in the understanding of the physical nature of a phenomenon being studied if the actual apparatus can be represented by an equivalent circuit which has the same properties. For this reason, an iron-core transformer is often represented by a special T-section (commonly called the equivalent circuit* of a transformer in textbooks on alternating-current machinery). As indicated in the next paragraph, a coupled circuit may be represented by one of two special T-sections.

In the analysis of coupled circuits, it is often convenient to convert the coupled circuit into one of two modified (or equivalent) coupled circuits:

(*a*) In one the secondary quantities are modified by some factors but the primary quantities are left unchanged.

(*b*) In the other the primary quantities are modified by some factors but the secondary quantities are left unaltered.

When treating the commercial iron-core transformer, authors of books on alternating-current machinery often designate the modified-secondary coupled circuit of type (*a*) as the equivalent coupled circuit with the secondary quantities reduced (or referred) to the primary side by reduction factors; and they designate the modified-primary coupled circuit of type (*b*) as the equivalent coupled circuit with the primary quantities reduced (or referred) to the secondary side by reduction factors. The special T-sections derived from these modified (or equivalent) coupled circuits are very useful. They not only assist in the understanding of coupled circuits but they also can be simplified for the ordinary iron-core transformer, as indicated in Arts. 12–5 and 12–6, and the numerical work involved in such problems can thereby be reduced.

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* When an induction motor is viewed as a transformer coupled circuit with the stator winding as primary and the rotor winding as secondary, it is also often represented by an equivalent circuit similar to that of a transformer. From such an equivalent circuit, or special T-section, the circle diagram of an induction motor is derived.

These modified coupled circuits and their equivalent T-sections will be treated in this chapter. Also, an attempt will be made to correct the wrong impression, which some students will have after they have taken courses in alternating-current machinery and communication circuits, that the transformer theory presented in books on alternating-current machinery is entirely different from the coupled circuit theory which they learned from books on communication circuits. It will be shown in Art. 12–4 that the inductances in the two series arms of the special T-section in Fig. 12–5(*b*) are the same as the leakage inductances (or leakage reactances) in Fig. 12–7(*b*), which are commonly used in problems involving commercial power transformers.

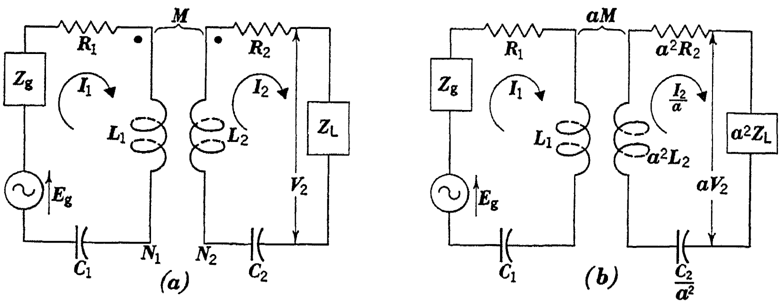


FIG. 12–1

12–1. Modified-Secondary Coupled Circuit.—A modified-secondary coupled circuit, Fig. 12–1(*b*), is an equivalent coupled circuit in which the secondary quantities of the original coupled circuit, Fig. 12–1(*a*), are modified by some factors but the primary quantities are left unchanged. Fig. 12–1(*a*) and Fig. 12–1(*b*) are equivalent as far as $\dot{I}_1$ is concerned. It is hoped that the material presented in this article will help to set up the special T-section or the modified-secondary T-section in a more understandable manner.

If the mutual impedance $\dot{Z}_m$ is multiplied by *any factor a* and the secondary impedance $\dot{Z}_2$ is multiplied by a^2, it can be shown that the primary current $\dot{I}_1$ will remain the same while the secondary current $\dot{I}_2$ will be transformed by the factor $\frac{1}{a}$ (the advantage of having the modified-secondary current in the form of

$\frac{\dot{I}_2}{a}$ will be considered in Arts. 12–5 and 12–6). This will now be demonstrated.

Consider the original coupled circuit, Fig. 12–1(*a*), where the impedances are defined as follows:

$\dot{Z}_1$ = all impedances in circuit 1, including that of the generator, with circuit 2 open;

$\dot{Z}_2$ = all impedances in circuit 2, including that of the load, with circuit 1 open;

$\dot{Z}_m = +j\omega M$.

It should be noted that the mutual impedance $\dot{Z}_m$ has been assumed to be a pure reactance. For simplicity, this assumption will be made throughout this chapter. When it is necessary to include the hysteresis and eddy-current losses of an iron-core transformer, a resistive component may be introduced into the mutual impedance, which becomes $\dot{Z}_m = R_m + j\omega M$.

Substituting $a\dot{Z}_m$ for $\dot{Z}_m$ and $a^2\dot{Z}_2$ for $\dot{Z}_2$ in equations (11–32) and (11–33), for the driving-point impedance and the transfer impedance, gives:

$$(\dot{Z}_1')_{\text{new}} = \dot{Z}_1 - \frac{a^2\dot{Z}_m^2}{a^2\dot{Z}_2} = \dot{Z}_1 - \frac{\dot{Z}_m^2}{\dot{Z}_2} = (\dot{Z}_1')_{\text{old}} \tag{12–1}$$

$$(\dot{Z}_{12}')_{\text{new}} = \frac{\dot{Z}_1 a^2\dot{Z}_2 - a^2\dot{Z}_m^2}{a\dot{Z}_m} = \frac{a(\dot{Z}_1\dot{Z}_2 - \dot{Z}_m^2)}{\dot{Z}_m} = a(\dot{Z}_{12}')_{\text{old}} \tag{12–2}$$

Since the driving-point impedance $\dot{Z}_1'$ has not been changed, as indicated in equation (12–1), the primary current will remain the same. Hence,

$$(\dot{I}_1)_{\text{new}} = \frac{\dot{E}_g}{(\dot{Z}_1')_{\text{new}}} = \frac{\dot{E}_g}{(\dot{Z}_1')_{\text{old}}} = (\dot{I}_1)_{\text{old}} \tag{12–3}$$

It is obvious from equation (12–2) that the transfer impedance $\dot{Z}_{12}'$ has been altered by the factor a and so the secondary current has been changed by the factor $\frac{1}{a}$. Thus,

$$(\dot{I}_2)_{\text{new}} = \frac{\dot{E}_g}{(\dot{Z}_{12}')_{\text{new}}} = \frac{\dot{E}_g}{a(\dot{Z}_{12}')_{\text{old}}} = \frac{(\dot{I}_2)_{\text{old}}}{a} \tag{12–4}$$

Fig. 12–1(*b*) shows a modified-secondary coupled circuit of Fig. 12–1(*a*). It can be seen that the secondary quantities are modified by the factors a and a^2, but the primary quantities remain the same as in the original circuit. The secondary quantities are changed in the following manner:

(*a*) Any current is changed by $\frac{1}{a}$, and

$$(I_2)_{\text{new}} = \frac{I_2}{a} \qquad (12\text{–}5)$$

(*b*) Any voltage is altered by a, and

$$(V_2)_{\text{new}} = aV_2 \qquad (12\text{–}6)$$

(*c*) Any resistance is modified by a^2, and

$$(R_2)_{\text{new}} = a^2R_2 \qquad (12\text{–}7)$$

(*d*) Any inductive reactance is changed by a^2, and

$$(X_{L2})_{\text{new}} = a^2X_{L2} \qquad (12\text{–}8)$$

Since $X_L = \omega L$, where L is the self-inductance of the coil, then

$$(L_2)_{\text{new}} = a^2L_2 \qquad (12\text{–}8a)$$

(*e*) Any capacitive reactance is modified by a^2, and

$$(X_{C2})_{\text{new}} = a^2X_{C2} \qquad (12\text{–}9)$$

Since $X_C = \frac{1}{\omega C}$, then

$$(C_2)_{\text{new}} = \frac{C_2}{a^2} \qquad (12\text{–}9a)$$

Attention is called here to the fact that, up to this point, no restriction has been placed upon the value of the factor a. In the case of an iron-core transformer, a is usually set equal to the turns ratio.

Example 12–1.—Fig. 12–2(*a*) shows a coupled circuit with two air-core coils having self-inductances L_1 and L_2. The known values are as follows:

$L_1 = 0.06$ henry	$\dot{E}_g = 10\underline{/0°}$
$L_2 = 0.01$ henry	$\dot{Z}_g = 3 + j4$
$M = 0.015$ henry	$\dot{Z}_L = 8 + j6$
$R_1 = 5$ ohms	$\omega = 100$ radians/sec
$R_2 = 1$ ohm	

Determine the modified-secondary coupled circuit for $a = 2$.

Solution.—According to equations (12–5) to (12–9), the secondary quantities of the original coupled circuit are modified by $a = 2$ and $a^2 = 4$. The modified-secondary coupled circuit for $a = 2$ is shown in Fig. 12–2(*b*). It should be noted that the primary quantities are left unchanged.

12–2. Modified-Secondary T-Section.—For lack of a better name, the special T-section derived from the modified-secondary coupled circuit shown in Fig. 12–1(*b*) shall be called the *modified-secondary T-section.* The advantages of such a T-section will be discussed in Arts. 12–5 and 12–6. It will be shown that the modified-secondary T-section is similar to the equivalent T-section of Fig. 12–1(*a*), as indicated in Fig. 12–3(*a*) and discussed in the chapter on "Mutual Induction and Coupled Circuits."

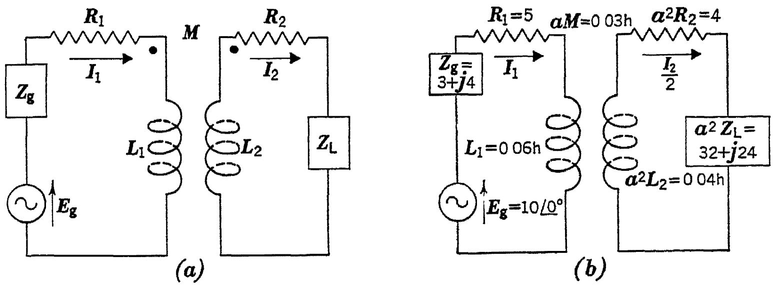


FIG. 12–2

For the winding and current senses specified in Fig. 12–1(*a*), Kirchhoff's emf equations are:

$$\dot{E}_g = \dot{Z}_1\dot{I}_1 - \dot{Z}_m\dot{I}_2 \tag{12–10}$$

$$0 = -\dot{Z}_m\dot{I}_1 + \dot{Z}_2\dot{I}_2 \tag{12–11}$$

where $\dot{I}_1$ and $\dot{I}_2$ are the actual primary and secondary currents. The equivalent T-section of Fig. 12–1(*a*), as derived previously on the basis of equations (12–10) and (12–11), is given in Fig. 12–3(*a*).

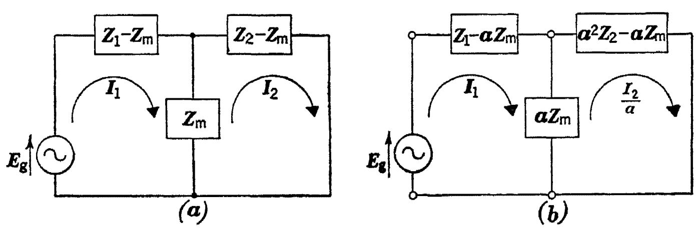


FIG. 12–3

As indicated in Art. 12–1, the modified-secondary coupled circuit is obtained from Fig. 12–1(*a*) by substituting $a\dot{Z}_m$ for $\dot{Z}_m$ and $a^2\dot{Z}_2$ for $\dot{Z}_2$. The modified-secondary T-section in Fig. 12–3(*b*) can then be determined from Fig. 12–3(*a*) by making the same substitutions.

It should be noted that in Fig. 12–3(*b*) $\dot{I}_1$ is the actual primary current; but, in general, $\frac{\dot{I}_2}{a}$ is not the actual secondary current. Fig. 12–3(*b*) will have more meaning when the characteristics of the branches are expressed in terms of *R*, *L*, *M*, and *a*. This will be done in the next article.

Example 12–2.—Give the modified-secondary T-section of the coupled circuit indicated in Example 12–1, for $a = 2$.

Solution.—With the winding and current senses specified in Fig 12–2(*a*), Kirchhoff's emf equations are [see equations (12–10) and (12–11)]:

$$\dot{E}_g = \dot{Z}_1\dot{I}_1 - \dot{Z}_m\dot{I}_2$$
$$0 = -\dot{Z}_m\dot{I}_1 + \dot{Z}_2\dot{I}_2$$

For this set of equations, Fig. 12–3(*b*) holds. The several impedances are:

$$\dot{Z}_1 = (3 + j4) + (5 + j6) = 8 + j10$$
$$\dot{Z}_2 = (1 + j1) + (8 + j6) = 9 + j7$$
$$\dot{Z}_m = +j1.5$$
$$\dot{Z}_1 - a\dot{Z}_m = (8 + j10) - (j3) = 8 + j7$$
$$a^2\dot{Z}_2 - a\dot{Z}_m = (36 + j28) - (j3) = 36 + j25$$

The modified-secondary T-section of the coupled circuit shown in Fig. 12–2 is indicated in Fig. 12–4.

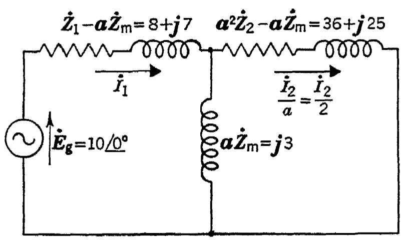


FIG. 12–4

12–3. Modified-Secondary T-Section With *R*, *L*, *M*, and *a*. To give more meaning to Fig. 12–3(*b*), the characteristics of the several branches will now be expressed in terms of *R*, *L*, *M*, and *a*.

Consider the case of a transformer connecting a load $\dot{Z}_L$ and a generator with an internal impedance $\dot{Z}_g$, as in Fig. 12–5(*a*). Since the purpose here is to show how a transformer can be represented by a modified-secondary T-section, the generator and the load will be kept separate from the rest of the circuit. Hence, $\dot{Z}_1$ and $\dot{Z}_2$ in Fig. 12–3 (*b*) now refer only to the transformer. Thus,

$$\dot{Z}_1 = \dot{Z}_p = R_p + j\omega L_p$$
$$\dot{Z}_2 = \dot{Z}_s = R_s + j\omega L_s$$
$$\dot{Z}_m = +j\omega M$$

where R_p = resistance of primary coil;
L_p = self-inductance of primary coil;
R_s = resistance of secondary coil;
L_s = self-inductance of secondary coil.

The series arms, as indicated in Fig. 12–5(*b*), are $(\dot{Z}_1 - a\dot{Z}_m) = \dot{Z}_p - a\dot{Z}_m$ and $(a^2\dot{Z}_2 - a\dot{Z}_m) = a^2\dot{Z}_s - a\dot{Z}_m$; or

$$(\dot{Z}_1 - a\dot{Z}_m) = R_p + j\omega(L_p - aM) \qquad \textbf{(12–12)}$$

$$(a^2\dot{Z}_2 - a\dot{Z}_m) = a^2R_s + j\omega(a^2L_s - aM) = a^2R_s + ja^2\omega\left(L_s - \frac{M}{a}\right) \qquad \textbf{(12–13)}$$

It should be noted that the value for the factor a in Fig. 12–5(*b*) is still unrestricted. For an iron-core transformer, it is convenient to set a equal to the turns ratio.

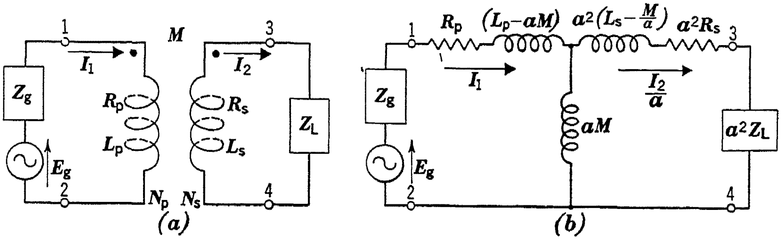


FIG. 12–5

In problems on iron-core transformers in books on alternating-current machinery it is customary to use leakage reactances which are derived from leakage inductances. The series arm inductances $(L_p - aM)$ and $\left(L_s - \frac{M}{a}\right)$ will be changed to leakage inductances in the next article.

Example 12–3.—The transformer coupled circuit shown in Fig. 12–6(*a*) has the following values:

$L_p = 8$ henrys	$\dot{E}_g = 1000\underline{/0°}$
$R_p = 70$ ohms	$R_g = 30$ ohms
$L_s = 2$ henrys	$R_L = 80$ ohms
$R_s = 20$ ohms	$a = 2$
$k = 0.98$	$\omega = 1000$ radians/sec

For simplicity, the iron losses of the transformer are assumed to be negligible. (*a*) Find the actual primary current $\dot{I}_1$ and the actual secondary current $\dot{I}_2$. (*b*) Determine the modified-secondary T-section with series arm inductances for the transformer alone.

Solution.—(*a*) The mutual impedance is

$$M = k\sqrt{L_pL_s} = 0.98\sqrt{8 \times 2} = 3.92$$

and

$$\dot{Z}_m = j\omega M = j3{,}920$$

The total impedance or self-impedance of the primary circuit, including R_g, is

$$\dot{Z}_1 = (30 + j0) + (70 + j8{,}000) = 100 + j8{,}000 = 8{,}000\underline{/89.3^\circ}$$

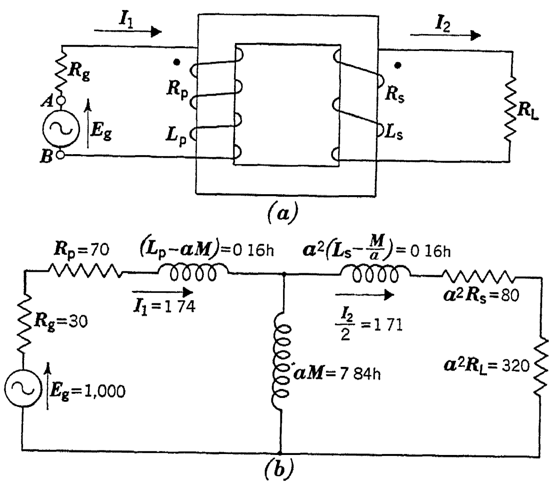


FIG. 12–6

The total impedance or self-impedance of the secondary circuit, including R_L, is

$$\dot{Z}_2 = (80 + j0) + (20 + j2{,}000) = 100 + j2{,}000 = 2{,}000\underline{/87.1^\circ}$$

The driving-point impedance at the terminals A and B is

$$\dot{Z}_1' = \dot{Z}_1 - \frac{\dot{Z}_m^2}{\dot{Z}_2} = (100 + j8{,}000) - \frac{(j3{,}920)^2}{100 + j2{,}000} = 575\underline{/32.7^\circ}$$

The actual primary current is

$$\dot{I}_1 = \frac{\dot{E}_g}{\dot{Z}_1'} = \frac{1{,}000\underline{/0^\circ}}{575\underline{/32.7^\circ}} = 1.74\underline{/-32.7^\circ}$$

The actual secondary current is

$$\dot{I}_2 = \frac{\dot{Z}_m}{\dot{Z}_2}\dot{I}_1 = \frac{3{,}920\underline{/90^\circ}}{2{,}000\underline{/87.1^\circ}} \times 1.74\underline{/-32.7^\circ} = 3.41\underline{/-29.8^\circ}$$

(*b*) It can be seen from Fig. 12–6(*a*) that Kirchhoff's emf equations have the forms of equations (12–10) and (12–11). By following the notation in Fig. 12–5(*b*), which holds for equations (12–10) and (12–11) and Fig. 12–6(*a*), we can represent the modified-secondary T-section for $a = 2$ and negligible iron losses as shown in Fig. 12–6(*b*).

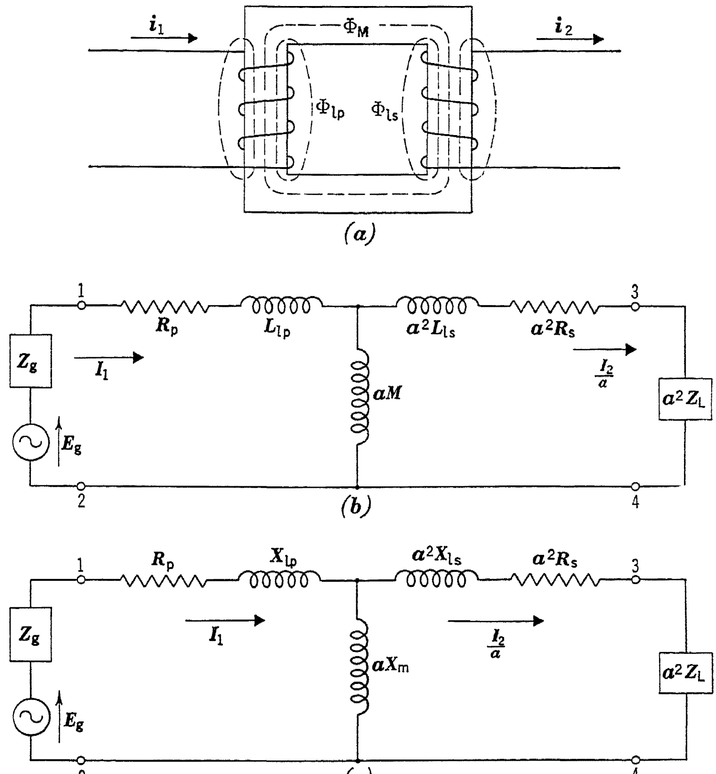


FIG. 12–7

12–4. Modified-Secondary T-Section With Leakage Reactances.—Fig. 12–7(*a*) shows a two-winding transformer where the total primary flux Φ_1 may be considered as consisting of the mutual flux component Φ_M and the leakage flux component Φ_{lp}; and, similarly, the total secondary flux Φ_2 may be considered as consisting of the mutual flux component Φ_M and the leakage flux component Φ_{ls}. These component fluxes induce voltages in the primary and secondary coils. When presenting the subject of iron-core transformers, authors of books on alternating-current machinery usually represent the induced voltages caused by leakage fluxes as equivalent IX drops or leakage reactance drops. This is permissible because the leakage fluxes (having a portion of the flux path in air), and consequently their induced voltages, are practically proportional to the currents which set up the leakage fluxes. It will be shown that the series arm inductances $(L_p - aM)$ and $\left(L_s - \frac{M}{a}\right)$ derived from the so-called classical coupled circuit theory may be considered as leakage inductances. For an iron-core transformer, this can be done quite easily when the factor a is set equal to the turns ratio.

First, it is desirable to have expressions for L_p, L_s, and M which will be used in the demonstration of the equivalence. If the permeability of the medium is constant, the self-inductance and mutual inductance may be defined as flux linkages per unit current. Thus,

$$L_p = \frac{N_p \Phi_1}{i_1} \tag{12–14}$$

$$L_s = \frac{N_s \Phi_2}{i_2} \tag{12–15}$$

$$M = \frac{N_s \Phi_M}{i_1} = \frac{N_p \Phi_M}{i_2} \tag{12–16}$$

where i_1 is the primary current and i_2 is the secondary current.

Then, for $a = \frac{N_p}{N_s}$, the series arm inductances may be shown to be leakage inductances with the aid of equations (12–14) to (12–16). Hence,

$$(L_p - aM) = \frac{N_p \Phi_1}{i_1} - \frac{N_p}{N_s} \times \frac{N_s \Phi_M}{i_1} = \frac{N_p(\Phi_1 - \Phi_M)}{i_1}$$

or

$$(L_p - aM) = \frac{N_p \Phi_{lp}}{i_1} = L_{lp} \tag{12–17}$$

$$\left(L_s - \frac{M}{a}\right) = \frac{N_s \Phi_2}{i_2} - \frac{N_s}{N_p} \times \frac{N_p \Phi_M}{i_2} = \frac{N_s(\Phi_2 - \Phi_M)}{i_2}$$

or

$$\left(L_s - \frac{M}{a}\right) = \frac{N_s \Phi_{ls}}{i_2} = L_{ls} \tag{12–18}$$

where $\Phi_{lp} = (\Phi_1 - \Phi_M)$ = primary leakage flux;
$\Phi_{ls} = (\Phi_2 - \Phi_M)$ = secondary leakage flux;
$L_{lp} = (L_p - aM)$ = primary leakage inductance;
$L_{ls} = \left(L_s - \frac{M}{a}\right)$ = secondary leakage inductance.

The primary and secondary leakage reactances are, respectively:

$$X_{lp} = \omega L_{lp} \quad (12\text{–}19)$$

$$X_{ls} = \omega L_{ls} \quad (12\text{–}20)$$

Hence, for a equal to the turns ratio, Fig. 12–5(b) may be represented by Fig. 12–7(b) with leakage inductances or by Fig. 12–7(c) with leakage reactances. The reader is reminded here that iron losses have been assumed to be zero or negligible in Fig. 12–7.

12–5. Differences in T-Sections.—An additional discussion concerning Fig. 12–3(a) and Fig. 12–3(b) might be helpful to clear up the differences between these two T-sections. Some advantages of the modified-secondary T-section in Fig. 12–3(b) or Fig. 12-7(b) will be brought out in the discussion.

In the equivalent T-section in Fig. 12-3(a), $\dot{I}_1$ and $\dot{I}_2$ are the actual currents in the primary and secondary circuits of the actual transformer. Normally, their magnitudes are different. As an example, the relative values may be studied in the case of a good commercial iron-core transformer where $a = \frac{N_p}{N_s} = 2$. The secondary current I_2 is approximately twice the primary current I_1. It is quite obvious then that the impedance $\dot{Z}_m$ of the shunt branch of Fig. 12–3(a), in general, cannot be neglected for simplification. The currents are determined from the following expressions, which usually require a good deal of algebraic work:

$$\dot{I}_1 = \frac{\dot{E}_g \dot{Z}_2}{\dot{Z}_1 \dot{Z}_2 - \dot{Z}_m^2} \quad (12\text{–}21)$$

$$\dot{I}_2 = \frac{\dot{E}_g \dot{Z}_m}{\dot{Z}_1 \dot{Z}_2 - \dot{Z}_m^2} = \frac{\dot{Z}_m}{\dot{Z}_2} \dot{I}_1 \quad (12\text{–}22)$$

For the modified-secondary T-section in Fig. 12-3(b), however, the impedance $a\dot{Z}_m$ of the shunt branch may be neglected under practical operating conditions of a power transformer. Thus, $k \approx 1$, $a = \frac{N_p}{N_s}$, and $a\dot{Z}_m$ is a large impedance compared with $(a^2\dot{Z}_2 - a\dot{Z}_m)$. In the operation of many power transformers, $a\dot{Z}_m$ is a large impedance compared with $(a^2\dot{Z}_2 - a\dot{Z}_m)$, or rather the absolute values are such that $|\dot{Z}_m| \gg |a\dot{Z}_2 - \dot{Z}_m|$, because the secondary resistance R_s is small compared with the secondary self-inductive reactance X_s and the magnitude of the load impedance $\dot{Z}_L$ is small compared with X_s. This will now be demonstrated. Since $\frac{L_p}{L_s} = \frac{X_p}{X_s} = \left(\frac{N_p}{N_s}\right)^2 = a^2$,

$$X_p = a^2 X_s$$

and

$$X_m = k\sqrt{X_p X_s} = kaX_s \approx aX_s$$

If $\dot{Z}_m = jX_m \approx jaX_s$ and $\dot{Z}_s = R_s + jX_s \approx jX_s$, then the condition for $|\dot{Z}_m| \gg |a\dot{Z}_2 - \dot{Z}_m|$ is

$$|jaX_s| \gg |a[jX_s + (R_L + jX_L)] - jaX_s|$$

or

$$|jaX_s| \gg |a(R_L + jX_L)|$$

Hence,

$$aX_s \gg a\sqrt{R_L^2 + X_L^2}$$

or

$$X_s \gg Z_L$$

If the impedance $a\dot{Z}_m$ of the shunt branch is so large that the shunt branch can be omitted, then the resultant circuit is a simple series circuit. For comparison, consider again the commercial iron-core transformer for which $a = \frac{N_p}{N_s} = 2$. Although the actual secondary current I_2 is approximately twice the primary current I_1, the value of $\frac{I_2}{a} = \frac{I_2}{2}$ is nearly equal to I_1. This may be interpreted to mean that the current through the shunt branch must be small or $a\dot{Z}_m$ must be large compared with $(a^2\dot{Z}_2 - a\dot{Z}_m)$. Hence, as far as I_1 and I_2 are concerned, the shunt branch may be omitted from the diagram for good measure; and $\dot{I}_1$ can be found very easily by solving the remaining series circuit in Fig. 12–3(*b*). If the branch with impedance aX_m can be omitted in Fig. 12–7(*c*) and $\dot{Z}_g = R_g + jX_g$ and $\dot{Z}_L = R_L + jX_L$, then $\dot{I}_1$ may be found by solving the simple series circuit. Thus,

$$\dot{I}_1 = \frac{\dot{E}_g}{(R_g + R_p + a^2R_s + a^2R_L) + j(X_g + X_{lp} + a^2X_{ls} + a^2X_L)} \quad (12\text{–}23)$$

and the approximate value of I_2 may be determined from the very simple relationship

$$I_2 = aI_1 \tag{12-24}$$

It may be concluded that, in general, the branch with impedance Z_m in Fig. 12-3(*a*) cannot be neglected unless the conditions are so special that $Z_m = \infty$ or $(\dot{Z}_2 - \dot{Z}_m) = 0$. On the other hand, the branch with impedance aZ_m in Fig. 12-3(*b*) may be neglected under *practical* conditions of commercial iron-core transformers. The possibility of simplifying a transformer to a simple series circuit under practical conditions and thereby making it much easier to find I_1 and I_2 may be considered a good reason for the use of modified T-sections.

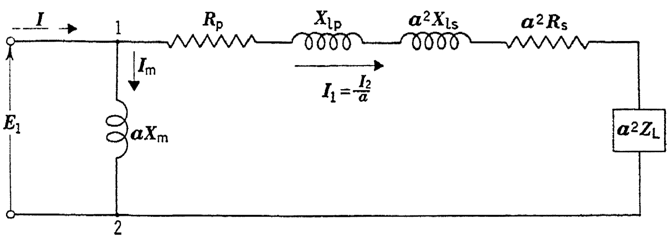


FIG. 12-8

12-6. Approximate Equivalent Circuit.—In dealing with iron-core transformers, it may not always be permissible to neglect entirely the branch with reactance aX_m in Fig. 12-7(*c*). Since the total voltage drop across the primary resistance R_p and the primary leakage reactance X_{lp} of Fig. 12-7(*c*) is usually small, it is a common practice in textbooks on alternating-current machinery to move the shunt branch to the terminals 1 and 2, as indicated in Fig. 12-8. This forms the so-called *approximate equivalent circuit of a power transformer*. The current flowing through the branch with reactance aX_m is the magnetizing current which sets up the mutual flux. This branch is sometimes called the magnetizing branch.

When the applied emf E_1 in Fig. 12-8 is held constant, the approximate equivalent circuit furnishes a simple method for finding $I_1 = \frac{I_2}{a}$ and also a satisfactory way of including the effect of the magnetizing current I_m.

If it is necessary to include the hysteresis and eddy-current losses of the iron core of a power transformer, a resistive component may be added to the mutual impedance $\dot{Z}_m = R_m + j\omega M$. However, the shunt branch is usually represented as a parallel circuit consisting of a pure resistive branch for the iron losses and a pure inductive branch for the magnetizing current. The total no-

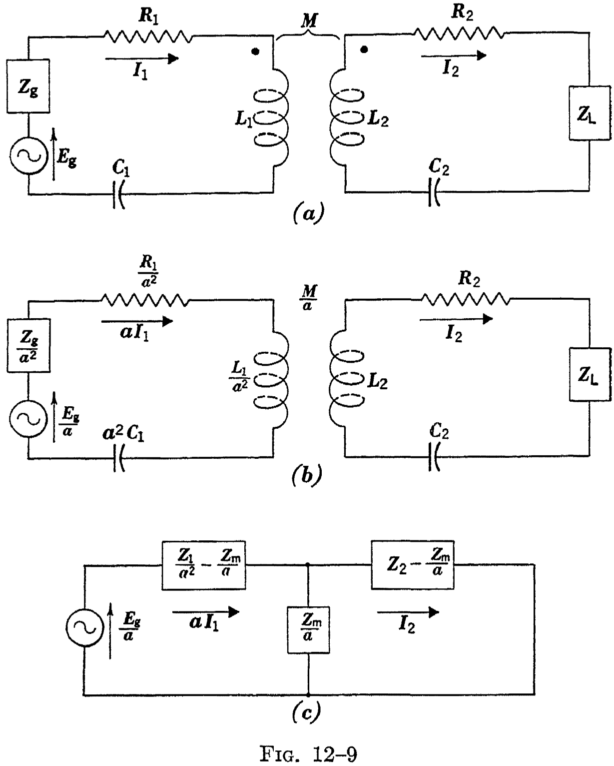


FIG. 12–9

load current is the vector sum of the currents flowing through these two branches. The student is referred to textbooks on alternating-current machinery for treatment of iron losses in iron-core transformers.

12–7. Modified-Primary Coupled Circuit.—A modified-primary coupled circuit, Fig. 12–9(*b*), is an equivalent coupled circuit in which the primary quantities of the original coupled circuit in

Fig. 12–9(a) are modified by some factors but the secondary quantities are left unaltered. Fig. 12–9(a) and Fig. 12–9(b) are equivalent as far as $\dot{I}_2$ is concerned. If $\dot{E}_g$ is multiplied by *any factor* $\frac{1}{a}$, $\dot{Z}_m$ is multiplied by $\frac{1}{a}$, and $\dot{Z}_1$ is multiplied by $\frac{1}{a^2}$, it can be shown that the secondary current $\dot{I}_2$ remains the same while the primary current $\dot{I}_1$ is changed by the factor a. This will now be demonstrated.

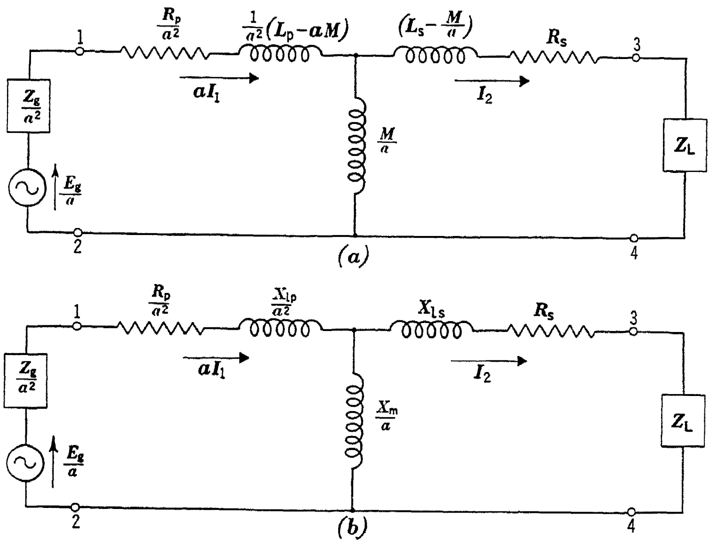


FIG. 12-10

The expressions for the actual primary and secondary currents in Fig. 12–9(a) are:

$$\dot{I}_1 = \frac{\dot{E}_g\dot{Z}_2}{\dot{Z}_1\dot{Z}_2 - \dot{Z}_m^2} \tag{12–25}$$

$$\dot{I}_2 = \frac{\dot{E}_g\dot{Z}_m}{\dot{Z}_1\dot{Z}_2 - \dot{Z}_m^2} \tag{12–26}$$

Substitution of $\frac{\dot{E}_g}{a}$ for $\dot{E}_g$, $\frac{\dot{Z}_m}{a}$ for $\dot{Z}_m$, and $\frac{\dot{Z}_1}{a^2}$ for $\dot{Z}_1$ in equations (12–25) and (12–26) gives:

$$(\dot{I}_1)_{\text{new}} = \frac{\dfrac{\dot{E}_g}{a} \times \dot{Z}_2}{\dfrac{\dot{Z}_1\dot{Z}_2}{a^2} - \dfrac{\dot{Z}_m^2}{a^2}} = a\left[\frac{\dot{E}_g\dot{Z}_2}{\dot{Z}_1\dot{Z}_2 - \dot{Z}_m^2}\right] = a(\dot{I}_1)_{\text{old}} \qquad (12\text{–}27)$$

$$(\dot{I}_2)_{\text{new}} = \frac{\dfrac{\dot{E}_g}{a} \times \dfrac{\dot{Z}_m}{a}}{\dfrac{\dot{Z}_1\dot{Z}_2}{a^2} - \dfrac{\dot{Z}_m^2}{a^2}} = \frac{\dot{E}_g\dot{Z}_m}{\dot{Z}_1\dot{Z}_2 - \dot{Z}_m^2} = (\dot{I}_2)_{\text{old}} \qquad (12\text{–}28)$$

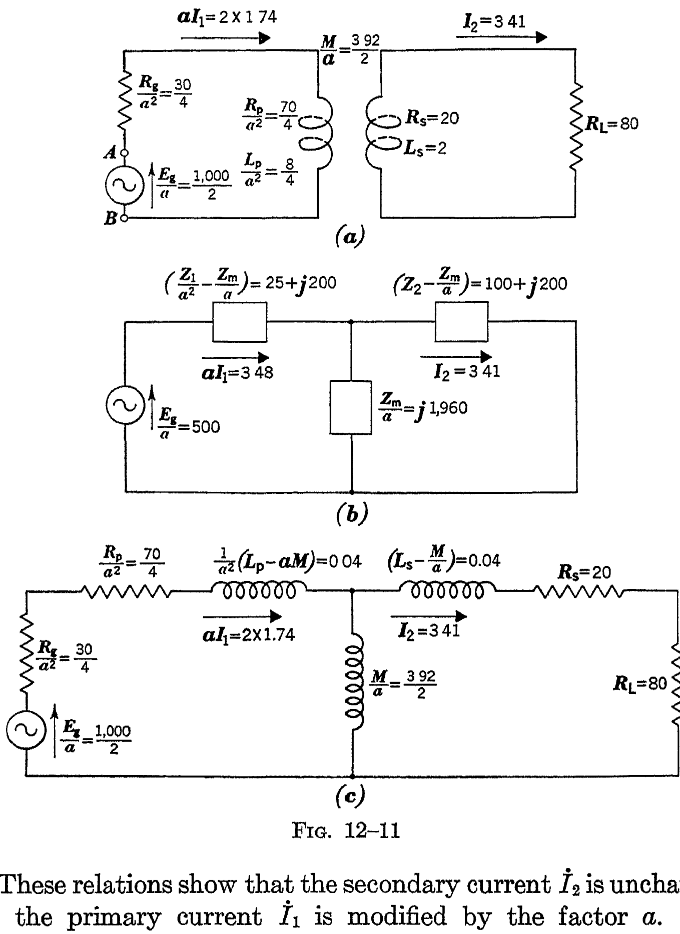


FIG. 12–11

These relations show that the secondary current $\dot{I}_2$ is unchanged and the primary current $\dot{I}_1$ is modified by the factor a. Fig. 12–9(b) shows the modified-primary coupled circuit where it can be seen that the primary quantities are modified by the factors a and a^2 and the secondary quantities are left unaltered. The modified-primary T-section is given in Fig. 12–9(*c*). It should be noted that, so far in this article, no restriction has been placed upon the factor a.

For the iron-core transformer where a is set equal to the turns ratio, Fig. 12–10(*a*) shows the modified-primary T-section with leakage inductances; and the modified-primary T-section with leakage reactances is given in Fig. 12–10(*b*).

Example 12–4.—For the conditions given in Example 12–3, determine the characteristics of the modified-primary coupled circuit and the modified-primary T-section.

Solution.—The modified-primary coupled circuit and its modified-primary T-section are shown in Figs. 12–11(*a*) and 12–11(*b*). The modified-primary T-section of the transformer with inductances in the branches is given in Fig. 12–11(*c*).

PROBLEMS

12–1. The coupled circuit shown in Fig. 12–12 has two coils with self-inductances L_1 and L_2. The known values are as follows:

$R_1 = 10$ ohms $\dot{E}_g = 10\underline{/0°}$
$L_1 = 0.4$ henry $Z_g = 0$
$R_2 = 5$ ohms $Z_L = 0$
$L_2 = 0.1$ henry $\omega = 500$ radians/sec
$M = 0.1$ henry

Determine the characteristics of the modified-secondary coupled circuit, similar to that shown in Fig. 12–1(*b*), for $a = 3$.

12–2. Repeat Problem 12–1 when $\dot{Z}_g = 300 + j400$ and $\dot{Z}_L = 600 + j800$.

12–3. The coupled circuit shown in Fig 12–1(*a*) has the following values:

$R_1 = 10$ ohms $M = 0.1$ henry
$L_1 = 0.4$ henry $\dot{E}_g = 10\underline{/0°}$
$C_1 = 200\ \mu f$ $\dot{Z}_g = 300 + j400$
$R_2 = 5$ ohms $\dot{Z}_L = 600 + j800$
$L_2 = 0.1$ henry $\omega = 500$ radians/sec
$C_2 = 100\ \mu f$

Determine the modified-secondary coupled circuit for $a = 2$.

12–4. Find the characteristics of the modified-secondary T-section for the circuit given in Problem 12–1.

12–5. Find the characteristics of the modified-secondary T-section for the conditions specified in Problem 12–3.

12–6. Determine the characteristics of the modified-secondary T-section, similar to that shown in Fig. 12–5(*b*), for Problem 12–1.

12–7. Derive expressions for the series arms, similar to those of equations (12–12) and (12–13), when the mutual impedance has the form $\dot{Z}_m = R_m + j\omega M$.

12–8. The coils in the coupled circuit shown in Fig. 12–13 have self-inductances L_p and L_s. The known values are as follows:

$R_p = 10$ ohms	$M = 0.1$ henry
$L_p = 0.4$ henry	$\dot{E}_g = 1\angle 0°$
$R_s = 2$ ohms	$\omega = 100$ radians/sec
$L_s = 0.1$ henry	

Determine the modified-secondary T-section with $\dot{Z}_1 - a\dot{Z}_m = 0$. This problem is to show that a can be a complex number.

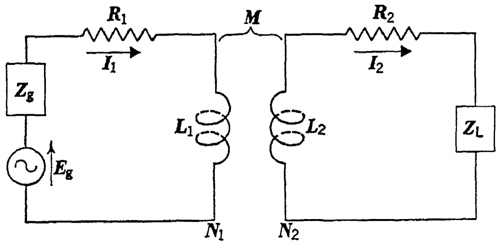


FIG. 12–12

12–9. Repeat Problem 12–8 with $L_s - \frac{M}{a} = 0$ instead of $\dot{Z}_1 - a\dot{Z}_m = 0$. This problem is to show that a may be so selected that the series-arm element representing the secondary leakage inductance is equal to zero.

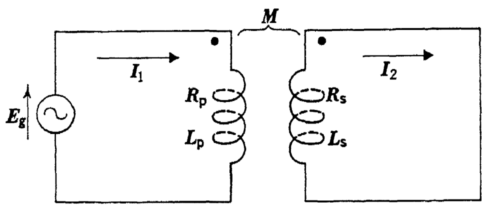


FIG. 12–13

12–10. The circuit elements shown in Fig. 12–14 have the following values, where L_p and L_s are the self-inductances of the coils.

$R_p = 10$ ohms	$a = 2$
$L_p = 0.4$ henry	$\dot{E}_g = 100\angle 0°$
$R_s = 2.5$ ohms	$R_g = 1{,}000$ ohms
$L_s = 0.1$ henry	$R_L = 50$ ohms
$k = 0.98$	$\omega = 500$ radians/sec

(*a*) Find the primary and secondary leakage inductances, L_{lp} and L_{ls}, of the transformer.

(b) Find the primary and secondary leakage reactances, X_{lp} and X_{ls}, of the transformers.

(c) Determine the modified-secondary T-section, similar to that shown in Fig. 12–5(b).

(d) Find I_1 and I_2.

12–11. The high side or primary of a 50-kva, 2300/230-volt, 60-cycle transformer consists of 500 turns. Under conditions approaching those of an ideal transformer, determine: (a) the number of turns in the secondary winding and (b) the rated high-side and low-side currents.

12–12. A 50-kva, 60-cycle, 440/220-volt transformer has the following values: For the high side or primary, $R_p = 0.04$ ohm and $X_{lp} = 0.125$ ohm; for the low side or secondary, $R_s = 0.0095$ ohm and $X_{ls} = 0.031$ ohm. (a) What are the equivalent values of R_s and X_{ls} when reduced or referred to the primary side? (b) What are the equivalent values of R_p and X_{lp} when reduced or referred to the secondary side?

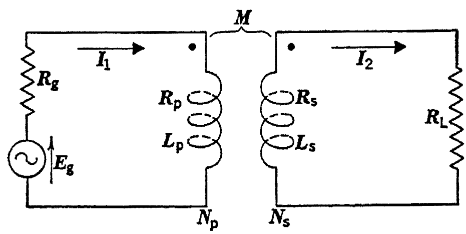


FIG. 12–14

12–13. Fig. 12–14 shows a coupled circuit with the following values:

$R_p = 10$ ohms	$a = 2$
$L_p = 4$ henrys	$\dot{E}_g = 100\underline{/0°}$
$R_s = 2.5$ ohms	$R_g = 1{,}000$ ohms
$L_s = 1$ henry	$R_L = 2$ ohms
$k = 0.98$	$\omega = 500$ radians/sec

(a) Determine the characteristics of the modified-secondary T-section with leakage reactances.

(b) Find $\dot{I}_1$ and $\dot{I}_2$ without neglecting the shunt branch.

(c) Find $\dot{I}_1$ and $\dot{I}_2$, neglecting the shunt branch entirely.

12–14. Reduce the circuit specified in Problem 12–13 to a circuit similar to that shown in Fig. 12–8. Find E_1, I_1, I_2, I_m, and I.

12–15. Refer to the circuit given in Problem 12–1 and determine: (a) the characteristics of the modified-primary coupled circuit for $a = 3$; (b) the characteristics of the modified-primary T-section with leakage reactances; (c) I_1 and I_2.

12–16. Refer to the circuit given in Problem 12–10 and determine: (a) the characteristics of the modified-primary T-section with leakage reactances; (b) I_1 and I_2.

12–17. If the reluctances of the magnetic paths for the primary and secondary fluxes are equal, then the ratio of the coil inductances will be proportional to the ratio of the square of the number of turns, or

$$\frac{L_p}{L_s} = \frac{N_p^2}{N_s^2}$$

For $a = \frac{N_p}{N_s}$, show that $(L_p - aM) = (1 - k)L_p$ and $\left(L_s - \frac{M}{a}\right) = (1 - k)L_s$, where k is the coefficient of coupling. Also, show that $L_p = a^2 L_s$ and $aM = kL_p$.

CHAPTER 13

IMPEDANCE TRANSFORMATION

Impedance transformation or impedance matching for maximum power transfer plays an important role in communication circuits. It has been shown in Art. 10–8 on Maximum Power Transfer that the maximum power is delivered by a generator with fixed internal impedance when the load impedance is adjusted to equal the conjugate of the generator impedance. In the application of some vacuum tubes, the internal impedance of the tube is considered a pure high resistance and so the apparent load impedance across the terminals of the vacuum tube should be a pure resistance of the same magnitude. The problem of impedance matching is one of the most important applications of parallel resonant circuits and of coupled circuits. Appreciable power outputs are obtained from some vacuum tubes with the use of parallel resonant circuits.

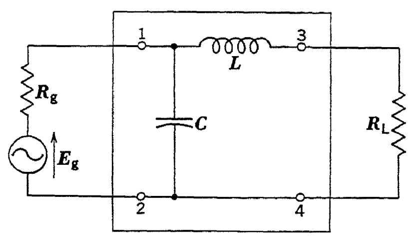


FIG. 13–1

It is the purpose of this chapter: (*a*) to give a method for transforming a small fixed load resistance into a larger resistance for matching; (*b*) to indicate a circuit which will transform a large fixed load resistance into a smaller resistance to match that of a generator; (*c*) to consider the iron-core transformer as a transforming device at audio frequencies; and (*d*) to discuss the use of tuned coupled circuits at radio frequencies for impedance matching.

13–1. Transforming Small *R* Into Larger *R*.—For purposes of impedance matching at a single frequency, a parallel circuit or its equivalent is often used to convert a fixed load or branch resistance into a definite higher resistance at parallel resonance (power factor = unity). This method of matching is useful when the internal resistance of a generator, such as that of some vacuum tubes, is high. Let it be required to convert a fixed load resistance R_L of Fig. 13–1 into a higher fixed resistance $R_{ar} = R_g$ at the terminals 1 and 2. It can be seen that the network to the right of 1 and 2 is a parallel circuit, as indicated in Fig. 13–2. Since R_L and $R_{ar} = R_g$ are fixed and given, the problem becomes one of determining the expressions for X_L and X_C, in terms of R_L and R_{ar}, to give the required matching.

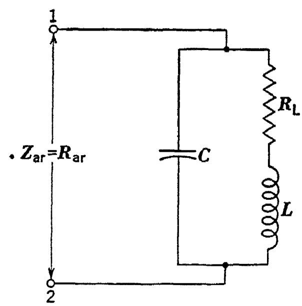


FIG. 13–2

The impedance of the parallel combination in Fig. 13–2 is

$$\dot{Z} = \frac{R_L X_C^2}{R_L^2 + (X_L - X_C)^2} + j\frac{X_C^2 X_L - X_C X_L^2 - R_L^2 X_C}{R_L^2 + (X_L - X_C)^2} \tag{13–1}$$

At anti-resonance or unity power factor resonance, this impedance becomes a pure resistance

$$R_{ar} = \frac{R_L X_C^2}{R_L^2 + (X_L - X_C)^2} \tag{13–2}$$

An expression for X_L will be determined first. Setting the j term of equation (13–1) equal to zero, and factoring out an X_C, we obtain:

$$X_C X_L - X_L^2 - R_L^2 = 0 \tag{13–3}$$

This gives

$$X_C X_L = R_L^2 + X_L^2 \tag{13–4}$$

$$(X_C - X_L) = \frac{R_L^2}{X_L} \tag{13-5}$$

$$(X_C - X_L)^2 = \frac{R_L^4}{X_L^2} \tag{13-6}$$

Substitution of the values from equations (13–4) and (13–6) in equation (13–2) gives:

$$R_{ar} = \frac{R_L X_C^2}{R_L^2 + \frac{R_L^4}{X_L^2}} = \frac{X_C^2 X_L{}^2}{R_L\ (R_L^2 + X_L^2)} = \frac{R_L^2 + X_L^2}{R_L} \tag{13-7}$$

which indicates that R_{ar} is always greater than R_L.

From the last equation,

$$X_L = \sqrt{R_{ar} R_L - R_L^2} \tag{13-8}$$

To find an expression for X_C, rewrite equation (13–4) in the following form:

$$X_C X_L = \frac{R_L^2 + X_L^2}{R_L} \times R_L = R_{ar} R_L \tag{13-9}$$

With the aid of equation (13–8),

$$X_C = \frac{R_{ar} R_L}{\sqrt{R_{ar} R_L - R_L^2}} = R_{ar} \sqrt{\frac{R_L}{R_{ar} - R_L}} \tag{13-10}$$

Equations (13–8) and (13–10) for finding X_L and X_C can be simplified when $R_{ar} \gg R_L$. Then,

$$X_L = \sqrt{R_L (R_{ar} - R_L)} \approx \sqrt{R_{ar} R_L} \tag{13-11}$$

$$X_C = \sqrt{\frac{R_{ar}^2 R_L}{R_{ar} - R_L}} \approx \sqrt{R_{ar} R_L} \tag{13-12}$$

With the parallel resonant frequency designated by f_{ar},

$$L \approx \frac{1}{2\pi f_{ar}} \sqrt{R_{ar} R_L} \tag{13-13}$$

$$C \approx \frac{1}{2\pi f_{ar} \sqrt{R_{ar} R_L}} \tag{13-14}$$

Example 13–1.—It is required to design a resistanceless transforming network to operate at a frequency of 10^6 cycles. The LC combination in Fig. 13–1 is to transform a fixed load resistance R_L of 50 ohms into a resistance of

5000 ohms across terminals 1 and 2 to match the fixed internal resistance R_g of the generator. Find the values of L and C.

Solution.—From equation (13–8),

$$X_L = \sqrt{5000 \times 50 - 50^2} = 498 \text{ ohms}$$

$$L = \frac{498}{2\pi \times 10^6} = 79.3\ \mu\text{h}$$

From equation (13–9),

$$X_C = \frac{5000 \times 50}{498} = 502 \text{ ohms}$$

$$C = \frac{1}{2\pi \times 10^6 \times 502} = 317\ 5\ \mu\mu\text{f}$$

Since $R_{ar} \gg R_L$, the values of X_L and X_C may be obtained from equations (13–11) and (13–12). Thus,

$$X_L = X_C = \sqrt{5000 \times 50} = 500 \text{ ohms}$$

13–2. Transforming Large R Into Small R.—Sometimes, it is desirable to transform a large fixed load resistance R into a smaller resistance R_{in} at a single frequency to match the fixed internal resistance R_g of a generator. The circuit shown in Fig. 13–4 may be used. This method of matching is useful when the internal resistance of a generator is low.

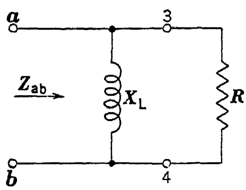


FIG. 13–3

The derivation of an expression for R_{in} follows. The impedance across the terminals a and b of Fig. 13–3 is

$$\dot{Z}_{ab} = \frac{jRX_L}{R + jX_L} = \frac{RX_L^2 + jR^2X_L}{R^2 + X_L^2} \tag{13–15}$$

where

$$R_{ab} = \frac{RX_L^2}{R^2 + X_L^2} = \text{Resistive component} \tag{13–16}$$

$$X_{ab} = \frac{R^2X_L}{R^2 + X_L^2} = \text{Reactive component} \tag{13–17}$$

The impedance $\dot{Z}_{ab}$ can be made to appear, at a single frequency, as a pure resistance $R_{in} = R_{ab}$ by connecting in series with the parallel combination, as shown in Fig. 13–4, a reactance

$$X_C = \frac{R^2 X_L}{R^2 + X_L^2} \qquad (13\text{–}17a)$$

This reactance X_C is equal in magnitude but opposite in sign to the reactive component X_{ab}. The circuit is then in series resonance and

$$R_{in} = R_{ab} = \frac{R X_L^2}{R^2 + X_L^2} \qquad (13\text{–}16a)$$

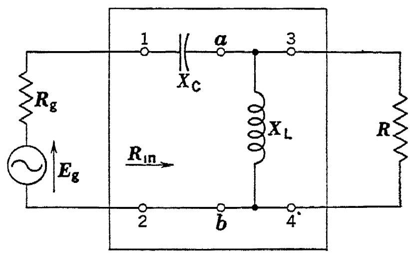


FIG. 13–4

Since R and $R_{in} = R_g$ are to be matched, the problem becomes one of determining X_L and X_C of Fig. 13–4 in terms of R and R_{in} for this match. From equation (13–16a),

$$X_L^2 = \frac{R_{in} R^2}{R - R_{in}}$$

or

$$X_L = R\sqrt{\frac{R_{in}}{R - R_{in}}} \qquad (13\text{–}18)$$

Substituting this value in equation (13–17a) and rearranging, we obtain:

$$X_C = \sqrt{R_{in}(R - R_{in})} \qquad (13\text{–}19)$$

To transform a large resistance R into a small resistance R_{in} by this method, the following facts should be observed. If $R \gg R_{in}$, then equations (13–18) and (13–19) become:

$$X_L \approx \sqrt{R R_{in}} \qquad (13\text{–}18a)$$

$$X_C \approx \sqrt{R R_{in}} \qquad (13\text{–}19a)$$

If $R \approx R_{in}$, then equations (13–18) and (13–19) reduce to $X_L \gg R$ and $X_C = 0$. That is, no transforming network is needed.

13–3. Iron-Core Transformer as an Impedance Matching Device.—The iron-core transformer is often used in audio-frequency circuits for matching of impedances. For such purposes it is desirable to design the transformer so that it will approximate the conditions of an ideal transformer* (defined in Art. 11–14). Fig. 13–5(*a*) shows such an iron-core transformer connecting a fixed

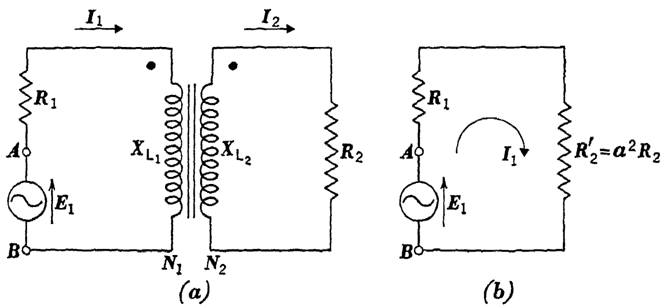


FIG. 13–5

load resistance R_2 to a generator having a fixed internal resistance R_1. The resistances of the coils are assumed to be negligible. The reactances due to the self-inductances of the coils are X_{L1} and X_{L2}. It will be shown that the reflected value of R_2 is a^2R_2, where $a = \frac{N_1}{N_2}$, and that the turns ratio for maximum power transfer is $a = \sqrt{\frac{R_1}{R_2}}$.

If the conditions of Fig. 13–5(*a*) are such that the actual transformer can be considered as an ideal transformer, then the following relations hold:

(*a*) R_2 will be transformed into the primary as $R_2' = a^2R_2$, Fig. 13–5(*b*).

(*b*) The driving-point impedance at the terminals *A* and *B* is

$$\dot{Z}_1' = R_1 + a^2R_2 \tag{13-20}$$

*It must not be assumed that all iron-core transformers used in the laboratory can be considered as ideal transformers.

(*c*) The primary current is

$$\dot{I}_1 = \frac{\dot{E}_1}{R_1 + a^2 R_2} \tag{13–21}$$

(*d*) The secondary current is

$$I_2 = aI_1 \tag{13–22}$$

It should be noted that the iron-core transformer gives a reflected resistance a^2R_2, which is *directly* proportional to R_2. Equations (13–21) and (13–22) indicate that, when R_2 is increased, both I_1 and I_2 will be decreased.

To match R_2 against R_1 for maximum power transfer, the turns ratio required is obtained in the following way. From Fig. 13–5(*b*),

$$a^2 R_2 = R_1$$

$$a = \frac{N_1}{N_2} = \sqrt{\frac{R_1}{R_2}} \tag{13–23}$$

Example 13–2.—Consider the circuit shown in Fig. 13–5(*a*) with the following values:

$\dot{E}_1 = 100\underline{/0^\circ}$ volts $\qquad R_1 = 20$ ohms

$\omega = 10{,}000$ radians/sec $\qquad R_2 = 5$ ohms

It is assumed that the iron-core transformer performs as an ideal transformer. (*a*) If the turns ratio $a = \frac{N_1}{N_2} = 4$, find the magnitudes of the primary and secondary currents, and also the power delivered to R_2. (*b*) Find the turns ratio of the transformer for maximum power transfer and the value of the power received by the load R_2.

Solution.—(*a*) From equation (13–20), the driving-point impedance at the terminals *A* and *B* is

$$\dot{Z}'_1 = R_1 + a^2 R_2 = 20 + (16 \times 5) = 100 \text{ ohms}$$

Hence,

$$I_1 = \frac{E_1}{Z'_1} = \frac{100}{100} = 1 \text{ amp}$$

$$I_2 = \frac{N_1}{N_2} I_1 = 4 \times 1 = 4 \text{ amp}$$

$$P_2 = I_2^2 R_2 = 4^2 \times 5 = 80 \text{ watts}$$

(*b*) From equation (13–23), the turns ratio for maximum power transfer is

$$a = \frac{N_1}{N_2} = \sqrt{\frac{R_1}{R_2}} = \sqrt{\frac{20}{5}} = 2$$

Then, for this turns ratio,

$$\dot{Z}_1' = 20 + (4 \times 5) = 40 \text{ ohms}$$

$$I_1 = \frac{100}{40} = 2.5 \text{ amp}$$

$$I_2 = aI_1 = 2 \times 2.5 = 5 \text{ amp}$$

$$P_2 = I_2^2 R_2 = 5^2 \times 5 = 125 \text{ watts}$$

It can be shown that, if a is greater or less than 2, then P_2 will be less than 125 watts.

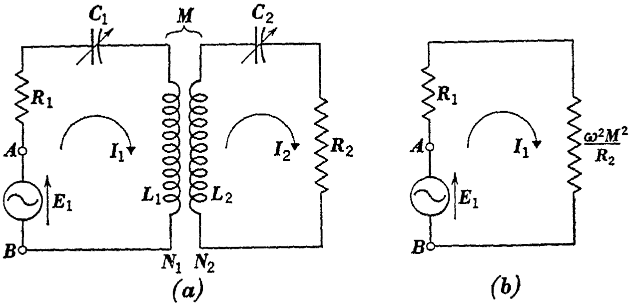


FIG. 13–6

13–4. Tuned Transformer Circuit as an Impedance Matching Device.—In many applications at radio frequencies, air-core transformers rather than iron-core transformers are used for matching purposes to obtain maximum power transfer. Some of the reasons are: The hysteresis and eddy-current losses of the iron core become too great at high frequencies; the cost of iron-core transformers is high; the transforming network needs to cover only a "narrow" band of frequencies at the radio range. The air-core transformer is not treated as an ideal transformer but is considered as part of a tuned coupled circuit. In Fig. 13–6(a), a fixed load resistance R_2 is matched against a fixed generator resistance R_1 by means of a tuned transformer circuit. The resistances of the air-core coils are assumed to be negligible. The self-inductances of the coils are L_1 and L_2. It will be shown that for this type of tuned coupled circuit the reflected load resistance is

$$R_2' = \frac{\omega^2 M^2}{R_2} \qquad (13\text{–}24)$$

and the condition for maximum power transfer is

$$\omega^2 M^2 = R_1 R_2 \qquad (13\text{–}25)$$

The expression for the reflected load resistance may be found in the following way. The driving-point impedance at the terminals A and B of Fig. 13–6(a) is

$$\dot{Z}_1' = R_1 + jX_1 - \frac{(j\omega M)^2}{R_2 + jX_2} \tag{13–26}$$

where $X_1 = X_{L1} - X_{C1}$ and $X_2 = X_{L2} - X_{C2}$.

When the primary reactance X_1 and the secondary reactance X_2 are tuned to zero at a single frequency, then

$$\dot{Z}_1' = R_1 + \frac{\omega^2 M^2}{R_2} \tag{13–27}$$

This equation states that the reflected load resistance $\frac{\omega^2 M^2}{R_2}$ in the tuned coupled circuit is *inversely* proportional to R_2. As far as $\dot{I}_1$ is concerned, Fig. 13–6(a) can be represented by Fig. 13–6(b).

The primary current is

$$\dot{I}_1 = \frac{\dot{E}_1}{\dot{Z}_1'} = \frac{\dot{E}_1}{R_1 + \frac{\omega^2 M^2}{R_2}} \tag{13–28}$$

which indicates that I_1 increases when R_2 is increased. This action is different from that in the case of an iron-core transformer.

The secondary current cannot be determined from the simple relationship given by equation (13–22), but it may be found from one of the following expressions:

$$\dot{I}_2 = \frac{\dot{E}_1 \dot{Z}_m}{\dot{Z}_1 \dot{Z}_2 - \dot{Z}_m^2} = \frac{\dot{E}_1 (j\omega M)}{R_1 R_2 + \omega^2 M^2} \tag{13–29}$$

or

$$\dot{I}_2 = \frac{\dot{Z}_m}{\dot{Z}_2} \dot{I}_1 = \frac{j\omega M}{R_2} \dot{I}_1 \tag{13–30}$$

For the condition of maximum power transfer to R_2, the proper value of M can be determined quite easily. Under the conditions when $X_1 = 0$ and $X_2 = 0$, it can be seen from Fig. 13–6(b) that

$$R_1 = \frac{\omega^2 M^2}{R_2} \tag{13–31}$$

or

$$\omega^2 M^2 = R_1 R_2 \tag{13–31a}$$

Since $M = k\sqrt{L_1 L_2}$, the value of M can be varied by changing k, L_1, and L_2. *The transformer is designed merely to provide the* *proper values of L_1, L_2, and coupling to obtain the required amount of mutual inductance M.* It can be seen that, in the design, the addition of turns in either the primary or the secondary will increase M. Such a change will affect the reflected resistance. If the turns could be added in either the primary or the secondary, that is, if $\frac{N_1}{N_2}$ could be greater or less than unity, then the turns ratio of the air-core transformer used has no significance under these circumstances. From the standpoint of efficiency, however, it is best to reduce the copper loss by having the fewer turns on the side which has the larger current flowing.

Example 13–3.—Consider the tuned coupled circuit shown in Fig. 13–6(*a*) with the following values:

$L_1 = 40\ \mu\text{h}$ $\qquad R_1 = 20$ ohms
$L_2 = 10\ \mu h$ $\qquad R_2 = 5$ ohms
$\omega = 10^6$ radians/sec $\qquad E_1 = 10$ volts

It is assumed that the resistances of the coils are negligible and that C_1 and C_2 are tuned to such values that $X_1 = X_{L1} - X_{C1} = 0$ and $X_2 = X_{L2} - X_{C2} = 0$.

(*a*) Find the power delivered to the load resistance R_2 when $k = 0.75$.

(*b*) Find the coefficient of coupling and the power transferred to R_2 when there is maximum power transfer.

Solution.—(*a*) The mutual inductance for $k = 0.75$ is

$$M = 0.75\sqrt{(40\times10^{-6})(10\times10^{-6})} = 15\times10^{-6} \text{ henry}$$

The driving-point impedance at the terminals A and B is, from equation (13–27),

$$\dot{Z}'_1 = 20 + \frac{(10^6\times15\times10^{-6})^2}{5} = 20 + 45 = 65 \text{ ohms}$$

The primary current is

$$I_1 = \frac{E_1}{Z'_1} = \frac{10}{65} = 0.154 \text{ amp}$$

The secondary current is not found from the turns ratio. It is

$$I_2 = \frac{Z_m}{Z_2} I_1 = \frac{\omega M I_1}{R_2} = \frac{15\times0.154}{5} = 0.462 \text{ amp}$$

The power P_2 delivered to R_2 is

$$P_2 = I_2^2 R_2 = 0.462^2\times5 = 1.07 \text{ watts}$$

(*b*) To find the coefficient of coupling for maximum power transfer to R_2, it is convenient to determine M from equation (13–31a). Thus,

$$M = \frac{\sqrt{R_1R_2}}{\omega} = \frac{\sqrt{20\times5}}{10^6} = 10\times10^{-6} \text{ henry}$$

Then

$$k = \frac{M}{\sqrt{L_1L_2}} = \frac{10\times10^{-6}}{\sqrt{(40\times10^{-6})(10\times10^{-6})}} = 0.5$$

Hence,

$$\dot{Z}_1' = 20 + \frac{(10^6 \times 10 \times 10^{-6})^2}{5} = 20 + 20 = 40 \text{ ohms}$$

$$I_1 = \frac{10}{40} = 0.25 \text{ amp}$$

$$I_2 = \frac{\omega M I_1}{R_2} = \frac{(10^6 \times 10 \times 10^{-6}) \times 0.25}{5} = 0.5$$

$$P_2 = I_2^2 R_2 = 0.5^2 \times 5 = 1.25 \text{ watts}$$

PROBLEMS

13–1. Derive expressions for X_L and X_C in terms of R_{ar} and R in the parallel resonant circuit shown in Fig. 13–7: (*a*) when R_{ar} is not very much greater than R; (*b*) when $R_{ar} \gg R$.

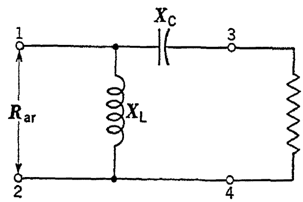


FIG. 13–7

13–2. Design a matching network similar to that shown in Fig. 13–1 to transform a load resistance of $R_L = 10$ ohms into an effective resistance of $R_{ar} = R_g = 1000$ ohms at a frequency of 10^6 cycles. Find the values of L and C.

13–3. Repeat Problem 13–2 if the load resistance $R_L = R = 10$ ohms is in the branch with C as shown in Fig. 13–7.

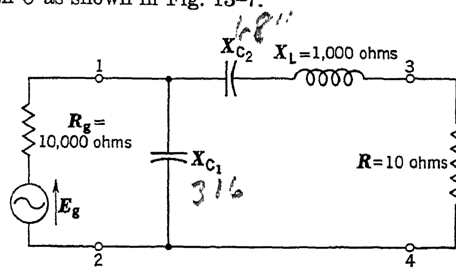


FIG. 13–8

13–4. Fig. 13–8 shows a 1,000,000-cycle generator having an internal resistance $R_g = 10{,}000$ ohms connected across the terminals of a matching network and a load resistance. Determine the values of X_{C1} and X_{C2} which will change the load presented to the terminals of the generator to an effective resistance $R_{ar} = 10{,}000$ ohms under the condition of parallel resonance.

13–5. Repeat Problem 13–4 with the load resistance R moved to the branch containing X_{C1} as shown in Fig. 13–9.

13–6. Fig. 13–10 shows a matching network and a load resistance. The branch containing $L_2 - C_2$ is in series resonance at 2×10^6 cycles and the parallel circuit across terminals 1 and 2 is in parallel resonance at 10^6 cycles. If, under the anti-resonant condition, the load $R = 50$ ohms is to be transformed into $R_{ar} = 10{,}000$ ohms, find L_1, L_2, and C_2.

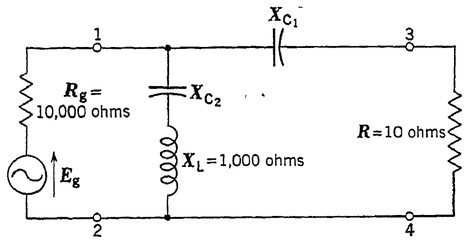


FIG. 13–9

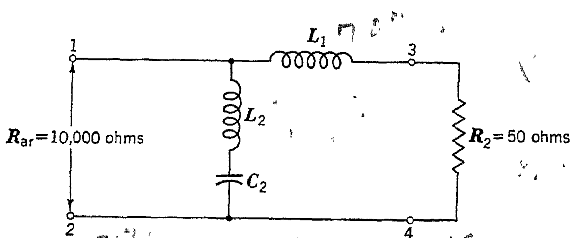


FIG. 13–10

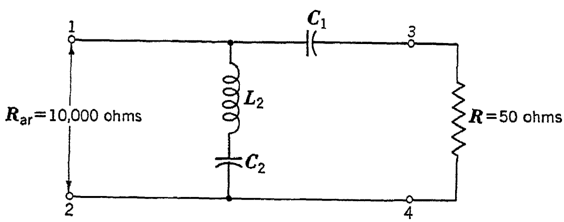


FIG. 13–11

13–7. Fig. 13–11 shows a matching network and a load resistance. The branch containing $L_2 - C_2$ is in series resonance at 10^6 cycles and the parallel circuit across terminals 1 and 2 is in parallel resonance at 2×10^6 cycles. It is required to transform the load resistance $R = 50$ ohms into $R_{ar} = 10{,}000$ ohms, under the anti-resonant condition, to match the internal resistance of a generator. Find C_1, C_2, and L_2.

13–8. It is desirable to use the circuit shown in Fig. 13–4 for the purpose of transforming a load resistance $R = 100$ ohms into a small resistance R_{in}. If $X_L = 20$ ohms at the specified frequency, find: (*a*) X_C such that Z_{in} will be a pure resistance R_{in}; (*b*) the value of R_{in} when the value of X_C found in part (*a*) is used in the circuit.

13–9. Refer to Fig. 13–4 If $R = 10{,}000$ ohms, $R_{in} = 100$ ohms, and $f = 159$ kilocycles, what are the values of L and C?

13–10. It is desirable to transform a large resistance R into a small resistance R_{in} by using the circuit shown in Fig. 13–12. Find the expressions for X_L and X_C in terms of R and R_{in}.

13–11. If, in Fig. 13–12, $R = 10{,}000$ ohms and $X_C = 50$ ohms, find: (*a*) X_L such that Z_{in} will be a pure resistance R_{in}; (*b*) the value of R_{in} when the value of X_L obtained in part (*a*) is used in the network.

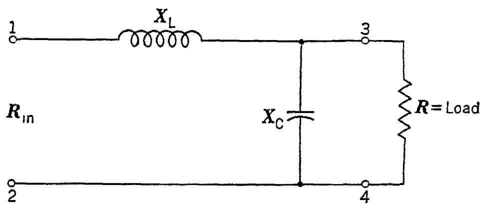


FIG. 13–12

13–12. If, in Fig. 13–5, the generator resistance is $R_1 = 10{,}000$ ohms and the load resistance is $R_2 = 100$ ohms, find: (*a*) the turns ratio of the "ideal" iron-core transformer for maximum power transfer; (*b*) the power delivered to the load R_2 in the secondary when $E_1 = 10$ volts.

13–13. An ideal transformer connects a T-network to a load, as shown in Fig. 13–13. If $\dot{Z}_L = 2.5 + j2.5$, determine. (*a*) the turns ratio of the ideal transformer for maximum power transfer to $\dot{Z}_L$; (*b*) the current flowing in $\dot{Z}_L$. (Hint: Replace the generator and T-network to the left of terminals 1 and 2 by an equivalent Thévenin generator.)

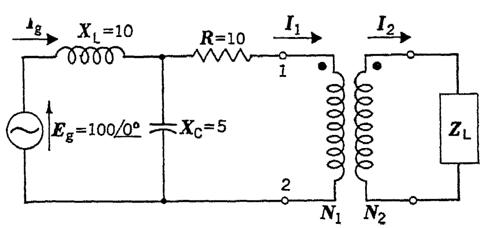


FIG. 13–13

13–14. A load $\dot{Z}_L$ is connected to an ideal transformer, as shown to the right of terminals 1 and 2 in Fig. 13–13. Assume that the turns ratio $\dfrac{N_1}{N_2} = 2$ and $\dot{Z}_L = 2.5 - j2.5$. If a generator is connected to the terminals 1 and 2 (without the T-network), what should be the vector impedance of the generator for maximum power transfer? Note that the generator impedance is varied in this case.

13–15. Consider the circuit shown in Fig. 13–5 with the following values:

$$\dot{E}_1 = 100\underline{/0°} \qquad a = \frac{N_1}{N_2} = 4$$

$$\omega = 5000 \text{ radians/sec} \qquad R_2 = 2 \text{ ohms}$$

The iron-core transformer is considered as an ideal transformer. (*a*) Find R_1 for the condition of maximum power transfer. (*b*) With the value obtained for R_1 in part (*a*), find I_1 and I_2. (*c*) What is the power delivered to R_2 in part (*b*)?

13–16. The tuned coupled circuit shown in Fig. 13–6(*a*) is used for maximum power transfer to R_2 with the conditions that $X_1 = X_{L1} - X_{C1} = 0$ and $X_2 = X_{L2} - X_{C2} = 0$. Also,

$$k = 0.4 \qquad R_2 = 2 \text{ ohms}$$
$$L_1 = 45\ \mu\text{h} \qquad E_1 = 10 \text{ volts}$$
$$L_2 = 20\ \mu\text{h} \qquad \omega = 10^6 \text{ radians/sec}$$

(*a*) What is the value of R_1 with a fixed R_2 of 2 ohms?
(*b*) What is the power delivered to R_2?

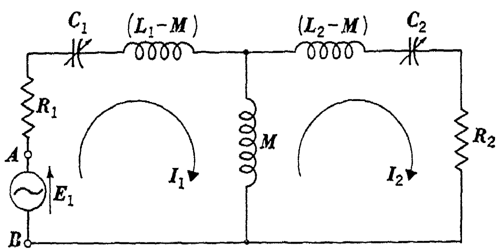


FIG. 13–14

13–17. Refer to Fig. 13–6(*a*). The circuit elements have the following values:

$$R_1 = 80 \text{ ohms} \qquad R_2 = 5 \text{ ohms} \qquad \dot{E}_1 = 10\underline{/0°}$$

$$L_1 = \frac{300}{2\pi}\ \mu\text{h} \qquad L_2 = \frac{150}{2\pi}\ \mu\text{h} \qquad f = 10^6 \text{ cps}$$

$$\frac{N_1}{N_2} = 2 \qquad M = \frac{20}{2\pi}\ \mu\text{h}$$

The condensers C_1 and C_2 are tuned to such values that $X_1 = 0$ and $X_2 = 0$. (*a*) Find $\dot{I}_1$, $\dot{I}_2$, and the power delivered to R_2. (*b*) Determine whether or not the power delivered in part (*a*) is a maximum; if not, find the coefficient of coupling for maximum power transfer.

13–18. If, in Fig. 13–6, $R_2 = 0$ and $X_1 = X_{L1} - X_{C1} = 0$ and $X_2 = X_{L2} - X_{C2} = 0$, find the expressions for $\dot{I}_1$ and $\dot{I}_2$. Explain your answers with the aid of Fig. 13–14.

CHAPTER 14

POLYPHASE CIRCUITS

In a single-phase circuit, the power delivered is pulsating. Even when the current and voltage are in phase, the power is zero twice in each cycle. When the power factor is less than unity, not only is the power zero four times in each cycle, but it is negative twice in each cycle. The pulsating nature of the power in single-phase circuits is objectionable for certain applications.

A polyphase circuit is somewhat like a multi-cylinder gasoline engine where the power delivered to the flywheel is practically steady. Although the power of any one phase may be negative at times, the total power of a three-phase system is constant if the load is balanced. This characteristic makes polyphase systems highly desirable, particularly for power loads.

In general, a single-phase induction motor as such has no starting torque, while a 5-hp, three-phase induction motor may be started by connecting it directly on to the line without starting equipment. Another consideration is the capacity of a machine. The rating of a motor, or generator, of a given physical size increases with the number of phases. The same machine operating three-phase has about 50 per cent greater rating than when operating single-phase. Polyphase machinery (induction motors, synchronous converters, etc.) has better operating characteristics than single-phase machinery.

A minor consideration in favor of the three-phase system is the fact that, in a three-phase power-transmission line having a given kilowatt capacity, a certain voltage between conductors, and a specified efficiency of transmission, the weight of copper required is only three-fourths that of a single-phase line. There are other considerations which favor the use of three-phase systems.

In practice, single-phase power is obtained quite generally from one phase of a three-phase system. That is, single-phase power can be obtained from any two "jacks" (or outlet terminals) of a three-phase outlet board. The balance of a three-phase system is maintained approximately by distributing the single-phase loads equally in the three phases.

In this chapter, we will consider the following: (*a*) two-phase and three-phase generators, (*b*) two-phase circuits, (*c*) three-phase systems with Y-connected loads, and (*d*) three-phase systems with Δ-connected loads.

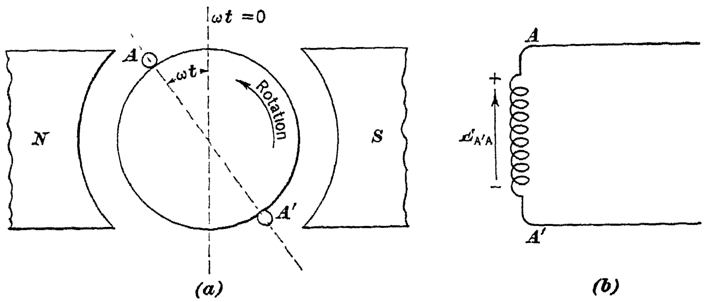


Fig. 14–1

14–1. Generation of Electromotive Forces in Coils on the Same Armature.—A concentrated coil $A'A$ shown in Fig. 14–1(*a*), or represented schematically in Fig. 14–1(*b*), rotates at a uniform angular velocity in a uniform magnetic field. The emf induced in the coil is, from equations (1–6) and (1–7),

$$e_{A'A} = E_{A'Am} \sin \omega t \tag{14–1}$$

where the maximum value is

$$E_{A'Am} = 2\pi f N \Phi_m$$

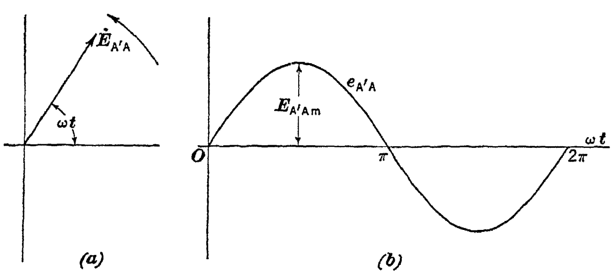


Fig. 14–2

Since this is a voltage at the source, it is considered as a voltage rise in the direction of the arrow, as indicated in Fig. 14–1(*b*). In this chapter, all voltages generated in coils or windings of generators will be considered as voltage rises (from minus polarity to plus polarity) in the direction of the arrows.

The sine wave representing the emf generated in coil $A'A$ is shown in Fig. 14–2(*b*). The corresponding revolving vector with its effective value is given in Fig. 14–2(*a*). The double-subscript notation will be used whenever there is an advantage in so doing.

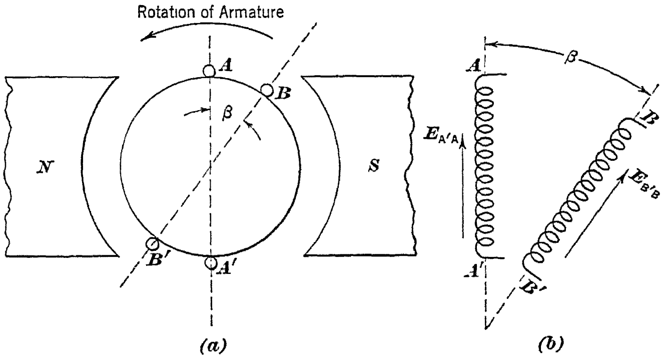


FIG. 14–3

In Fig. 14–3, two concentrated coils $A'A$ and $B'B$ are placed β degrees apart on the same armature. If it is assumed that terminal A corresponds to terminal B and the rotation of the armature is counter-clockwise. the generated emf of coil $A'A$ will lead that of coil $B'B$ by a time angle β. The time angle and the space angle

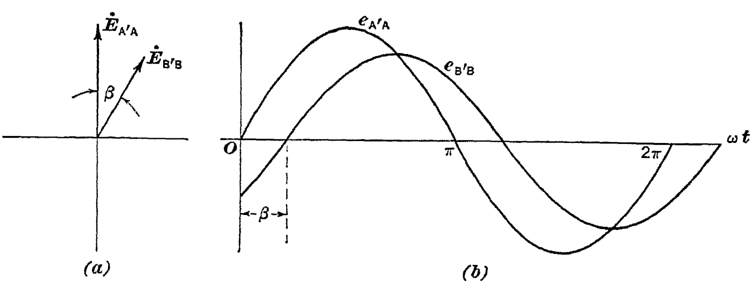


FIG. 14–4

are equal in a two-pole machine. The two sets of equations for the generated voltages are:

$$e_{A'A} = E_{A'Am} \sin \omega t \quad (14\text{–}2a)$$

and

$$e_{B'B} = E_{B'Bm} \sin (\omega t - \beta) \quad (14\text{–}2b)$$

or

$$e_{B'B} = E_{B'Bm} \sin \omega t \quad (14\text{–}3a)$$

and

$$e_{A'A} = E_{A'Am} \sin (\omega t + \beta) \quad (14\text{–}3b)$$

In general, $E_{A'Am}$ may be different from $E_{B'Bm}$, since the coils may have different numbers of turns. The sine waves representing equations (14–2a) and (14–2b) are shown in Fig. 14–4(*b*). The vector diagram for the effective values of the voltages is given in Fig. 14–4(*a*). It can be seen from Fig. 14–4 that the two emfs have the same frequency and the emf $e_{A'A}$ leads the emf $e_{B'B}$ by β degrees. By following this analysis, it is possible to obtain relations between sinusoidal emfs in any number of coils mounted on the armature and separated from each other by different angles.

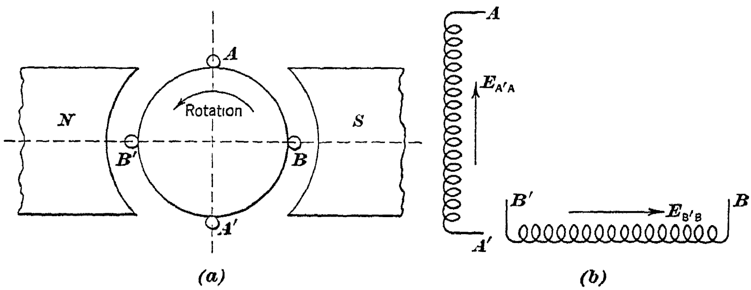


FIG. 14–5

14–2. Polyphase Alternators.—A polyphase alternator differs from a single-phase alternator only in the number of its armature windings. A single-phase alternator has a single armature winding. A two-phase alternator has two independent windings. In general, there are as many independent windings as there are phases. As a rule, these windings are displaced from one another by equal angles which are determined by the number of phases. For example, the six windings of a six-phase alternator are separated from each other by an angle of $\frac{360}{6} = 60$ electrical degrees. This statement does not hold for the so-called two-phase alternator which has two similar armature windings 90 electrical degrees apart.

Owing to practical advantages, all commercial alternators except those of small size have stationary armatures (commonly referred to as stators) and rotating field poles. With the exception of those driven by high-speed steam turbines, alternators usually have more than two poles.

For simplicity, the internal or winding impedances of all polyphase generators will be considered to be negligible in this chapter unless otherwise stated.

14–3. Two-Phase Generators.—Three-phase systems are superseding two-phase systems, although there are still some two-phase systems in use. For this reason, only a brief discussion of two-phase systems will be given. If two similar windings, as shown in Fig. 14–5, are so spaced as to give two equal voltages 90 degrees apart in time, a two-phase alternator results.

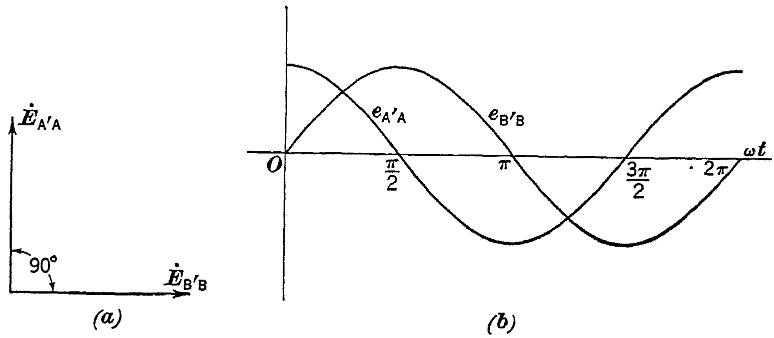


FIG. 14–6

The emf waves of the windings are shown in Fig. 14–6(*b*). For the given rotation, the voltage of $A'A$ leads that of $B'B$ by 90°. When one voltage is a maximum, the other is zero. Fig. 14–6(*a*) shows these voltages represented vectorially. Since the windings are similar, the maximum values will be equal and the effective values also will be equal. Let the emf generated in $B'B$ be

$$e_{B'B} = E_m \sin \omega t \qquad (14\text{–}4)$$

Then

$$e_{A'A} = E_m \sin (\omega t + 90°) \qquad (14\text{–}5)$$

14–4. Principles Involved in the Solution of Two-Phase Systems.—In practice, the two phases of a two-phase alternator generally have no electrical connection within the machine. The four ends of the two windings are brought out separately through the frame. If the four leads coming out from the windings are connected to the load, as shown in Fig. 14–7(*a*), the system is known as a two-phase, four-wire system. If one end of winding $A'A$ is connected to one end of winding $B'B$, as shown in Fig. 14–9(*a*), then the system is known as a two-phase, three-wire system.

In the solution of a two-phase problem, each phase obeys the rules of a single-phase circuit. For the two-phase, four-wire system, it is only necessary to solve two single-phase circuits with two equal applied voltages 90° apart. In solving a two-phase, three-wire problem, Fig. 14–9(*a*), the currents $\dot{I}_A$ and $\dot{I}_B$ are obtained separately as in single-phase circuits. The current $\dot{I}_N$ in the third wire is found by applying Kirchhoff's Current Law at a junction point.

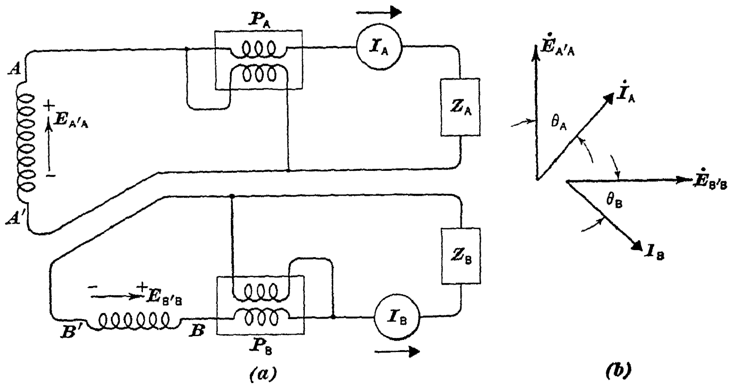


FIG. 14–7

14-5. Two-Phase, Four-Wire System.—In the two-phase, four-wire system, each phase is electrically distinct from the other, as shown in Fig. 14–7(*a*). The two emfs are equal in magnitude and differ in time phase by 90 degrees. The system voltage is the effective value of the winding voltages, or $E_{A'A} = E_{B'B}$. The winding $A'A$ feeds the load $\dot{Z}_A$, and the winding $B'B$ feeds the load $\dot{Z}_B$. These loads may be separate single-phase loads or they may be combined magnetically as in the windings of two-phase motors.

In solving for $\dot{I}_A$ or the current in phase A, it may be helpful to consider that the lower circuit is not present; thus, when $\dot{E}_{A'A}$ is the emf of the generator (which is equal to the voltage drop across Z_A in the direction of the current arrow),

$$\dot{I}_A = \frac{\dot{E}_{A'A}}{\dot{Z}_A} = \frac{\dot{E}_{A'A}}{R_A + jX_A} \qquad (14\text{–}6)$$

Similarly, if the lower circuit is solved as a single-phase circuit, the current in phase B is

$$\dot{I}_B = \frac{\dot{E}_{B'B}}{\dot{Z}_B} = \frac{\dot{E}_{B'B}}{R_B + jX_B} \qquad (14\text{–}7)$$

The vector diagram is given in Fig. 14–7(*b*). The emf vectors are not connected to emphasize the fact that the phases can be considered separately. The emfs, however, are 90° apart since both are furnished by the same machine.

In a two-phase, four-wire system, the total power may be measured by two wattmeters, one in each phase, regardless of the loads. The arithmetical sum of the wattmeter readings will always give the total power delivered by the alternator. As in single-phase circuits, the power supplied to Z_A in phase A is

$$P_A = E_{A'A} I_A \cos \theta_{I_A}^{E_{A'A}} = E_{A'A} I_A \cos \theta_A \qquad (14\text{–}8)$$

and the power supplied to Z_B in phase B is

$$P_B = E_{B'B} I_B \cos \theta_{I_B}^{E_{B'B}} = E_{B'B} I_B \cos \theta_B \qquad (14\text{–}9)$$

where

$$\theta_{I_A}^{E_{A'A}} = \theta_A = \tan^{-1} \frac{X_A}{R_A}$$

$$\theta_{I_B}^{E_{B'B}} = \theta_B = \tan^{-1} \frac{X_B}{R_B}$$

In other words, the notation $\theta_{I_A}^{E_{A'A}}$ signifies the angle between the voltage-rise vector $\dot{E}_{A'A}$ and the current vector $\dot{I}_A$.

The total power is

$$P = P_A + P_B \qquad (14\text{–}10)$$

Example 14–1.—In a two-phase, four-wire, 110-volt system, a lighting load (unity power factor) $\dot{Z}_A = R = 10$ is connected across the terminals of phase A and an impedance load $\dot{Z}_B = 3 + j4$ is connected across the terminals of phase B. Find the current flowing in each phase, the power delivered to each load, and the total power supplied by the two-phase, four-wire system.

Solution.—Assume that the rotation of the generator is such that the voltage $\dot{E}_{A'A}$ of phase A leads the voltage $\dot{E}_{B'B}$ of phase B by 90°. Since this is a 110-volt system, the effective value of the coil voltages is 110 volts. Arbitrarily choose an instant of time such that the generated voltage rise of phase B is

$$\dot{E}_{B'B} = 110\underline{/0°}$$

Then the corresponding voltage rise of phase A is

$$\dot{E}_{A'A} = 110\underline{/90°}$$

From equations (14-6) and (14-7),

$$\dot{I}_A = \frac{\dot{E}_{A'A}}{\dot{Z}_A} = \frac{110\underline{/90^\circ}}{10\underline{/0^\circ}} = 11\underline{/90^\circ}$$

$$\dot{I}_B = \frac{\dot{E}_{B'B}}{\dot{Z}_B} = \frac{110\underline{/0^\circ}}{5\underline{/53.2^\circ}} = 22\underline{/-53.2^\circ}$$

The power factor angles are:

$$\theta_A = 0^\circ$$
$$\theta_B = \tan^{-1}\tfrac{4}{3} = 53.2^\circ$$

The wattmeters read:

$$P_A = E_{A'A} I_A \cos\theta_A = 110 \times 11 \times \cos 0^\circ = 1210 \text{ watts}$$
$$P_B = E_{B'B} I_B \cos\theta_B = 110 \times 22 \times \cos 53.2^\circ = 1452 \text{ watts}$$

The total power is

$$P = P_A + P_B = 1210 + 1452 = 2662 \text{ watts}$$

or $$P = I_A^2 R_A + I_B^2 R_B = 11^2 \times 10 + 22^2 \times 3 = 2662 \text{ watts}$$

The vector diagram is shown in Fig 14-8.

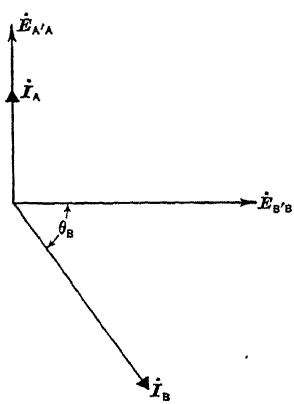


FIG. 14-8

14-6. Balanced Polyphase System.—A polyphase system (or a polyphase circuit) is said to be *balanced* when the conditions in all phases are the same, that is, when the voltages across all phases are the same, the currents flowing in all phases are the same, and the power factors of all phases are the same. When any one of these three quantities is not the same in all the phases, the system is said to be *unbalanced*.

Thus, if the impedances $\dot{Z}_A$ and $\dot{Z}_B$ connected to a two-phase alternator as shown in Fig. 14–7(*a*) are equal, as in the case of a two-phase motor, the system is said to be a balanced two-phase, four-wire system. The ammeter readings will be the same, the power factors will be the same, and the wattmeter readings will be the same. The total power is twice that of one phase.

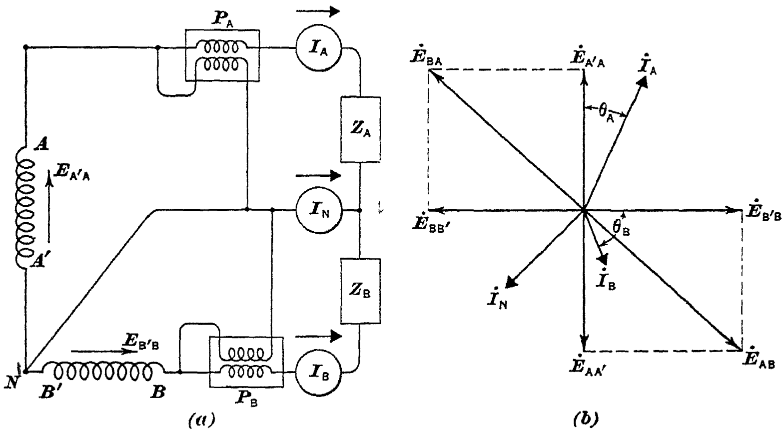


FIG. 14–9

14–7. Two-Phase, Three-Wire System.—To reduce the number of wires as well as the weight of copper, terminals A' and B' are sometimes connected together and a common wire is brought out from the point N as indicated in Fig. 14–9(*a*). For the general case where $\dot{Z}_A$ and $\dot{Z}_B$ are different, the voltage and current relations are developed as follows.

(*a*) *Voltage Relations:* For a two-phase alternator, the generated emfs or voltage rises are:

$$e_{B'B} = E_m \sin \omega t$$
$$e_{A'A} = E_m \sin (\omega t + 90°)$$

The voltage rise from A to B is equal to the sum of the voltage rise from A to A' and the voltage rise from B' to B. Thus,

$$\begin{aligned} e_{AB} &= e_{AA'} + e_{B'B} = e_{B'B} - e_{A'A} \\ &= E_m[\sin \omega t - \sin (\omega t + 90°)] \\ &= E_m(\sin \omega t - \cos \omega t) = E_m\sqrt{2}\left[\frac{1}{\sqrt{2}} \sin \omega t - \frac{1}{\sqrt{2}} \cos \omega t\right] \end{aligned}$$

or

$$e_{AB} = \sqrt{2} E_m \sin (\omega t - 45°) \qquad \textbf{(14–11)}$$

The voltage rise from B to A is the negative of e_{AB}, or

$$e_{BA} = -e_{AB} = \sqrt{2}E_m \sin(\omega t + 135°) \tag{14–12}$$

In the complex notation, the generated emfs of the two phases for the instant shown in the vector diagram in Fig. 14–9(b) are:

$$\dot{E}_{B'B} = E_{B'B}\underline{/0°} = E_{B'B} + j0 \tag{14–13}$$

$$\dot{E}_{A'A} = E_{A'A}\underline{/90°} = 0 + jE_{A'A} \tag{14–14}$$

The voltage rise from line A to line B is found by adding the voltage $\dot{E}_{AA'}$ and the voltage $\dot{E}_{B'B}$ vectorially. Thus,

$$\dot{E}_{AB} = \dot{E}_{AA'} + \dot{E}_{B'B} = \dot{E}_{B'B} - \dot{E}_{A'A} = E_{B'B} - jE_{A'A} \tag{14–15}$$

The voltage $\dot{E}_{AB}$ is the vector difference of the voltages $\dot{E}_{B'B}$ and $\dot{E}_{A'A}$ and lags 45° behind $\dot{E}_{B'B}$, since $E_{B'B} = E_{A'A}$.

(b) *Current Relations:* The current relations may be obtained by considering the two phases as two separate single-phase circuits with applied emfs 90° apart. From Ohm's Law,

$$i_B = \frac{E_m}{Z_B} \sin(\omega t - \theta_B) \tag{14–16}$$

$$i_A = \frac{E_m}{Z_A} \sin(\omega t + 90° - \theta_A) \tag{14–17}$$

where θ_B and θ_A are the power factor angles of the loads $\dot{Z}_B$ and $\dot{Z}_A$, respectively.

Before determining the current i_N in the common wire, it is necessary to decide upon a positive sense for the currents flowing in the wires. It is immaterial whether the positive sense of current flow is toward the point N or away from the point N. One direction should be chosen to keep us straight on the signs when we add currents at a junction point. For our purpose consider the direction away from the point N as positive, as in Fig. 14–9(a).

By applying Kirchhoff's Current Law at point N, we obtain

$$i_A + i_B + i_N = 0 \tag{14–18}$$

Hence,

$$i_N = -(i_A + i_B)$$

or

$$i_N = -E_m\left[\frac{1}{Z_A}\sin(\omega t + 90° - \theta_A) + \frac{1}{Z_B}\sin(\omega t - \theta_B)\right] \tag{14–19}$$

In complex notation, the current vectors shown in Fig. 14-9(*b*) are:

$$\dot{I}_A = \frac{\dot{E}_{A'A}}{\dot{Z}_A} = \frac{E_{A'A}\angle 90^\circ}{\dot{Z}_A\angle \theta_A} = I'_A + jI''_A \qquad (14\text{–}20)$$

$$\dot{I}_B = \frac{\dot{E}_{B'B}}{\dot{Z}_B} = \frac{E_{B'B}\angle 0^\circ}{Z_B\angle \theta_B} = I'_B - jI''_B \qquad (14\text{–}21)$$

$$\dot{I}_N = -(\dot{I}_A + \dot{I}_B) = -(I'_A + I'_B) + j(I''_B - I''_A) \qquad (14\text{–}22)$$

where I'_A and I'_B are the horizontal components of the currents and I''_A and I''_B are the vertical components of the currents.

If the system is balanced, or $\dot{Z}_A = \dot{Z}_B$, the currents flowing in the outer two wires are equal in magnitude and 90° apart. The current flowing in the common wire is equal to the vector sum of two equal currents 90° apart. That is, the three current vectors form a right triangle and the magnitude of $\dot{I}_N$ is equal to the hypotenuse. Thus,

$$I_N = \sqrt{2}\, I_A = \sqrt{2}\, I_B \qquad (14\text{–}23)$$

Example 14–2.—The two single-phase loads of Example 14–1 are connected to a two-phase, three-wire, 110-volt system. Find the current $\dot{I}_N$.

Solution.—Consider each phase as a single-phase circuit, as in Example 14–1. Then

$$\dot{I}_A = 11\angle 90^\circ = 0 + j11$$

$$\dot{I}_B = 22\angle -53.2^\circ = 13.2 - j17.6$$

When Kirchhoff's Current Law is applied at a junction point,

$$\begin{aligned}\dot{I}_N &= -(\dot{I}_A + \dot{I}_B)\\ &= -13.2 + j6.6 = 14.72\angle 153.4^\circ\end{aligned}$$

Example 14–3.—A balanced two-phase load $\dot{Z}_A = \dot{Z}_B = \dot{Z} = 3 + j4$ is connected to a two-phase, three-wire, 100-volt system. Find the vector currents in the three line wires. Draw the vector diagram.

Solution.—Let the generated emf $\dot{E}_{B'B}$ be the reference vector. Then

$$\dot{E}_{B'B} = 100\angle 0^\circ$$

$$\dot{E}_{A'A} = 100\angle 90^\circ$$

By solving each phase as a single-phase circuit, we get:

$$\dot{I}_B = \frac{\dot{E}_{B'B}}{\dot{Z}} = \frac{100\angle 0^\circ}{5\angle 53.2^\circ} = 20\angle -53.2^\circ = 12 - j16$$

$$\dot{I}_A = \frac{\dot{E}_{A'A}}{\dot{Z}} = \frac{100\angle 90^\circ}{5\angle 53.2^\circ} = 20\angle 36.8^\circ = 16 + j12$$

From Kirchhoff's Current Law, the current in the third wire is

$$\dot{I}_N = -(\dot{I}_A + \dot{I}_B) = -28 + j4 = 28.28\underline{/171.9^\circ}$$

In magnitude,

$$I_A = I_B = 20 \text{ amp}$$

$$I_N = \sqrt{2}\, I_A = \sqrt{2}\, I_B = 28.28 \text{ amp}$$

The vector diagram is shown in Fig. 14–10.

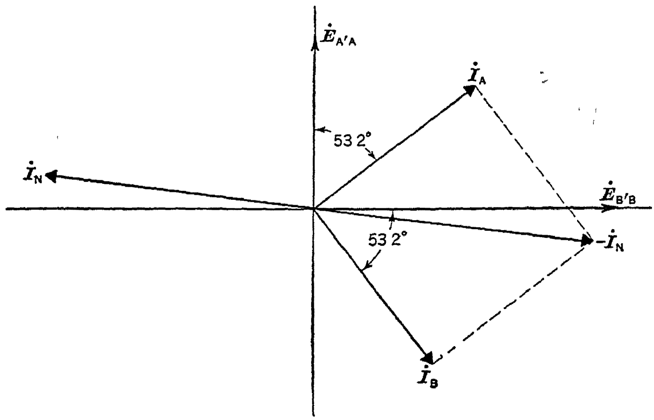


Fig. 14–10

Since the power factor angle of each branch of the balanced two-phase load is 53.2°, the power factor* of the balanced two-phase load is said to be equal to cos 53.2°. The power supplied by the generator is equal to twice that supplied to one phase. It is

$$P = 2E_{B'B}I_B \cos \theta_{I_B}^{E_{B'B}}$$
$$= 2 \times 100 \times 20 \cos 53.2^\circ = 2400 \text{ watts}$$

Example 14–4.—A balanced two-phase load $\dot{Z}_A = \dot{Z}_B = \dot{Z} = 2.57 + j1.93$ consumes 5000 watts (5 kw) from a two-phase, three-wire, 100-volt alternator. What is the magnitude of the current in each phase of the a-c generator?

Solution.—Since the load is balanced, each phase consumes one-half of the total watts, or 2500 watts. The power factor angle for the entire load, which is the same for each phase, is $\theta = \tan^{-1} \frac{1.93}{2.57} = 36.8^\circ$. From the power equation for single-phase circuits, the magnitude of the current is

$$I = \frac{P}{E \cos \theta} = \frac{2500}{100 \cos 36.8^\circ} = 31.25 \text{ amp}$$

* The power factor of an unbalanced two-phase load or an unbalanced three-phase load will not be considered in this text.

14–8. Three-Phase Generation.—In Fig. 14–11 three identical windings, $A'A$, $B'B$, and $C'C$, are placed 120 electrical degrees apart on the same armature. They rotate at a uniform angular velocity in the same magnetic field. Since the windings are similar, the amplitudes of the generated emfs are

$$E_{A'Am} = E_{B'Bm} = E_{C'Cm} = E_m$$

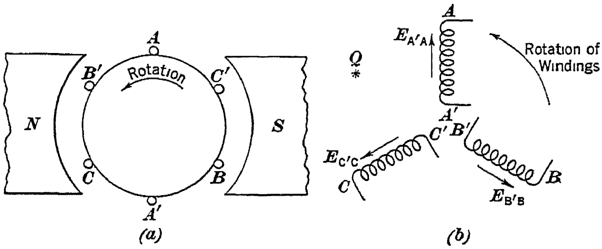


FIG. 14–11

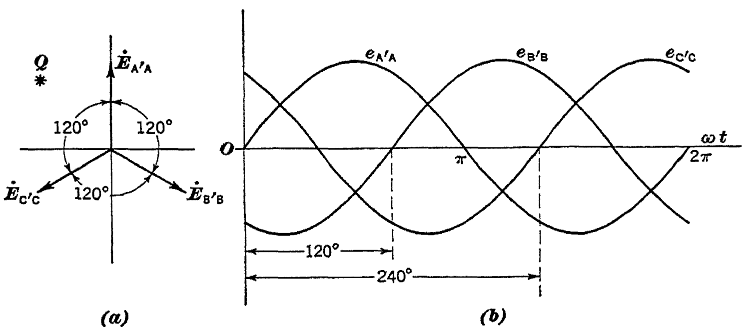


FIG. 14–12

If A, B, and C are considered to be the corresponding ends of the windings, then for the rotation indicated the emfs generated in the three windings are:

$$e_{A'A} = E_m \sin \omega t \tag{14–24}$$

$$e_{B'B} = E_m \sin (\omega t - 120°) = E_m \sin (\omega t + 240°) \tag{14–25}$$

$$e_{C'C} = E_m \sin (\omega t - 240°) = E_m \sin (\omega t + 120°) \tag{14–26}$$

The waves representing the three emfs are shown in Fig. 14–12(b). An examination of the diagram shows that for any particular instant the algebraic sum of the three emfs is zero. When one voltage is zero, the other two have 86.6 per cent of their maximum values and have opposite signs. When any one emf wave is at its maximum, the sign of each of the others is opposite to that of the maximum and the magnitude of each is 50 per cent of its maximum value. The vectors representing effective values of these voltages are shown in Fig. 14–12(*a*); they are equal in magnitude and 120° apart.

Phase Sequence: Phase sequence is defined as the order in which the vectors representing the voltages are considered in time phase with respect to each other. In Figs. 14–11(*b*) and 14–12(*a*), the phase sequence is *ABC*. An observer at point *Q* will see *A* first, then *B*, and finally *C*. If the windings in Fig. 14–11(*b*) are rotated in the opposite direction, the sequence of the voltages will be *ACB*, and two of the vectors in Fig. 14–12(*a*) must be interchanged (see Figs. 14–17 and 14–18). The student must not conclude that the sequence *ACB* is obtained from Fig. 14–12(*a*) by rotating the vectors in the clockwise direction, since the conventional direction of rotation of vectors is counter-clockwise.

The importance of phase sequence on line currents will be demonstrated when these voltages are applied to unbalanced three-phase loads.

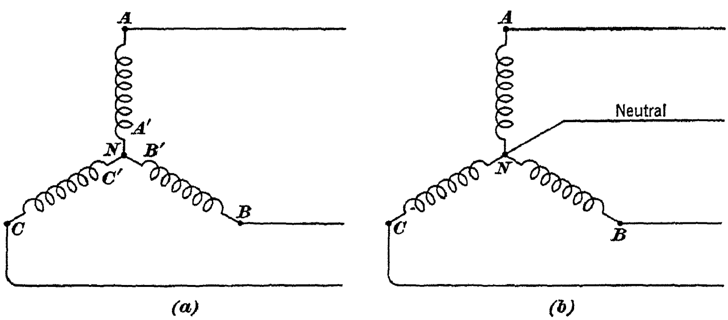


FIG. 14–13

14–9. Star or Y-Connected Alternator.—If the three windings in Fig. 14–11(*b*) are connected electrically at *A′*, *B′*, and *C′*, the machine becomes a Y-connected, three-phase alternator delivering three equal voltages that are 120° apart at the open ends. A three-phase, three-wire system is shown in Fig. 14–13(*a*). The point *N* is called the neutral point. It is only necessary to have three wires lead to the external circuit, although the neutral wire from N is sometimes carried along, making a three-phase, four-wire system, as in Fig. 14–13(*b*).

The emfs of the three windings are given by equations (14–24) to (14–26). These three generated voltages are called the *winding voltages* or the *coil* voltages*. The *line-to-line voltages*, or simply the *line voltages*, are equal to the differences of the coil voltages. Thus, for $e_{NA} = e_{A'A}$, $e_{NB} = e_{B'B}$, and $e_{NC} = e_{C'C}$,

$$e_{BA} = e_{BN} + e_{NA} = e_{NA} - e_{NB} \tag{14–27}$$

$$e_{AC} = e_{AN} + e_{NC} = e_{NC} - e_{NA} \tag{14–28}$$

$$e_{CB} = e_{CN} + e_{NB} = e_{NB} - e_{NC} \tag{14–29}$$

Substituting values of e_{NA} and e_{NB} from equations (14–24) and (14–25), respectively, gives the line voltage e_{BA}. Thus,

$$\begin{aligned} e_{BA} &= E_m \sin \omega t - E_m \sin (\omega t + 240°) \\ &= E_m \sin \omega t - E_m \sin \omega t \cos 240° - E_m \cos \omega t \sin 240° \\ &= E_m \sin \omega t - E_m\left[-\frac{1}{2} \sin \omega t - \frac{\sqrt{3}}{2} \cos \omega t\right] \end{aligned}$$

or

$$e_{BA} = E_m\left[\frac{3}{2} \sin \omega t + \frac{\sqrt{3}}{2} \cos \omega t\right] = \sqrt{3}\, E_m \sin (\omega t + 30°) \tag{14–30}$$

From equations (14–24) and (14–26),

$$\begin{aligned} e_{AC} &= E_m \sin (\omega t + 120°) - E_m \sin \omega t \\ &= E_m[\sin \omega t \cos 120° + \cos \omega t \sin 120° - \sin \omega t] \end{aligned}$$

or

$$e_{AC} = E_m\left[-\frac{3}{2} \sin \omega t + \frac{\sqrt{3}}{2} \cos \omega t\right] = \sqrt{3}\, E_m \sin (\omega t + 150°) \tag{14–31}$$

Also, from equations (14–25) and (14–26),

$$\begin{aligned} e_{CB} &= E_m \sin (\omega t + 240°) - E_m \sin (\omega t + 120°) \\ &= E_m[\sin \omega t \cos 240° + \cos \omega t \sin 240° - \sin \omega t \cos 120° \\ &\qquad - \cos \omega t \sin 120°] \end{aligned}$$

or

$$e_{CB} = E_m[-\sqrt{3} \cos \omega t] = \sqrt{3}\, E_m \sin (\omega t + 270°) \tag{14–32}$$

When the coil voltages are equal in magnitude and differ in time phase by 120°, the line voltages are spaced 120° apart and

* The expression "phase voltages" will not be used, for the sake of clearness. In Fig. 14–13(*a*), phase voltages may mean the three coil voltages $e_{A'A}$, $e_{B'B}$, and $e_{C'C}$ or the three line voltages e_{BA}, e_{AC}, and e_{CB}.

are equal in magnitude. Such a system is sometimes called a balanced and symmetrical three-phase system of voltages. The waves representing the coil voltages and line voltages are shown in Fig. 14–14. It can be seen from the coil voltage equations (14–24) to (14–26) and the line voltage equations (14–30) to (14–32) that the magnitude of the line voltages is equal to $\sqrt{3}$ times the magnitude of the coil (or line-to-neutral) voltages. Thus, we may write the equation

$$E_{\text{line}} = \sqrt{3}\, E_{\text{coil}} \tag{14–33}$$

An examination of Fig. 14–14 shows that, for any particular instant of time, the following conditions exist:

(*a*) The algebraic sum of the three coil voltages is zero.

(*b*) The algebraic sum of the three line voltages is zero.

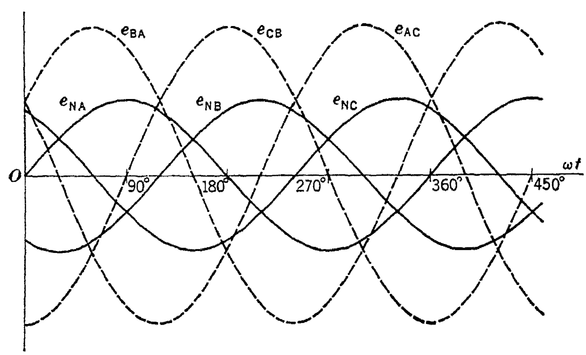


FIG. 14–14

In Fig. 14–15 are shown the vectors representing the coil and line voltages. The line voltage $\dot{E}_{BA}$ is obtained by adding the voltage from B to N and the voltage from N to A vectorially. Thus, we may write the equation

$$\dot{E}_{BA} = \dot{E}_{BN} + \dot{E}_{NA} = \dot{E}_{NA} - \dot{E}_{NB} \tag{14–34}$$

Similarly,

$$\dot{E}_{AC} = \dot{E}_{AN} + \dot{E}_{NC} = \dot{E}_{NC} - \dot{E}_{NA} \tag{14–35}$$

$$\dot{E}_{CB} = \dot{E}_{CN} + \dot{E}_{NB} = \dot{E}_{NB} - \dot{E}_{NC} \tag{14–36}$$

It also can be seen from the vector diagram that the magnitude of the line voltages is equal to $\sqrt{3}$ times the magnitude of the coil (or line-to-neutral) voltages since, for example, $\dot{E}_{NB}$ and $\dot{E}_{CN}$ are 30° from $\dot{E}_{CB}$.

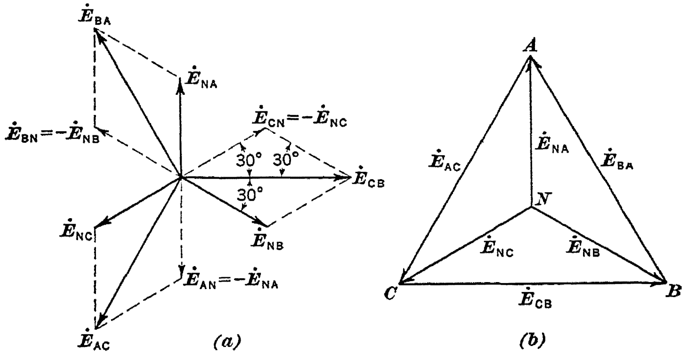


FIG. 14–15

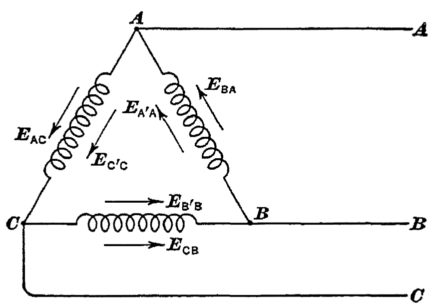


FIG. 14–16

14–10. Δ-Connected Alternator.—When the three windings are arranged, as in Fig. 14–16, so that A is connected to C', C is connected to B', and B is connected to A', the machine is said to be a Δ-connected generator. The line voltages will be the same as the generated coil voltages. Thus,

$$\left.\begin{aligned} e_{BA} &= e_{A'A} = E_m \sin \omega t \\ e_{AC} &= e_{C'C} = E_m \sin (\omega t + 120°) \\ e_{CB} &= e_{B'B} = E_m \sin (\omega t - 120°) \end{aligned}\right\} \qquad (14\text{–}37)$$

At first glance the three coils (each containing a source of emf) appear to form a short circuit. The actual conditions existing in this closed circuit may be shown by considering the three voltages in series. The three voltages as given by equations (14–37) are equal in magnitude and differ in phase by 120°. Hence, the algebraic sum at every instant is zero.

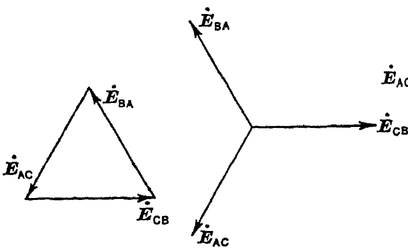


FIG. 14–17

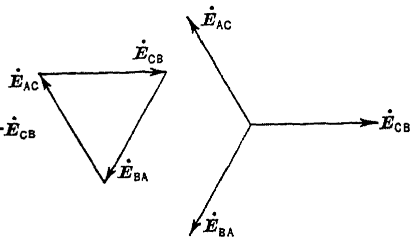


FIG. 14–18

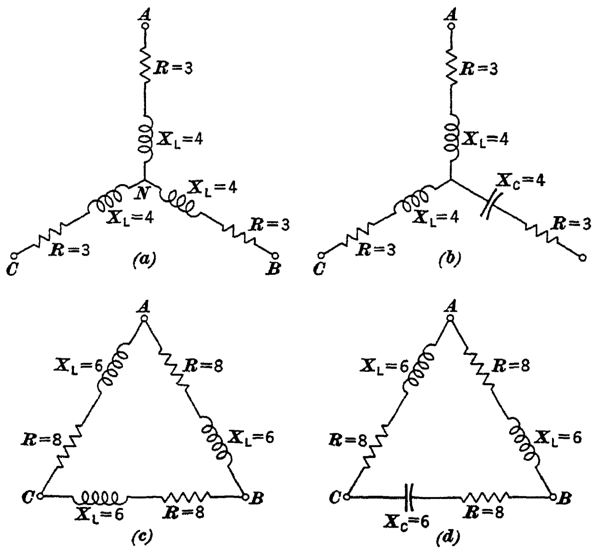


FIG. 14–19

The vector diagram of the three voltages for the sequence *ABC* is shown in Fig. 14–17. If the windings are rotated in the opposite direction, the sequence is *ACB* and the vector diagram for this case is shown in Fig. 14–18.

14–11. Balanced Three-Phase Loads.—If three single-phase loads forming a three-phase load, or three-phase induction motor loads, have equal resistive components and equal and identical reactive components, then the three-phase load is said to be *balanced*. Fig. 14–19(*a*) or Fig. 14–19(*c*) shows a balanced three-phase load which consists of three identical single-phase loads. The load given in Fig. 14–19(*b*) or Fig. 14–19(*d*) is not a balanced three-phase load since the reactive components are not alike in the three legs.

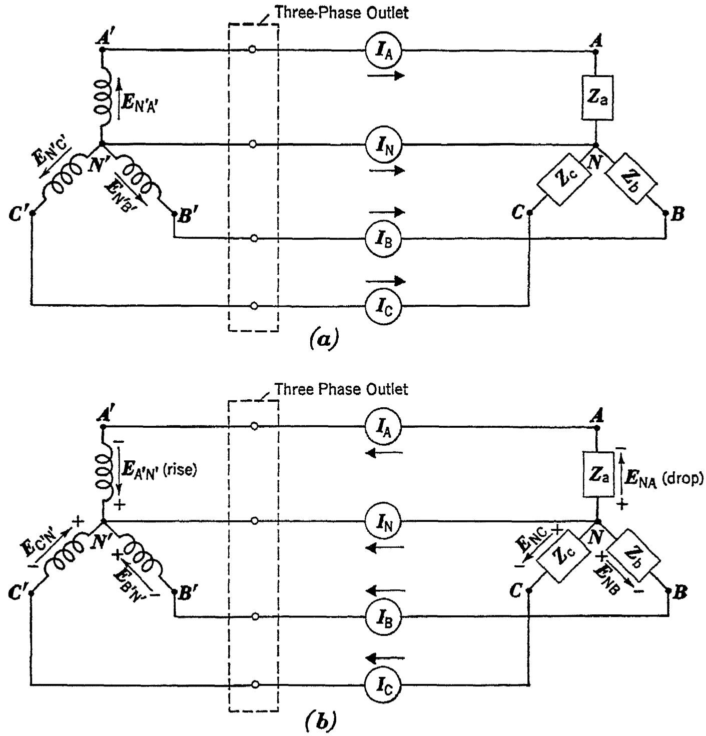


FIG. 14–20

14–12. Notation in the Solution of Three-Phase Circuits.—A three-phase circuit with generator and load is shown in Fig. 14–20(*a*). The generator terminals are marked *A'*, *B'*, and *C'*, simply to distinguish them from the load terminals *A*, *B*, and *C*. A three-phase system is usually designated as a three-phase, four-wire, 110-volt system or a three-phase, three-wire, 110-volt system. The given voltage is the line-to-line voltage, and not the line-to-neutral voltage. A three-phase system may also be indicated as a three-phase, four-wire, 4000/2300-volts, 60-cycle system, in which the line-to-line voltage is 4000 and the line-to-neutral voltage is 2300. Such a system can be changed, by means of transformers, to three-phase, four-wire, 110-volts and three-phase, four-wire, 220 volts.

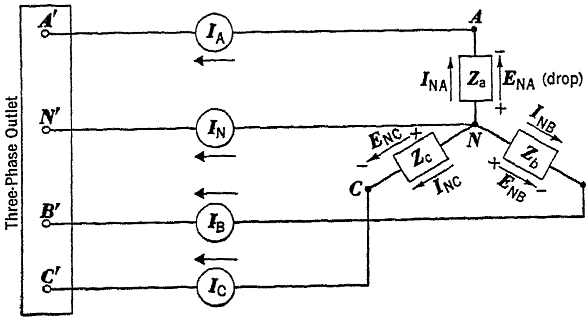


FIG. 14–21

Since the choice of positive sense for voltages or currents is quite arbitrary, it may be more convenient to use the notation in Fig. 14–20(*b*). In this diagram, the generated emfs $E_{A'N'}$, $E_{B'N'}$, and $E_{C'N'}$ are considered to be voltage rises from A' to N', B' to N', and C' to N', respectively; and the load voltages E_{NA}, E_{NB}, and E_{NC} are considered to be voltage drops from N to A, N to B, and N to C, respectively. It should be noted that, for example, the polarity from the neutral wire $N'N$ to the line wire $A'A$ is from plus to minus for both the generator voltage rise $\dot{E}_{A'N'}$ and the load voltage drop $\dot{E}_{NA}$ (note that the head of an arrow that indicates the positive sense of a load voltage drop has a minus polarity). The choice of positive senses for the line currents and the voltage drops at the load in Fig. 14–20(*b*) is convenient when the generator is left out of the diagram, as in Fig. 14–21. The voltages E_{NA}, E_{NB}, and E_{NC} must be interpreted as voltage drops* (from plus polarity to minus polarity) in the direction of the current arrows or in the direction of the arrows which indicate the positive senses of the voltage drops. A vector diagram similar to that of Fig. 14–15(*b*) may be used for the voltage drops E_{NA}, E_{CB}, etc. This is a natural selection when considering how the circuit and vector diagrams are drawn for these problems.

* Some authors prefer to use the symbol V for voltage drops.

14–13. Principles Involved in the Solution of Three-Phase Problems.—A three-phase circuit may be treated as three single-phase circuits. From Ohm's Law, the current flowing in a load is equal to the voltage drop across the load divided by the impedance of the load. Currents at any junction point may be determined by applying Kirchhoff's Current Law. Thus, it may be said that the problem of a three-phase circuit resolves itself to one of finding the voltage drop across each branch of the three-phase load. To obtain the quantitative values and the phase relations of the voltages and the currents in the various branches of a three-phase circuit, the following procedure may be adopted.

(*a*) If the phase sequence of the line voltage drops is not given and the reference vector is not specified, it is convenient to choose $\dot{E}_{CB} = E_{CB}\underline{/0°}$, Fig. 14–15 or Fig. 14–17, as a reference vector and to use the sequence *ABC*. Then,

$$\dot{E}_{CB} = E_{CB}\underline{/0°};\ \dot{E}_{BA} = E_{BA}\underline{/120°};\ \dot{E}_{AC} = E_{AC}\underline{/240°}$$

(*b*) It is convenient to use the same positive sense for both the voltage drop and the current in each branch, as shown in Fig. 14–21.

(*c*) Determine the voltage drop across each branch of the three-phase load, such as $\dot{E}_{NA}$, $\dot{E}_{NB}$, and $\dot{E}_{NC}$; and find the current in each branch of the load. Thus,

$$\dot{I}_{NA} = \frac{\dot{E}_{NA}}{\dot{Z}_{NA}};\ \dot{I}_{NB} = \frac{\dot{E}_{NB}}{\dot{Z}_{NB}};\ \dot{I}_{NC} = \frac{\dot{E}_{NC}}{\dot{Z}_{NC}}$$

(*d*) Kirchhoff's Current Law applies at every junction point.

14–14. Three-Phase, Four-Wire System, Unbalanced Y-Connected Load.—Three single-phase loads $\dot{Z}_a$, $\dot{Z}_b$, and $\dot{Z}_c$ are connected in wye to a three-phase, four-wire system, as shown in Fig. 14–21. To find the four currents it is first necessary to determine the voltage drops $\dot{E}_{NA}$, $\dot{E}_{NB}$, and $\dot{E}_{NC}$. Then, by applying Ohm's Law, the branch currents can be found. From the physical set-up, the branch currents and the line currents are the same. Finally, the current in the neutral can be obtained by applying Kirchhoff's Current Law at the point *N*.

Since the common point *N* at the load is "fixed" by the neutral of the generator, the magnitude of the voltages from *N* to the lines is equal to the magnitude of the line voltages divided by $\sqrt{3}$ [from equation (14–33)]. Hence,

$$E_{NA} = E_{NB} = E_{NC} = \frac{E_{CB}}{\sqrt{3}} = \frac{E_{BA}}{\sqrt{3}} = \frac{E_{AC}}{\sqrt{3}} \qquad (14\text{–}38)$$

If it is assumed that the vector diagram in Fig. 14–15 represents this system of voltage drops, the vector equations for the voltage drops across the single-phase loads are:

$$\left.\begin{array}{l} \dot{E}_{CB} = E_{CB}\underline{/0^\circ};\ \dot{E}_{BA} = E_{BA}\underline{/120^\circ};\ \dot{E}_{AC} = E_{AC}\underline{/240^\circ} \\ \dot{E}_{NA} = E_{NA}\underline{/90^\circ};\ \dot{E}_{NB} = E_{NB}\underline{/-30^\circ};\ \dot{E}_{NC} = E_{NC}\underline{/210^\circ} \end{array}\right\} \qquad (14\text{–}39)$$

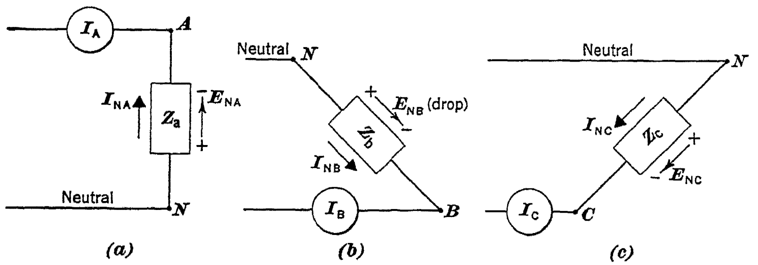


FIG. 14–22

When the voltage drop across each leg of the Y-connected load is known, the determination of the line currents $\dot{I}_A$, $\dot{I}_B$, and $\dot{I}_C$ can further be simplified by breaking up the three-phase load into three single-phase loads as shown in Fig. 14–22. The load current $\dot{I}_{NA}$, which from the physical nature of the circuit is the same as the line current $\dot{I}_A$, is equal to the voltage drop across the load $\dot{Z}_a$ divided by the impedance. Thus, the current for the single-phase circuit in Fig. 14–22(a) is, by Ohm's Law,

$$\dot{I}_A = \dot{I}_{NA} = \frac{\dot{E}_{NA}}{\dot{Z}_A} \qquad (14\text{–}40)$$

Similarly, from Figs. 14–22(b) and (c),

$$\dot{I}_B = \dot{I}_{NB} = \frac{\dot{E}_{NB}}{\dot{Z}_B} \qquad (14\text{–}41)$$

$$\dot{I}_C = \dot{I}_{NC} = \frac{\dot{E}_{NC}}{\dot{Z}_C} \qquad (14\text{–}42)$$

The arrows in Fig. 14–21 indicate only the positive sense and not the direction of current flow at all times. The student must not conclude that the currents $\dot{I}_A$, $\dot{I}_B$, $\dot{I}_C$, and $\dot{I}_N$ are flowing away from point N and then wonder where the currents come from (see Example 14-6). Application of Kirchhoff's Current Law at point N gives

$$\dot{I}_N+\dot{I}_A+\dot{I}_B+\dot{I}_C=0$$

or

$$\dot{I}_N=-(\dot{I}_A+\dot{I}_B+\dot{I}_C) \qquad (14\text{–}43)$$

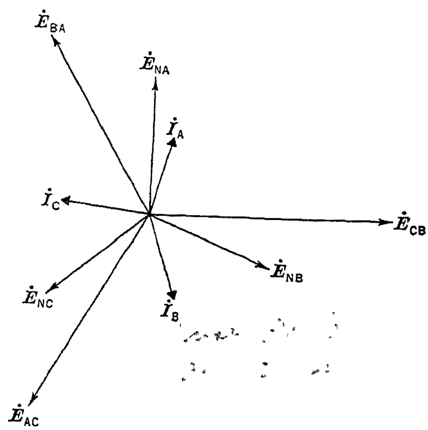


FIG. 14–23

The vector diagram is shown in Fig. 14–23. For simplicity, $\dot{I}_N$ is left out of the diagram.

Example 14–5.—Three unequal single-phase loads, $\dot{Z}_a=5+j0$, $\dot{Z}_b=3+j4$ and $\dot{Z}_c=8-j6$, are connected in wye to a three-phase, four-wire, 110-volt system, as shown in Fig. 14–24. It is required to find the line currents for the sequence ABC and to draw the vector diagram.

Solution.—Since this is a three-phase, four-wire circuit, the voltage at the point N is fixed. The magnitudes of all the line-to-neutral voltages are the same and are equal to the magnitude of the line-to-line voltage divided by $\sqrt{3}$. When the vector diagram in Fig. 14–15 is used to represent the system of voltage drops, it is recommended that $\dot{E}_{CB}$ be used as a reference vector. This reduces the mental effort in visualizing the vectors in the vector diagram and the corresponding voltage drops in the circuit. Thus, for sequence ABC, the expressions for the voltage drops are:

$$\dot{E}_{CB}=110\underline{/0^\circ} \qquad \dot{E}_{NA}=63.5\underline{/90^\circ}$$

$$\dot{E}_{BA}=110\underline{/120^\circ} \qquad \dot{E}_{NB}=63.5\underline{/-30^\circ}$$

$$\dot{E}_{AC}=110\underline{/240^\circ} \qquad \dot{E}_{NC}=63.5\underline{/-150^\circ}$$

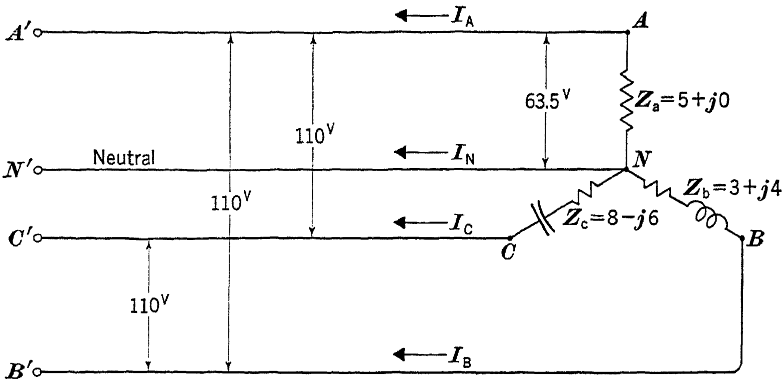


FIG. 14–24

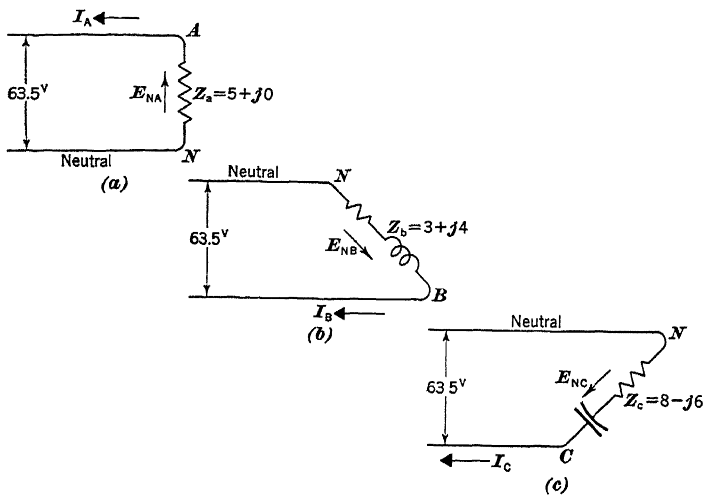


FIG. 14–25

The line currents $\dot{I}_A$, $\dot{I}_B$, and $\dot{I}_C$ may be obtained by solving the three single-phase circuits as shown in Fig. 14–25. They are:

$$\dot{I}_A = \dot{I}_{NA} = \frac{\dot{E}_{NA}}{\dot{Z}_a} = \frac{63.5\angle 90^\circ}{5\angle 0^\circ} = 12.7\angle 90^\circ = 0 + j12.7$$

$$\dot{I}_B = \dot{I}_{NB} = \frac{\dot{E}_{NB}}{\dot{Z}_b} = \frac{63.5\angle -30^\circ}{5\angle 53.2^\circ} = 12.7\angle -83.2^\circ = 1.54 - j12.57$$

$$\dot{I}_C = \dot{I}_{NC} = \frac{\dot{E}_{NC}}{\dot{Z}_c} = \frac{63.5\angle -150^\circ}{10\angle -36.8^\circ} = 6.35\angle -113.2^\circ = -2.48 - j5.84$$

From Kirchhoff's Current Law at point N in Fig. 14–24, the current flowing in the neutral wire is

$$\dot{I}_N = -(\dot{I}_A + \dot{I}_B + \dot{I}_C)$$
$$= -[(0+j12.7)+(1.54-j12.57)+(-2.84-j5.84)] = 5.78\underline{/80.6^\circ}$$

The complete vector diagram is shown in Fig. 14–26.

It can be seen from Figs. 14–25 and 14–26 that the current in the load Z_a is in phase with the voltage drop across Z_a, the current in Z_b lags behind the voltage drop across Z_b, and the current in Z_c leads the voltage drop across Z_c. This is what we learn in Chapter 5 on Series Circuits.

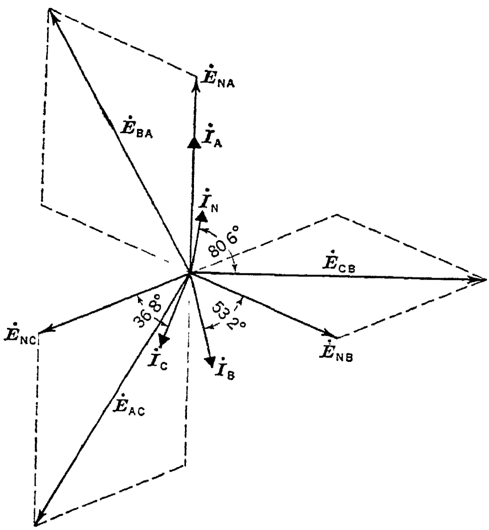


FIG. 14–26

14–15. Three-Phase, Four-Wire System, Balanced Y-Connected Load.—When the three single-phase loads in Fig. 14–21 are equal, or $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = \dot{Z}$, the vector equations for the voltages will remain the same as those given in equation (14–39). This is so since there is no shift in the voltage at the neutral point. The three line currents $\dot{I}_A$, $\dot{I}_B$, and $\dot{I}_C$ are equal in magnitude and separated from each other by an angle of 120°. The neutral current $\dot{I}_N$ is zero at all times, and the neutral conductor may be removed without disturbing the voltage and current relations.

Example 14–6.—The three impedances in Fig. 14–24 are made equal to each other, or $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = 3+j4$. They form a balanced three-phase load with a lagging power factor of cos 53.2° = 0.6. It is desired to find the line currents.

Solution.—If the same sequence and the same voltage-drop expressions as given in Example 14–5 are used, the line currents are:

$$\dot{I}_A = \frac{\dot{E}_{NA}}{\dot{Z}_a} = \frac{63.5\underline{/90^\circ}}{5\underline{/53.2^\circ}} = 12.7\underline{/36.8^\circ} = 10.15 + j7.62$$

$$\dot{I}_B = \frac{\dot{E}_{NB}}{\dot{Z}_b} = \frac{63.5\underline{/-30^\circ}}{5\underline{/53.2^\circ}} = 12.7\underline{/-83.2^\circ} = 1.54 - j12.57$$

$$\dot{I}_C = \frac{\dot{E}_{NC}}{\dot{Z}_c} = \frac{63.5\underline{/-150^\circ}}{5\underline{/53.2^\circ}} = 12.7\underline{/-203.2^\circ} = -11.69 + j4.95$$

From Kirchhoff's Current Law at point N, the current in the neutral wire is

$$\begin{aligned}\dot{I}_N &= -(\dot{I}_A + \dot{I}_B + \dot{I}_C)\\ &= -[(10.15 + j7.62) + (1.54 - j12.57) + (-11.69 + j4.95)] = 0\end{aligned}$$

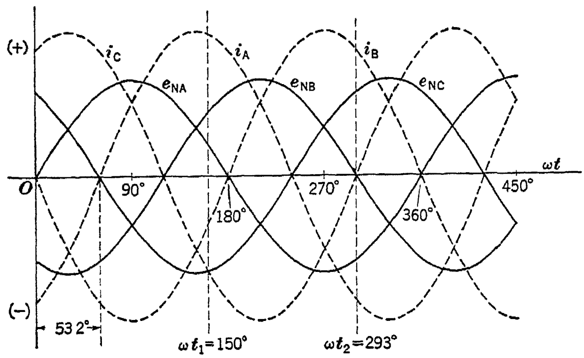


FIG. 14–27

For emphasis, attention is again called to the fact that, when the Y-connected load is balanced, the neutral current $\dot{I}_N$ is zero. It is immaterial whether the neutral wire is connected to the load or not. The voltage at the point N is not changed, since no neutral current flows.

The waves representing the three coil voltages and the three line currents are shown in Fig. 14–27. The algebraic sum of i_A, i_B, and i_C at every instant is zero since the load is balanced. At time t_1 when $\omega t_1 = 150°$, the current $i_A = +17.85$ is positive, or in Fig. 14–24 the current in line A is flowing from A to A'. At the same time t_1, the currents $i_B = -7.06$ and $i_C = -10.79$ are both negative, or in Fig. 14–24 the currents i_B and i_C are flowing from B' to B and C' to C, respectively. At time t_2 when $\omega t_2 = 293°$, there is no current flowing in line $C'C$; i_B is flowing from B to B' and i_A is now flowing from A' to A.

14–16. Power in Y-System.—Consider the circuit shown in Fig. 14–21. The average powers consumed by the loads $\dot{Z}_a$, $\dot{Z}_b$, and $\dot{Z}_c$, respectively, are:

$$P_a = E_{NA} I_{NA} \cos \theta_{I_{NA}}^{E_{NA}} = I_{NA}^2 R_A \qquad (14\text{–}44)$$

$$P_b = E_{NB} I_{NB} \cos \theta_{I_{NB}}^{E_{NB}} = I_{NB}^2 R_B \qquad (14\text{–}45)$$

$$P_c = E_{NC} I_{NC} \cos \theta_{I_{NC}}^{E_{NC}} = I_{NC}^2 R_C \qquad (14\text{–}46)$$

The total power is the arithmetical sum of the three, or

$$P = P_a + P_b + P_c \qquad (14\text{–}47)$$

When the three-phase load is *balanced*, the line currents are equal and the power factors are the same. The power consumed by each branch is the same, and the total power is equal to 3 times the power taken by each branch. Hence,

$$P = 3P_a = 3E_{NA} I_{NA} \cos \theta_{I_{NA}}^{E_{NA}} \qquad (14\text{–}48)$$

Since the line current is equal to the coil current and the coil voltage is equal to the line voltage divided by $\sqrt{3}$, equation (14–48) may be written as follows:

$$P = \sqrt{3}\ E_{\text{line}}\, I_{\text{line}} \cos \theta_{I_{NA}}^{E_{NA}} \qquad (14\text{–}49)$$

where $\cos \theta_{I_{NA}}^{E_{NA}}$ is usually considered to be the power factor of the balanced Y-connected load.

Equation (14–49) states that the total power taken by a balanced three-phase load is equal to $\sqrt{3}$ times the product of the line voltage, line current, and power factor of the load. The power factor angle $\theta_{I_{NA}}^{E_{NA}}$ of a balanced three-phase load is the angle between the coil voltage and the coil current and not the angle between the line-to-line voltage and the line current.

It can be seen from equation (14–48) that the total volt-amperes of a balanced three-phase system is 3 times the number of volt-amperes per phase. Thus,

$$\text{Total volt-amperes} = 3E_{NA} I_{NA} = 3E_{\text{coil}} I_{\text{coil}} \qquad (14\text{–}48a)$$

In terms of line voltage and line current,

$$\text{Total volt-amperes} = \sqrt{3}\ E_{\text{line}}\, I_{\text{line}} \qquad (14\text{–}49a)$$

Example 14–7.—Determine the total power consumed in the balanced three-phase load of Example 14–6.

Solution.—The line voltage is given as 110 volts and the line currents are 12.7 amp. The power factor is equal to cos 53.2°. Hence, substitution of values in equation (14–49) gives

$$P = \sqrt{3} \times 110 \times 12.7 \times 0.6 = 1450.8 \text{ watts}$$

14–17. Three-Phase, Three-Wire Systems With Y-Connected Loads.—(*a*) *Balanced Y-Connected Load:* It has been shown in Example 14–6 that for a balanced load the neutral current is zero and the voltage of the common point N is the voltage of the neutral of the system. The line-to-neutral voltage is equal to the line voltage divided by $\sqrt{3}$. Thus, the solution of a balanced Y-load connected to a three-phase, three-wire system is identical with that given for Example 14–6.

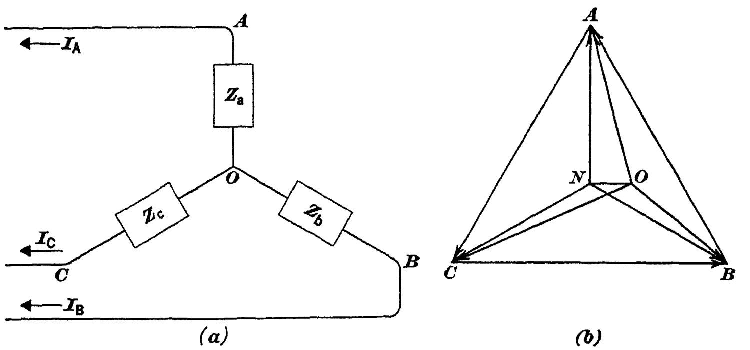


FIG. 14–28

(*b*) *Unbalanced Y-Connected Load:** Consider a Y-connected load, as shown in Fig. 14–28(*a*), connected to a three-phase, three-wire system. When the single-phase loads are unequal, or $\dot{Z}_a \neq \dot{Z}_b \neq \dot{Z}_c$, the magnitudes of the three load voltages will, in general, be unequal. If, for instance, $\dot{Z}_c$ is a very small impedance, the voltage across it will be small (since current is limited by the other two impedances) and the voltage of the point O will be almost the same as the voltage of the point C. It must always be true, however, that the vectors of load voltages add to give the corresponding line-to-line voltages, and a convenient means of expressing this fact graphically is to draw the triangle of Fig. 14–28(*b*). Line-to-line voltages form the outside of the triangle, while load voltages are drawn to the vertices from some point such as O;

* See Appendix E and Appendix F.

with this construction it is evident that relations similar to equations (14–34) to (14–36) in which the subscript N is replaced by O are always satisfied. The special case of balanced line-to-line voltages and balanced loads results in line-to-neutral voltages that are also balanced and that radiate from the point N at the center of the triangle.

In general the point O is displaced from the neutral position N. As the impedance $\dot{Z}_c$ approaches zero, the point O approaches C; and, as the impedance Z_c becomes larger, the point O moves away from C.

By varying the single-phase loads, the point O may be made to lie at any point within or without the triangle ABC. Owing to this "unstable" voltage condition, that is, since the voltages from the common point O to the three lines are not equal, unbalanced Y-connected loads are seldom used in three-phase, three-wire systems.

In the general case, the point O will be displaced from the neutral position of the supply system by some voltage $\dot{E}_{NO}$. In order to calculate the line currents it is necessary to know the voltage across each of the single-phase loads. These voltages from line to the common point O may be found by considering the vector diagram shown in Fig. 14–28(b). It is obvious from the triangles ANO, BNO, and CNO that

$$\dot{E}_{AN}+\dot{E}_{NO}+\dot{E}_{OA}=0 \tag{14–50}$$

$$\dot{E}_{BN}+\dot{E}_{NO}+\dot{E}_{OB}=0 \tag{14–51}$$

$$\dot{E}_{CN}+\dot{E}_{NO}+\dot{E}_{OC}=0 \tag{14–52}$$

or

$$\dot{E}_{OA}=\dot{E}_{NA}-\dot{E}_{NO} \tag{14–53}$$

$$\dot{E}_{OB}=\dot{E}_{NB}-\dot{E}_{NO} \tag{14–54}$$

$$\dot{E}_{OC}=\dot{E}_{NC}-\dot{E}_{NO} \tag{14–55}$$

If it is assumed that the line-to-line voltages are balanced, the magnitude of the voltages $\dot{E}_{NA}$, $\dot{E}_{NB}$, and $\dot{E}_{NC}$ to neutral can be obtained from the relation $E_{\text{coil}}=\dfrac{E_{\text{line}}}{\sqrt{3}}$. Their phase relations can be determined from the vector diagram. Thus, it is evident that the load voltages $\dot{E}_{OA}$, $\dot{E}_{OB}$, and $\dot{E}_{OC}$ can be found if $\dot{E}_{NO}$ is known. The determination of $\dot{E}_{NO}$ will now be considered.

The current flowing in each load is equal to the voltage across that load multiplied by the admittance of the load. Thus,

$$\dot{I}_A = \dot{I}_{OA} = \dot{E}_{OA}\dot{Y}_a = \dot{E}_{NA}\dot{Y}_a - \dot{E}_{NO}\dot{Y}_a \tag{14-56}$$

$$\dot{I}_B = \dot{I}_{OB} = \dot{E}_{OB}\dot{Y}_b = \dot{E}_{NB}\dot{Y}_b - \dot{E}_{NO}\dot{Y}_b \tag{14-57}$$

$$\dot{I}_C = \dot{I}_{OC} = \dot{E}_{OC}\dot{Y}_c = \dot{E}_{NC}\dot{Y}_c - \dot{E}_{NO}\dot{Y}_c \tag{14-58}$$

Since this is a three-phase, three-wire system, the vector sum of the three line currents is zero, or

$$\dot{I}_A + \dot{I}_B + \dot{I}_C = 0$$

Hence,

$$\dot{E}_{NA}\dot{Y}_a + \dot{E}_{NB}\dot{Y}_b + \dot{E}_{NC}\dot{Y}_c - \dot{E}_{NO}(\dot{Y}_a + \dot{Y}_b + \dot{Y}_c) = 0$$

from which

$$\dot{E}_{NO} = \frac{\dot{E}_{NA}\dot{Y}_a + \dot{E}_{NB}\dot{Y}_b + \dot{E}_{NC}\dot{Y}_c}{\dot{Y}_a + \dot{Y}_b + \dot{Y}_c} \tag{14-59}$$

When the voltages $\dot{E}_{NA}$, $\dot{E}_{NB}$, and $\dot{E}_{NC}$ are known and $\dot{E}_{NO}$ has been calculated, it is a simple matter to find $\dot{E}_{OA}$, $\dot{E}_{OB}$, and $\dot{E}_{OC}$ by the use of equations (14–53) to (14–55). The line currents are given by equations (14–56) to (14–58).

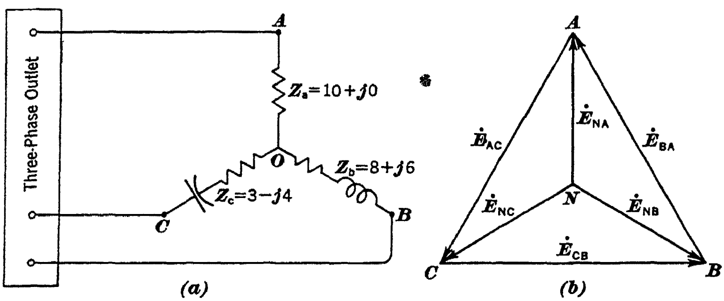


FIG. 14–29

Example 14–8.—Three unequal impedances, $\dot{Z}_a = 10 + j0$, $\dot{Z}_b = 8 + j6$, and $\dot{Z}_c = 3 - j4$, are connected in wye to a three-phase, three-wire, 220-volt system, Fig. 14–29(a). It is required to find the three line currents and the total power taken by this load. Also, draw the complete vector diagram.

Solution.—The sequence of the system voltages will be assumed as ABC. The voltage relations are then as shown in Fig. 14–29(b). Thus, the vector expressions for the voltage drops are:

$$\dot{E}_{CB} = 220\underline{/0°} \qquad \dot{E}_{NA} = 127\underline{/90°}$$

$$\dot{E}_{BA} = 220\underline{/120°} \qquad \dot{E}_{NB} = 127\underline{/-30°}$$

$$\dot{E}_{AC} = 220\underline{/240°} \qquad \dot{E}_{NC} = 127\underline{/-150°}$$

From the circuit, the admittances of the single-phase loads are:

$$\dot{Y}_a = \frac{1}{\dot{Z}_a} = \frac{1}{10\underline{/0^\circ}} = 0.1\underline{/0^\circ}$$

$$\dot{Y}_b = \frac{1}{\dot{Z}_b} = \frac{1}{10\underline{/36.8^\circ}} = 0.1\underline{/-36.8^\circ}$$

$$\dot{Y}_c = \frac{1}{\dot{Z}_c} = \frac{1}{5\underline{/-53.2^\circ}} = 0.2\underline{/53.2^\circ}$$

The voltage $\dot{E}_{NO}$ can be found by substituting values in equation (14–59). Thus, it is

$$\frac{(127\underline{/90^\circ} \times 0.1\underline{/0^\circ}) + (127\underline{/-30^\circ} \times 0.1\underline{/-36.8^\circ}) + (127\underline{/-150^\circ} \times 0.2\underline{/53\ 2^\circ})}{0.1\underline{/0^\circ} + 0.1\underline{/-36.8^\circ} + 0.2\underline{/53.2^\circ}}$$

or
$$\dot{E}_{NO} = 76.8\underline{/-103.8^\circ} = -18.5 - j74.6$$

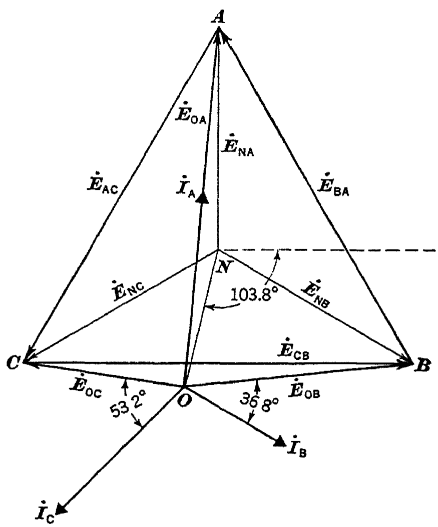


FIG. 14–30

From equations (14–53) to (14–55), the load voltage drops are:

$$\dot{E}_{OA} = \dot{E}_{NA} - \dot{E}_{NO} = 127\underline{/90^\circ} - 76.8\underline{/-103.8^\circ} = 202.5\underline{/84.8^\circ}$$

$$\dot{E}_{OB} = \dot{E}_{NB} - \dot{E}_{NO} = 127\underline{/-30^\circ} - 76.8\underline{/-103.8^\circ} = 128.9\underline{/5^\circ}$$

$$\dot{E}_{OC} = \dot{E}_{NC} - \dot{E}_{NO} = 127\underline{/-150^\circ} - 76.8\underline{/-103.8^\circ} = 92.2\underline{/173.0^\circ}$$

With the load voltages now known, it is a simple matter to solve for the line currents. They are:

$$\dot{I}_A = \dot{I}_{OA} = \dot{E}_{OA}\dot{Y}_a = 202.5\underline{/84.8^\circ} \times 0.1\underline{/0^\circ} = 20.25\underline{/84.8^\circ}$$

$$\dot{I}_B = \dot{I}_{OB} = \dot{E}_{OB}\dot{Y}_b = 128.9\underline{/5.0^\circ} \times 0.1\underline{/-36.8^\circ} = 12.89\underline{/-31.8^\circ}$$

$$\dot{I}_C = \dot{I}_{OC} = \dot{E}_{OC}\dot{Y}_c = 92.2\underline{/173.0^\circ} \times 0.2\underline{/53^\circ} = 18.44\underline{/226.0^\circ}$$

The vector diagram is shown in Fig. 14–30. Here is a case where the point O lies without the triangle ABC.

The powers taken by the load impedances are:

$$P_a = E_{OA}I_{OA} \cos \theta_{I_{OA}}^{E_{OA}} = 202.5 \times 20.25 \cos 0^\circ = 4100 \text{ watts}$$

$$P_b = E_{OB}I_{OB} \cos \theta_{I_{OB}}^{E_{OB}} = 128.9 \times 12.89 \cos 36.8^\circ = 1320 \text{ watts}$$

$$P_c = E_{OC}I_{OC} \cos \theta_{I_{OC}}^{E_{OC}} = 92.2 \times 18.44 \cos 53.2^\circ = 1010 \text{ watts}$$

The total power taken by the unbalanced three-phase load is

$$P = P_a + P_b + P_c = 6430 \text{ watts}$$

14–18. The Consideration of Phase Sequence in Unbalanced Three-Phase Circuits.—The question of phase sequence is not an important one in a balanced three-phase circuit. In the general case, however, where an unbalanced three-phase load is connected to a three-phase system, a change in phase sequence will change the magnitudes of the line currents. Thus, it is possible to have two sets of ammeter readings in the lines of a three-phase circuit, the values depending on the phase sequence of the applied voltages. This sometimes accounts for the difference between the calculated values and the actual readings obtained in the laboratory. The effect of the phase sequence on the magnitudes of the line currents may be illustrated by solving Example 14–8 for the sequence ACB.

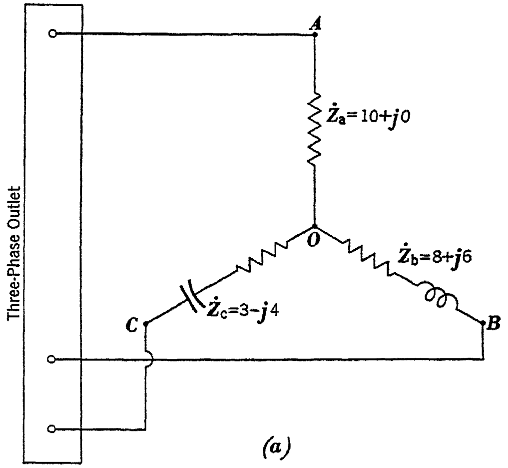
Example 14–9.—Solve Example 14–8 for the sequence ACB.

Solution.—If two leads at the three-phase outlet are interchanged, the sequence of the voltage drops across the load terminals is changed from ABC in Fig. 14–29 to ACB in Fig. 14–31. This change in sequence may be made by drawing the line voltage triangle as shown in Fig. 14–31(b). If it is assumed that $\dot{E}_{CB} = 220\underline{/0^\circ}$, as before, then:

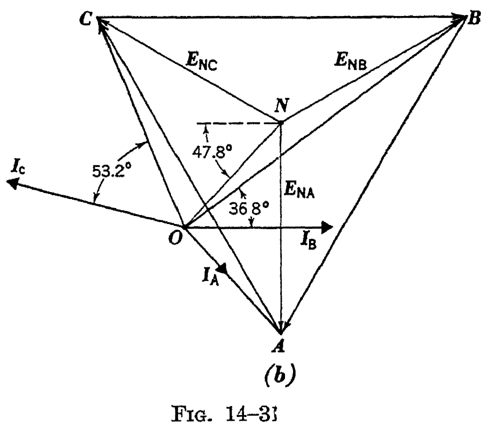
$$\dot{E}_{CB} = 220\underline{/0^\circ} \qquad \dot{E}_{NA} = 127\underline{/-90^\circ}$$

$$\dot{E}_{BA} = 220\underline{/-120^\circ} \qquad \dot{E}_{NB} = 127\underline{/30^\circ}$$

$$\dot{E}_{AC} = 220\underline{/120^\circ} \qquad \dot{E}_{NC} = 127\underline{/150^\circ}$$



(a)



(b)

FIG. 14-31

From equation (14-59), the voltage $\dot{E}_{NO}$ is

$$\frac{(127\underline{/-90^\circ}\times 0.1\underline{/0^\circ})+(127\underline{/30^\circ}\times 0.1\underline{/-36.8^\circ})+(127\underline{/150^\circ}\times 0.2\underline{/53.2^\circ})}{0.1\underline{/0^\circ}+0.1\underline{/-36.8^\circ}+0.2\underline{/53.2^\circ}}$$

or

$$\dot{E}_{NO}=84.2\underline{/227.8^\circ}$$

Substituting values in equations (14-53) to (14-55), we obtain:

$$\dot{E}_{OA}=127\underline{/-90^\circ}-84.2\underline{/227.8^\circ}=85.9\underline{/-48.9^\circ}$$

$$\dot{E}_{OB}=127\underline{/30^\circ}-84.2\underline{/227.8^\circ}=208\underline{/37^\circ}$$

$$\dot{E}_{OC}=127\underline{/150^\circ}-84.2\underline{/227.8^\circ}=137\underline{/113^\circ}$$

From equations (14–56) to (14–58), the line currents are:

$$\dot{I}_A = \dot{I}_{OA} = \dot{E}_{OA}\dot{Y}_a = 85.9\underline{/-48.9°} \times 0.1\underline{/0°} = 8.59\underline{/-48.9°}$$

$$\dot{I}_B = \dot{I}_{OB} = \dot{E}_{OB}\dot{Y}_b = 208\underline{/37°} \times 0.1\underline{/-36.8°} = 20.8\underline{/0.2°}$$

$$\dot{I}_C = \dot{I}_{OC} = \dot{E}_{OC}\dot{Y}_c = 137\underline{/113°} \times 0.2\underline{/53.2°} = 27.4\underline{/166.2°}$$

The vector diagram for the sequence ACB is given in Fig. 14–31(b). The powers consumed by the single-phase loads are:

$$P_a = E_{OA}I_{OA} \cos \theta_{I_{OA}}^{E_{OA}} = 85.9 \times 8.59 \cos 0° = 735 \text{ watts}$$

$$P_b = E_{OB}I_{OB} \cos \theta_{I_{OB}}^{E_{OB}} = 208 \times 20.8 \cos 36.8° = 3445 \text{ watts}$$

$$P_c = E_{OC}I_{OC} \cos \theta_{I_{OC}}^{E_{OC}} = 137 \times 27.4 \cos 53.2° = 2250 \text{ watts}$$

The total power is, as before,

$$P = P_a + P_b + P_c = 6430 \text{ watts}$$

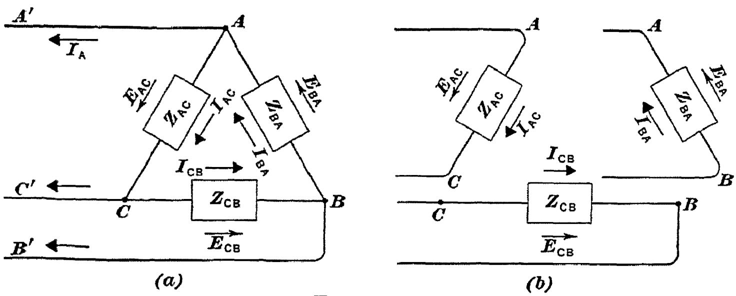


FIG. 14–32

It can be seen by comparing Figs. 14–30 and 14–31(b) that the point O has shifted its position to another point without the triangle. The branch load voltages for sequence ACB are different from those for sequence ABC. Likewise, the line currents obtained with the sequence ACB are different from the currents obtained with the sequence ABC. The powers taken by the individual branch impedances are different for the two sequences, but the total power consumed remains the same.

14–19. Three-Phase, Three-Wire Systems With Δ-Connected Loads.—In Fig. 14–32(a) are shown three impedances $\dot{Z}_{BA}$, $\dot{Z}_{AC}$, and $\dot{Z}_{CB}$ connected in delta to a three-phase, three-wire system. The voltage across each single-phase load is, by inspection, the known line-to-line voltage. There should be no difficulty in finding the current flowing in each branch of the delta by Ohm's Law, whether the Δ-connected load is balanced or unbalanced. The line currents $\dot{I}_A$, $\dot{I}_B$, and $\dot{I}_C$ are obtained by applying Kirchhoff's Current Law at points A, B, and C, respectively.

14–20. Unbalanced Δ-Connected Load.—When the three impedances of Fig. 14–32(*a*) are unequal, the branch currents in the delta will be unequal. It is evident that the line currents will also be unequal.

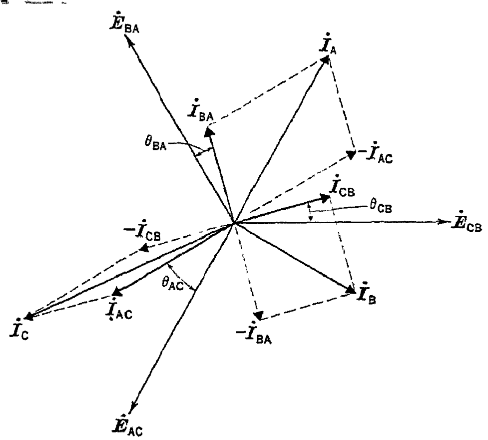


Fig. 14–33

In the solution of a three-phase circuit containing an unbalanced Δ-connected load, the voltage drops across the single-phase loads $\dot{E}_{CB}$, $\dot{E}_{BA}$, and $\dot{E}_{AC}$ are known. The branch currents can then be found very easily by solving the three single-phase circuits in Fig. 14–32(*b*). By applying Kirchhoff's Current Law at the junction points A, B, and C, the line currents can be determined. Thus, for sequence ABC, the branch currents are:

$$\dot{I}_{CB} = \frac{\dot{E}_{CB}}{\dot{Z}_{CB}} \qquad (14\text{–}60)$$

$$\dot{I}_{BA} = \frac{\dot{E}_{BA}}{\dot{Z}_{BA}} \qquad (14\text{–}61)$$

$$\dot{I}_{AC} = \frac{\dot{E}_{AC}}{\dot{Z}_{AC}} \qquad (14\text{–}62)$$

With the positive senses as indicated in Fig. 14–32(a), the application of Kirchhoff's Current Law will give the following expressions for the line currents:

$$\dot{I}_A = \dot{I}_{BA} - \dot{I}_{AC} \tag{14–63}$$

$$\dot{I}_B = \dot{I}_{CB} - \dot{I}_{BA} \tag{14–64}$$

$$\dot{I}_C = \dot{I}_{AC} - \dot{I}_{CB} \tag{14–65}$$

A vector diagram is given in Fig. 14–33.

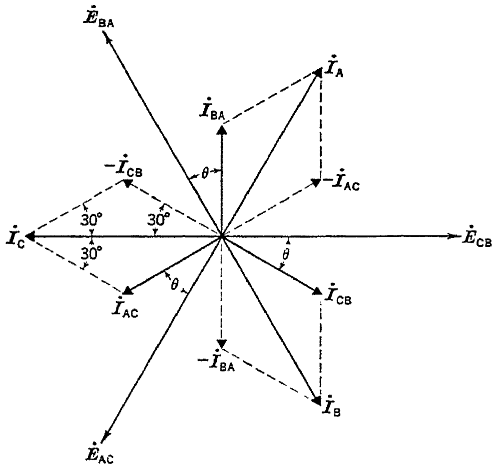


FIG. 14–34

14–21. Balanced Δ-Connected Load.—If the three impedances in Fig. 14–32(a) are equal, the load is said to be a balanced Δ-connected load. The circuit is called a balanced three-phase circuit. Since the applied voltages are equal in magnitude and 120° apart, the branch currents will be equal in magnitude and 120° apart. Their vector expressions are as given by equations (14–60) to (14–62). Each branch current makes the same power factor angle θ with the voltage across that branch. The vectors are shown in Fig. 14–34. The line currents are equal in magnitude and 120° apart and are given by equations (14–63) to (14–65). It will be observed from the vector diagram that the length of the line current vectors is $\sqrt{3}$ times the length of the branch current vectors. Thus, in a balanced Δ-system, the line voltage is equal to the branch voltage, or $E_{\text{line}} = E_{\text{branch}}$; but the line current is $\sqrt{3}$ times the branch current, or $I_{\text{line}} = \sqrt{3}\ I_{\text{branch}}$.

14–22. Power in Δ-System.—The powers consumed by the branches of a Δ-connected load are:

$$P_{BA} = E_{BA} I_{BA} \cos \theta_{I_{BA}}^{E_{BA}} = I_{BA}^2 R_{BA} \qquad (14\text{–}66)$$

$$P_{AC} = E_{AC} I_{AC} \cos \theta_{I_{AC}}^{E_{AC}} = I_{AC}^2 R_{AC} \qquad (14\text{–}67)$$

$$P_{CB} = E_{CB} I_{CB} \cos \theta_{I_{CB}}^{E_{CB}} = I_{CB}^2 R_{CB} \qquad (14\text{–}68)$$

The total power is the arithmetical sum of the three, or

$$P = P_{BA} + P_{AC} + P_{CB} \qquad (14\text{–}69)$$

When the load is *balanced*, the total power is equal to 3 times the power of one branch, or

$$P = 3 E_{BA} I_{BA} \cos \theta_{I_{BA}}^{E_{BA}} \qquad (14\text{–}70)$$

Since the line voltage is equal to the branch voltage and the line current is equal to $\sqrt{3}$ times the branch current, equation (14–70) may be written as follows:

$$P = \sqrt{3}\, E_{\text{line}}\, I_{\text{line}} \cos \theta_{I_{BA}}^{E_{BA}} \qquad (14\text{–}71)$$

where $\cos \theta_{I_{BA}}^{E_{BA}}$ is usually considered to be the power factor of the balanced Δ-connected load.

Equation (14–71) is the same as equation (14–49) for the balanced Y-system. The power factor angle $\theta_{I_{BA}}^{E_{BA}}$ of the balanced Δ-connected load is the power factor angle of the single-phase load.

From equation (14–70) the total volt-amperes of a balanced Δ-system is 3 times the number of volt-amperes per phase, or

$$\text{Total volt-amperes} = 3 E_{BA}\, I_{BA} = 3 E_{\text{coil}}\, I_{\text{coil}} \qquad (14\text{–}70a)$$

The total volt-amperes can also be expressed in terms of the line voltage and the line current. From equation (14–71),

$$\text{Total volt-amperes} = \sqrt{3}\, E_{\text{line}}\, I_{\text{line}} \qquad (14\text{–}71a)$$

Example 14–10.—Consider a balanced Δ-connected load, $\dot{Z}_{BA} = \dot{Z}_{AC} = \dot{Z}_{CB} = 3 - j4$, connected to a three-phase, three-wire, 110-volt system. Find the line currents and the total power taken by the entire load. Draw the vector diagram.

Solution.—For convenience, let the line voltage drops of sequence ABC be:

$$\dot{E}_{CB} = 110\underline{/0°}$$

$$\dot{E}_{BA} = 110\underline{/120°}$$

$$\dot{E}_{AC} = 110\underline{/240°}$$

By applying Ohm's Law to each of the three single-phase loads, the branch currents are found to be:

$$\dot{I}_{CB} = \frac{\dot{E}_{CB}}{\dot{Z}_{CB}} = \frac{110\underline{/0°}}{5\underline{/-53.2°}} = 22\underline{/53.2°}$$

$$\dot{I}_{BA} = \frac{\dot{E}_{BA}}{\dot{Z}_{BA}} = \frac{110\underline{/120°}}{5\underline{/-53.2°}} = 22\underline{/173.2°}$$

$$\dot{I}_{AC} = \frac{\dot{E}_{AC}}{\dot{Z}_{AC}} = \frac{110\underline{/240°}}{5\underline{/-53.2°}} = 22\underline{/293\ 2°}$$

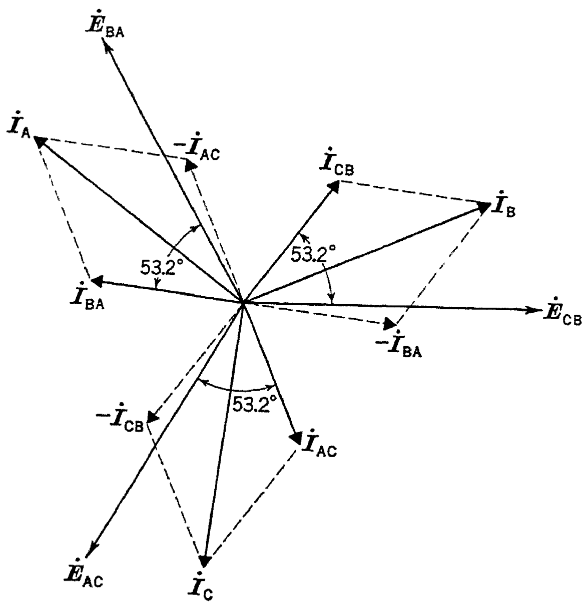


FIG. 14–35

With positive senses indicated in Fig. 14–32(*a*), the line currents as given by equations (14–63) to (14–65) are:

$$\dot{I}_A = \dot{I}_{BA} - \dot{I}_{AC} = (-21.83 + j2.68) - (8.6 - j20.26) = 38.1\underline{/143°}$$

$$\dot{I}_B = \dot{I}_{CB} - \dot{I}_{BA} = (13.2 + j17.6) - (-21.83 + j2.68) = 38.1\underline{/23°}$$

$$\dot{I}_C = \dot{I}_{AC} - \dot{I}_{CB} = (8.6 - j20.26) - (13.2 + j17.6) = 38.1\underline{/263°}$$

The total power is

$$P = \sqrt{3} \times 110 \times 38.1 \times \cos 53.2° = 4360 \text{ watts}$$

The vector diagram is shown in Fig. 14–35

14–23. Three-Wire, Y-Circuits With Line Impedances.—Consider the three-phase circuit indicated in Fig. 14–36. It is assumed that the internal, or winding, impedances of the alternator windings are equal to each other and that the impedance of each line wire is $\dot{Z}_m$. The generated coil emfs or voltage rises of the three-phase alternator are $\dot{E}_{N'A'}$, $\dot{E}_{N'B'}$, and $\dot{E}_{N'C'}$. Two methods of obtaining the line currents will be given. The first makes use of the development of Art. 14–17 and the "displaced neutral" convention; and the second solves the same problem directly from Kirchhoff's Laws.

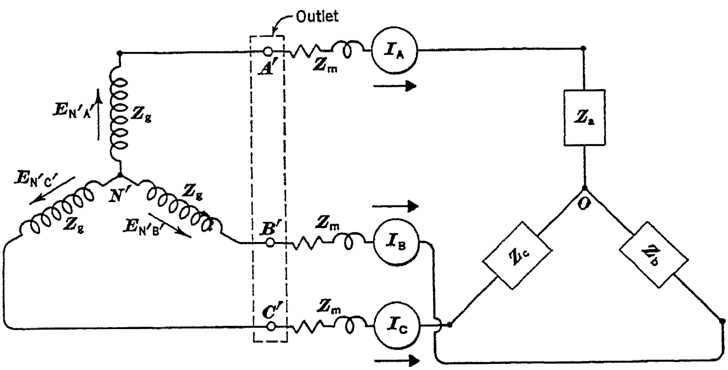


FIG. 14–36

Method A: In this method it is assumed that the winding impedance of the generator is negligible, and the circuit is considered as a Y-load connected to a three-phase outlet with terminals A', B', and C' (where the terminal voltages are assumed to be balanced at all times). That is, the line impedance $\dot{Z}_m$ and $\dot{Z}_a$ are grouped together to form one leg of the Y; and, similarly, for the other legs of the Y. If the impedances $\dot{Z}_a$, $\dot{Z}_b$, and $\dot{Z}_c$ are equal and the line impedances are equal, the problem resolves itself into a balanced Y-load connected to a three-phase, three-wire system. If the load and line impedances are all different, the circuit becomes an unbalanced Y-load connected to a three-phase, three-wire system. The solutions of these two cases are given in Art. 14–17.

Method B: In this method,* the internal, or winding, impedance $\dot{Z}_g$ of the generator is included in the consideration of the problem. The procedure is to write Kirchhoff's emf equations for the different closed paths and the current equation at the junction point O; and then solve the equations simultaneously for the line currents. Thus, if $\dot{E}_{N'A'}$, $\dot{E}_{N'B'}$, and $\dot{E}_{N'C'}$ are considered to be the internal generated voltage rises (equal to the line-to-neutral voltages at no load), then application of Kirchhoff's Emf Law and Current Law gives:

$$\dot{E}_{N'A'}-\dot{E}_{N'B'}=\dot{I}_A(\dot{Z}_g+\dot{Z}_m+\dot{Z}_a)-\dot{I}_B(\dot{Z}_b+\dot{Z}_m+\dot{Z}_g) \quad (14\text{–}72)$$

$$\dot{E}_{N'C'}-\dot{E}_{N'A'}=\dot{I}_C(\dot{Z}_g+\dot{Z}_m+\dot{Z}_c)-\dot{I}_A(\dot{Z}_a+\dot{Z}_m+\dot{Z}_g) \quad (14\text{–}73)$$

$$\dot{E}_{N'B'}-\dot{E}_{N'C'}=\dot{I}_B(\dot{Z}_g+\dot{Z}_m+\dot{Z}_b)-\dot{I}_C(\dot{Z}_c+\dot{Z}_m+\dot{Z}_g) \quad (14\text{–}74)$$

$$\dot{I}_A+\dot{I}_B+\dot{I}_C=0 \quad (14\text{–}75)$$

Solving equation (14–75) and any two of the other three equations simultaneously will give the desired line currents.

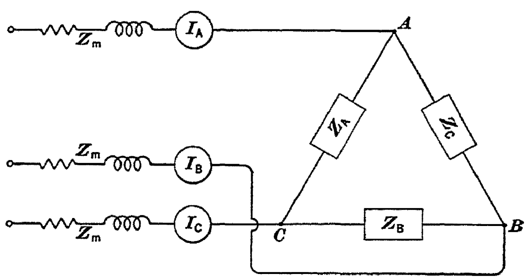


FIG. 14–37

14–24. Three-Wire, Δ-Circuits With Line Impedances.—A three-phase circuit with a Δ-connected load and line impedances is shown in Fig. 14–37. The line currents can be determined by applying Kirchhoff's Laws to the circuit as outlined in Method B in Art. 14–23. It may be more convenient, however, first to convert the Δ-connected load into a Y-connected load, as outlined in the Y-Δ transformation of Chapter 10 on Network Theorems. Then with this conversion, the solution of the line currents follows that outlined in either method of Art. 14–23.

* See Appendix E.

PROBLEMS

14–1. A lighting load $R=10$ ohms and an inductive coil for which $\dot{Z}=12+j16$ are connected to a two-phase, four-wire, 110-volt supply. The lighting load is connected to phase A, and the coil is connected to phase B. Find: (*a*) the currents flowing in each phase; (*b*) the power taken by each load and the total power delivered by the source. Draw the vector diagram.

14–2. A balanced two-phase load $\dot{Z}_A=\dot{Z}_B=8+j6$ is connected to a two-phase, four-wire, 220-volt system. Calculate the current and power in each phase. Draw the vector diagram.

14–3. A two-phase alternator is rated at 1,000 kva, 2300 volts, 60 cycles. What is the current rating of each winding?

14–4. A two-phase, three-wire generator is rated 15 kva, 220 volts, 60 cycles. (*a*) What is the current rating for each winding? (*b*) What are the three voltages measured at the terminal board of the machine when magnitude and phase are considered?

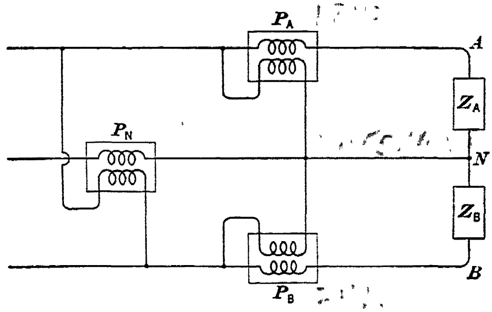


FIG. 14–38

14–5. Two single-phase loads, $\dot{Z}_A=10+j0$ and $\dot{Z}_B=8-j6$, are connected to a two-phase, three-wire, 110-volt system. Determine the three line currents and draw the vector diagram. Also, find the power taken by each load and the total taken by the combination.

14–6. A balanced two-phase load $\dot{Z}_A=\dot{Z}_B=4+j3$ is connected to a two-phase, three-wire, 100-volt generator. What are the currents in the three line wires? Draw the vector diagram. Also, find the power delivered by the generator.

14–7. Two single-phase loads, $\dot{Z}_A=2+j3$ and $\dot{Z}_B=4-j2$, are connected to a two-phase, three-wire, 110-volt system, as shown in the diagram of Fig. 14–38. The wattmeters are so connected that they all read up scale. Determine the reading of each wattmeter. Draw the vector diagram.

14–8. Three single-phase loads are connected to a two-phase, three-wire, 220-volt system, as shown in Fig. 14–39. Load $\dot{Z}_{AN}=9+j12$ is connected to the lines A and N; load $\dot{Z}_{BN}=24-j10$, to the lines B and N; and load $\dot{Z}_{AB}=10+j40$, to the lines A and B. Find: (*a*) the three line currents; (*b*) the power taken by each load and the total power dissipated by the combination. Draw the vector diagram.

14–9. Find the readings of the wattmeters connected in the circuit of Problem 14–8 when the middle wire is opened (*a*) at point *P* and (*b*) at point *Q*.

14–10. Three impedances, $\dot{Z}_a = 4 - j3$, $\dot{Z}_b = 7 + j3$, and $\dot{Z}_c = 2 + j5$, are connected in wye to a three-phase, four-wire, 115-volt system. The neutral of the system is connected to the common point of the load. Determine the three line currents and the current flowing in the neutral. Draw the vector diagram.

14–11. Three single-phase loads, $\dot{Z}_a = 3 + j0$, $\dot{Z}_b = 2 + j3$, and $\dot{Z}_c = 1 + j2$, are connected in wye to a three-phase, four-wire, 100-volt system. The common point of the load is connected to the neutral of the system. Determine the readings of four ammeters connected in the four conductors. Draw the vector diagram.

14–12. Repeat Problem 14–11 for the opposite phase sequence. Compare the line currents with those obtained in Problem 14–11.

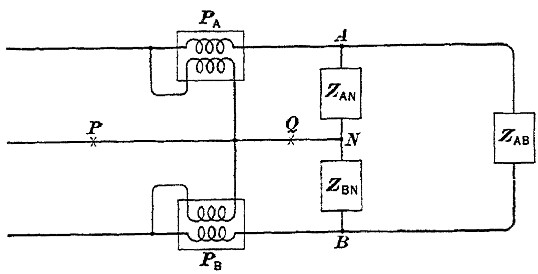


FIG. 14–39

14–13. Three equal resistances $R = 10$ are connected in wye to a three-phase, four-wire, 100-volt system. Find the vector line currents and draw the vector diagram. Also find the total power taken by using equation (14–49).

14–14. A balanced three-phase load $\dot{Z}_Y = 3 + j4$ is connected to a three-phase, four-wire, 110-volt system. The neutral is connected to the common point of the load. Find the vector currents in the four wires.

14–15. Repeat Problem 14–14 with the neutral open. Then find the total power taken by the balanced load by using equation (14–49).

14–16. A balanced Y-connected load $\dot{Z} = 6 - j8$ is connected to a three-phase, three-wire, 220-volt system. Calculate the line currents and draw the vector diagram. Also, find the power taken by this load by using equation (14–49).

14–17. Repeat Problem 14–16 for the opposite phase sequence. How do the magnitudes of the currents compare with those in Problem 14–16?

14–18. Three resistances, $R_a = 5$, $R_b = 3$, and $R_c = 7$, are connected in wye to a three-phase, three-wire, 110-volt system. Find the line currents for the phase sequence *ABC*. Draw the vector diagram.

14–19. Repeat Problem 14–18 for the sequence *ACB*.

14–20. Three impedances, $\dot{Z}_a = 10 + j0$, $\dot{Z}_b = 8 + j6$, and $\dot{Z}_c = 6 - j8$, are connected in wye to a three-phase, three-wire, 220-volt system. Assuming the sequence *ABC*, determine the three vector line currents and draw the vector diagram.

14–21. Repeat Problem 14–20 for the sequence ACB.

14–22. The three impedances given in Problem 14–10 are connected in delta to the same source. The impedance $\dot{Z}_a$ is between lines A and B; $\dot{Z}_b$ is between lines B and C; and $\dot{Z}_c$ is between lines C and A. Find: (a) the current flowing in each single-phase load; (b) the three line currents; (c) the power taken by each load and the power supplied to the combination. Draw the vector diagram.

14–23. A load consisting of three resistances, $R_a = 2$, $R_b = 4$, and $R_c = 6$, is connected to a three-phase, three-wire, 100-volt system. Compare the power and the line currents when the resistances are connected in wye and in delta.

14–24. A balanced Δ-load $R = 10$ is connected to a three-phase, three-wire, 100-volt system. Calculate the vector line currents and draw the vector diagram. Also calculate the total power by using equation (14–71).

14–25. Repeat Problem 14–24 for a balanced Δ-load consisting of three equal impedances $\dot{Z} = 3 + j4$.

14–26. Repeat Problem 14–24 for a balanced Δ-load formed by three equal single-phase loads $\dot{Z} = 3 - j4$.

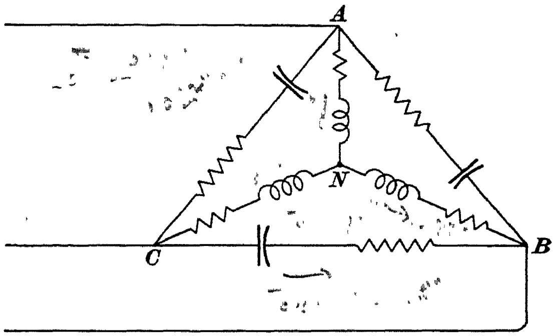


FIG. 14–40

14–27. A balanced Δ-load $\dot{Z} = 2 + j7$ is connected to a three-phase, three-wire, 100-volt system. Calculate the vector line currents and the power when one of the lines is opened.

14–28. A balanced Y-connected load $\dot{Z}_Y = 3 + j4$ and a balanced Δ-connected load $\dot{Z}_\Delta = 8 - j6$ are connected in parallel to a three-phase, three-wire, 110-volt system, as shown in Fig. 14–40. Find the three vector line currents and draw the vector diagram.

14–29. Four single-phase loads are connected to a three-phase, four-wire, 100-volt source, as shown in Fig. 14–41. Using $\dot{E}_{CB}$ as a reference vector, find: (a) the vector currents in the different loads; (b) the vector line and neutral currents. Draw the vector diagram.

14–30. If a voltmeter is connected between the points N and D in Problem 14–29, what will it read? What are the vector voltage from D to E and that from E to F?

14–31. A balanced Y-connected load $\dot{Z}_Y = 6 + j8$ is receiving power from a three-phase, three-wire, 220-volt alternator. The impedance of each line wire from generator to load is $\dot{Z}_L = 1 + j2$. Find: (*a*) the vector line currents; (*b*) the vector line voltages at the load.

14–32. Repeat Problem 14–31 with the balanced load $\dot{Z}_\Delta = 6 + j8$ connected in delta.

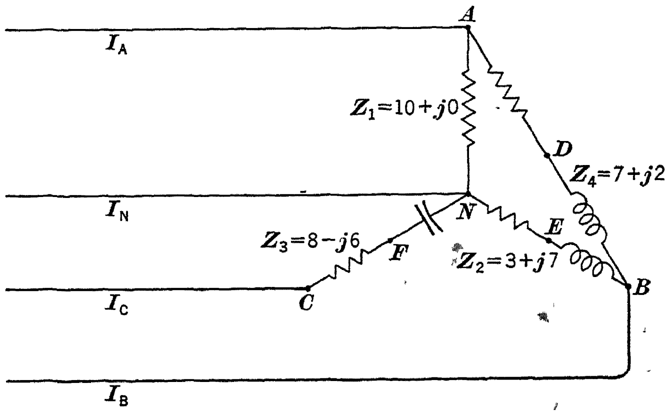


FIG. 14–41

14–33. A coil having a resistance of 5 ohms and an inductive reactance of 8 ohms is connected in series with a non-inductive resistor of 10 ohms across the lines A and B of a three-phase, three-wire, 110-volt system. What is the reading of a voltmeter connected between the third line C and the junction point of the coil and the resistor?

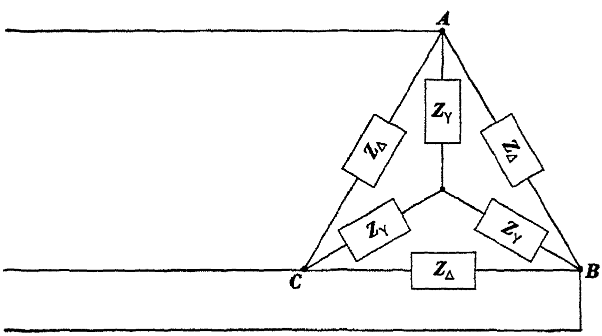


FIG. 14–42

14–34. The following statement is made: "An unbalanced Δ-connected load is connected to a three-phase, three-wire system. If the ratio of x to r is the same for all the branches, a change in phase sequence of the applied voltages will not change the magnitudes of the line currents." Assume the system voltage to be 100 volts, and show whether or not the statement applies to the following cases

(*a*) The Δ-load consists of

$$\dot{Z}_{AB} = 5\angle 0°, \quad Z_{BC} = 10\angle 0°, \quad \text{and} \quad \dot{Z}_{CA} = 20\angle 0°$$

(*b*) The three branches of the delta are

$$\dot{Z}_{AB} = 5\angle 36.8°, \quad \dot{Z}_{BC} = 10\angle 36.8°, \quad \text{and} \quad \dot{Z}_{CA} = 20\angle 36.8°$$

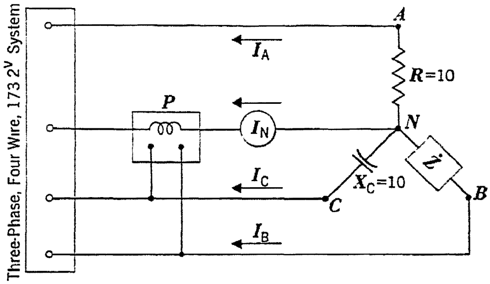


FIG. 14–43

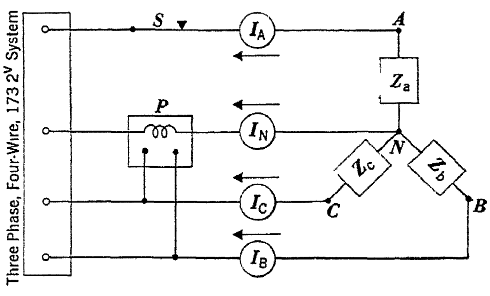


FIG. 14–44

14–35. Will the statement in Problem 14–34 apply to a Y-connected load? Check by assuming that the impedances given in the preceding problem are connected in wye.

14–36. Fig. 14–42 shows a balanced Y-load and a balanced Δ-load connected to a balanced three-phase system. If $\dot{Z}_Y = R = 10\angle 0°$ and $\dot{Z}_\Delta = 10\angle -90°$, determine the power factor of the combination.

14–37. Repeat Problem 14–36 if $\dot{Z}_Y = 4 - j3$ and $\dot{Z}_\Delta = 6 + j8$.

14–38. In Fig. 14–43, the impedance $\dot{Z} = 10\angle\theta$ is either a pure R, a pure L, or a pure C. For the condition indicated in the diagram, $I_N = 7.32$ amp and $P = 0$. Determine the type of element for $\dot{Z}$.

14–39. Each of the single-phase loads $\dot{Z}_a$, $\dot{Z}_b$, and $\dot{Z}_c$ in Fig 14–44 is a single element; that is, each may be a pure R, a pure L, or a pure C. The sequence of the three-phase, four-wire, 173.2-volt system is ABC, and the voltage drop $\dot{E}_{CB} = 173.2\underline{/0°}$. With the switch S closed, the readings of the meters are: $I_A = I_B = I_C = 10$ amp; $I_N = 7.32$ amp; $P = 0$. With the switch S open, the readings of the meters are: $I_B = I_C = 10$ amp; $I_N = 17.32$ amp; $P = 0$. Determine the value and type of each of the single elements.

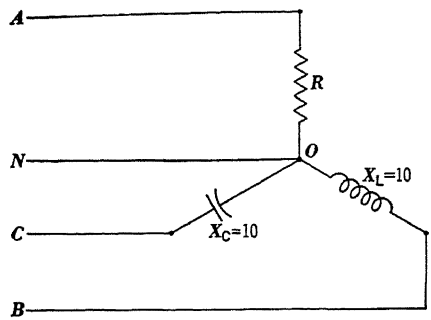


FIG. 14–45

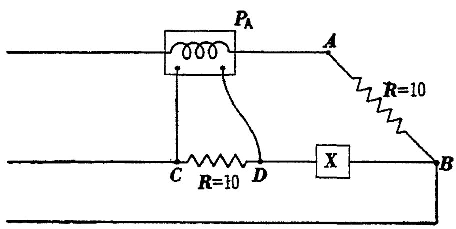


FIG. 14–46

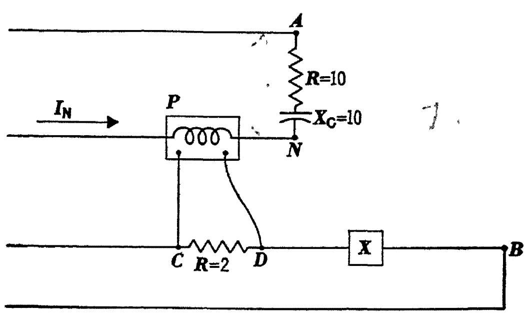


FIG. 14–47

14–40. Fig. 14–45 shows a Y-load connected to a balanced three-phase, four-wire, 100-volt system. Determine the value of R such that the voltages from O to the lines remain the same when the neutral wire is disconnected from O.

14–41. A balanced Y-load takes 30 kw at a lagging power factor of 0.5 from a balanced three-phase system. What total power will be taken from the same balanced three-phase source if the three impedances of the Y-load are reconnected as a balanced Δ-load?

14–42. Two single-phase loads are connected to a balanced three-phase, 100-volt system, as shown in Fig. 14–46. The unknown element X is a pure reactance. Determine the type and value of X such that the wattmeter P_A will read zero. In your analysis, use sequence ABC and $\dot{E}_{CB}$ as reference.

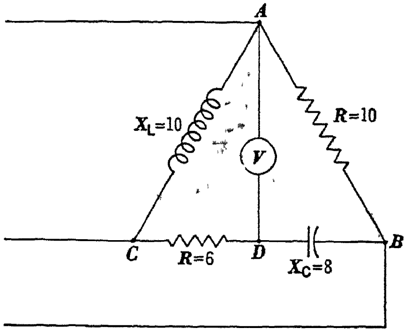


Fig. 14–48

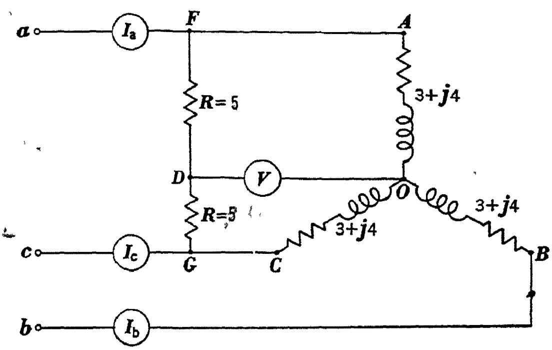


Fig. 14–49

14–43. Fig. 14–47 shows two single-phase loads connected to a three-phase, four-wire, 173.2-volt system. The impedance across the terminals C and B is $\dot{Z}_{CB} = 2 \pm jX$. Find the value and type of the element X such that the wattmeter P will read zero. Use sequence ABC and $\dot{E}_{CB}$ as reference. $X = X_c =$

14–44. Fig. 14–48 shows three single-phase loads connected in delta to a three-phase, 100-volt system. The sequence is ABC and the reference voltage is $\dot{E}_{CB}$. Find the voltmeter reading connected across A and D. $V = 41$ ∠

14–45. A balanced Y-load and a single-phase load are connected to a three-phase, 173.2-volt system, as shown in Fig. 14–49. The sequence is ABC and $\dot{E}_{CB} = 173.2\angle 0°$. (*a*) Find the reading of the voltmeter connected between O and D. (*b*) Find the readings of the ammeters.

CHAPTER 15

POWER MEASUREMENTS IN THREE-PHASE CIRCUITS

15–1. Three-Wattmeter Method.—The power delivered by a three-phase supply to either a Y-connected or a Δ-connected load may be measured by having a wattmeter in each of the single-phase (or branch) loads, as shown in Fig. 15–1.

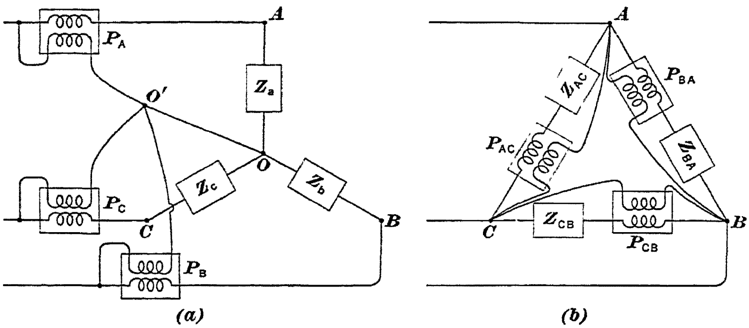


FIG. 15–1

In Fig. 15–1(*a*), each wattmeter reads the product of the voltage across its potential coil, the current flowing through its current coil, and the cosine of the phase difference angle of these two quantities. Thus,

$$P = EI \cos \theta_I^E$$

For clearness, the following notation will be used in this chapter. All voltages at the load must be interpreted as voltage drops unless otherwise stated. In the double-subscript notation, the positive sense of a current or that of a voltage drop is from the first subscript to the second subscript. For example, the positive sense of the current $\dot{I}_{OA}$ is from O to A, and the positive sense (*i. e.*, from plus polarity to minus polarity) of the voltage drop $\dot{E}_{OA}$ is also from O to A.

Wattmeter P_A reads the power taken by the single-phase load $\dot{Z}_a$, or

$$P_A = E_{OA} I_{OA} \cos \theta_{I_{OA}}^{E_{OA}} = I_{OA}^2 R_a \qquad (15\text{–}1)$$

Similar expressions for the power taken by the single-phase loads $\dot{Z}_b$ and $\dot{Z}_c$ are:

$$P_B = E_{OB} I_{OB} \cos \theta_{I_{OB}}^{E_{OB}} = I_{OB}^2 R_b \qquad (15\text{–}2)$$

$$P_C = E_{OC} I_{OC} \cos \theta_{I_{OC}}^{E_{OC}} = I_{OC}^2 R_c \qquad (15\text{–}3)$$

For a Δ-connected load, as indicated in Fig. 15–1(*b*), the wattmeters read the power taken by the single-phase loads $\dot{Z}_{BA}$, $\dot{Z}_{AC}$, and $\dot{Z}_{CB}$. The readings are:

$$P_{BA} = E_{BA} I_{BA} \cos \theta_{I_{BA}}^{E_{BA}} = I_{BA}^2 R_{BA} \qquad (15\text{–}4)$$

$$P_{AC} = E_{AC} I_{AC} \cos \theta_{I_{AC}}^{E_{AC}} = I_{AC}^2 R_{AC} \qquad (15\text{–}5)$$

$$P_{CB} = E_{CB} I_{CB} \cos \theta_{I_{CB}}^{E_{CB}} = I_{CB}^2 R_{CB} \qquad (15\text{–}6)$$

The total power absorbed by the three-phase load, connected either in wye or in delta, is obviously the arithmetical sum* of the powers taken by the different branches. Thus,

$$P_Y = P_A + P_B + P_C \qquad (15\text{–}7)$$

$$P_\Delta = P_{BA} + P_{AC} + P_{CB} \qquad (15\text{–}8)$$

Equations (15–7) and (15–8) hold whether the three-phase loads are balanced or unbalanced. If the loads connected in wye or in delta are balanced (irrespective of the power factor of the polyphase load), then

$$P_A = P_B = P_C \qquad (15\text{–}9)$$

$$P_{BA} = P_{AC} = P_{CB} \qquad (15\text{–}10)$$

Also,

$$P_Y = 3P_A \qquad (15\text{–}11)$$

$$P_\Delta = 3P_{BA} \qquad (15\text{–}12)$$

Hence, for a balanced three-phase load, a single wattmeter may be used to obtain the power consumed by a load, as indicated by equations (15–11) and (15–12).

* The arithmetical sum is used here because none of the wattmeter readings can be negative when the meters are connected correctly.

15–2. Three-Wattmeter Method With Point O' "Floating." It is not always possible in the case of a Y-connected load to get at the common point O which is required for the connections of the wattmeters shown in Fig. 15–1(*a*). Neither is it always practical to break into a Δ-connected load as shown in Fig. 15–1(*b*). When this is the case, three wattmeters may be connected as shown in Fig. 15–2, with the point O' floating (that is, not connected to anything), to measure power absorbed by a three-phase load. The *algebraic* sum* of the three wattmeter readings gives the true power taken by the polyphase load, or

$$P = P_A + P_B + P_C \tag{15–13}$$

Equation (15–13) holds for any degree of unbalancing or for any wave form, whether the load is Y-connected or Δ-connected.

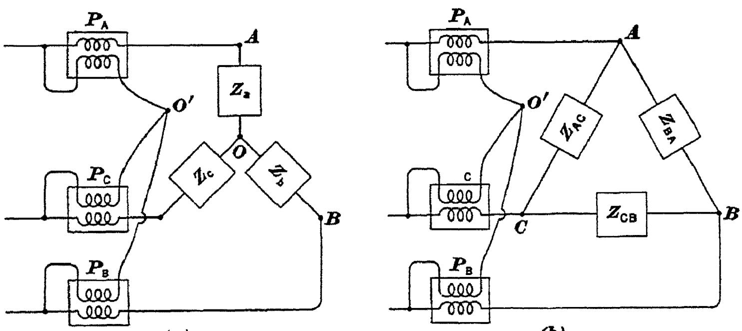


FIG. 15–2

If the meter coils of the three wattmeters in Fig. 15–2 are identical, the three readings will be alike when the load is balanced. The point O' is at the same potential as point O and $e_{OO'} = 0$.

If the meter coils are different, as is possible in wattmeters used in the laboratory, the point O' will establish itself at the point of equilibrium of the voltage drops across the potential coil circuits, as explained under Y-connected loads in three-phase, three-wire systems. In this case, the three wattmeter readings will not be alike, even when the polyphase load is balanced. The *algebraic* sum of their indications, however, is the total power consumed.

* A wattmeter reading is negative when the angle θ_I^E is greater than 90°.

If the common point O of the Y-connected load in Fig. 15–2(a) is connected to the neutral of the system and this neutral wire carries current, then equation (15–13) does not give the true power when the three wattmeters are not alike and O' is floating; this is so because $e_{OO'} \neq 0$ and $i_{OA}+i_{OB}+i_{OC} \neq 0$. This will be seen in the next article where equation (15–19) holds only when equation (15–16) is valid. The algebraic sum of the three wattmeter readings may be made to give the true power by connecting O' to the neutral. This obviously is equivalent to having a wattmeter in each branch of the Y-connected load.

15–3. Proof of the Three-Wattmeter Method With Point O' "Floating."—Consider Fig. 15–2(a). The instantaneous power in any single-phase circuit is defined as the product of the instantaneous values of current and voltage. Then, for three single-phase loads connected in wye, the total instantaneous power is

$$p = e_{OA}i_{OA} + e_{OB}i_{OB} + e_{OC}i_{OC} \tag{15–14}$$

Since this is a three-phase, three-wire system,

$$i_{OA} + i_{OB} + i_{OC} = 0 \tag{15–15}$$

Let $e_{OO'}$ be the instantaneous potential difference between O and O'. Then, it is obvious that

$$e_{OO'}(i_{OA} + i_{OB} + i_{OC}) = 0 \tag{15–16}$$

or

$$e_{OO'}i_{OA} + e_{OO'}i_{OB} + e_{OO'}i_{OC} = 0 \tag{15–16a}$$

Subtracting equation (15–16) from equation (15–14) gives:

$$p = (e_{OA} - e_{OO'})i_{OA} + (e_{OB} - e_{OO'})i_{OB} + (e_{OC} - e_{OO'})i_{OC} \tag{15–17}$$

But,

$$e_{O'O} + e_{OA} = e_{OA} - e_{OO'} = e_{O'A}$$
$$e_{O'O} + e_{OB} = e_{OB} - e_{OO'} = e_{O'B}$$
$$e_{O'O} + e_{OC} = e_{OC} - e_{OO'} = e_{O'C}$$

Therefore, equation (15–17) may be written as follows:

$$p = e_{O'A}i_{OA} + e_{O'B}i_{OB} + e_{O'C}i_{OC} \tag{15–18}$$

The average power is

$$P = \frac{1}{T}\int_0^T p\,dt$$

or

$$P = \frac{1}{T}\int_0^T e_{O'A}i_{OA}\,dt + \frac{1}{T}\int_0^T e_{O'B}i_{OB}\,dt + \frac{1}{T}\int_0^T e_{O'C}i_{OC}\,dt \tag{15–19}$$

This is the sum of the readings of the three actual wattmeters. It should be noted that equation (15–19) is true only when equation (15–16) is true.

The proof of the case shown in Fig. 15–2(*b*) is obvious when the Δ-connected load is transformed into a Y-connected load, as outlined in Chapter 10 on Network Theorems.

15–4. Two-Wattmeter Method.—It will be seen that point O' may be at *any* potential without altering equations (15–19) and (15–13). When the point O', Fig. 15–2, is connected to the line C, the reading of the wattmeter P_C is zero because the potential across the potential coil of that wattmeter is zero. Equation (15–13) becomes

$$P = P_A + P_B \tag{15–20}$$

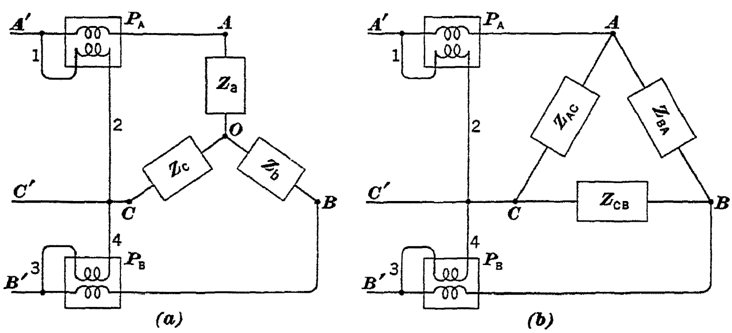


FIG. 15–3

Similarly, if the point O' is connected to line B or to line A, the power delivered to the load has one of the following values:

$$\left.\begin{aligned} P &= P_A + P_C \\ P &= P_B + P_C \end{aligned}\right\} \tag{15–20a}$$

Consider the case when the point O' is connected to line C. Since the wattmeter P_C reads zero, it may be taken out of the circuit. The two wattmeters left in the circuit, as in Fig. 15–3, will give the total power dissipated in the load if their readings are added properly (see Art. 15–7). This is known as the two-wattmeter method of measuring power delivered by a three-phase, three-wire system. It can be noted from Fig. 15–3(*a*) that wattmeter P_A does not read the power delivered to Z_a and wattmeter

P_B does not give the power delivered to Z_b. The two-wattmeter method (algebraic sum of P_A and P_B) gives the true power taken from a three-phase, three-wire system without regard to the balance of the load or to the wave form.

If the Y-connected load in Fig. 15–3(a) is connected to a three-phase, four-wire system, then three wattmeters are necessary to measure the total power correctly, except when the neutral wire carries no current. *In general, to read the total power correctly at least* $(n-1)$ *wattmeters* are necessary in an n-wire system.*

15–5. Proof of the Two-Wattmeter Method for Measuring Power Taken by a Y-Connected Load in a Three-Phase, Three-Wire System.—The instantaneous power taken by a Y-connected load, Fig. 15–3(a), is

$$p = e_{OA}i_{OA} + e_{OB}i_{OB} + e_{OC}i_{OC} \qquad (15\text{–}21)$$

Since $e_{CA} = e_{CO} + e_{OA}$,

$$e_{OA} = e_{CA} - e_{CO} = e_{CA} + e_{OC}$$

Also, $e_{CB} = e_{CO} + e_{OB}$, or

$$e_{OB} = e_{CB} - e_{CO} = e_{CB} + e_{OC}$$

Then, by substitution of these values in equation (15–21), the power expression becomes:

$$p = (e_{CA} + e_{OC})i_{OA} + (e_{CB} + e_{OC})i_{OB} + e_{OC}i_{OC}$$

or

$$p = e_{CA}i_{OA} + e_{CB}i_{OB} + e_{OC}(i_{OA} + i_{OB} + i_{OC}) \qquad (15\text{–}22)$$

In a three-phase, three-wire system the algebraic sum of the three instantaneous line currents is zero. Since i_{OA}, i_{OB}, and i_{OC} are also line currents, then equation (15–22) reduces to

$$p = e_{CA}i_{OA} + e_{CB}i_{OB} \qquad (15\text{–}23)$$

The average power is found in the usual manner. Thus,

$$P = \frac{1}{T}\int_0^T e_{CA}i_{OA}\,dt + \frac{1}{T}\int_0^T e_{CB}i_{OB}\,dt \qquad (15\text{–}24)$$

When the point O' is connected to line C, $\frac{1}{T}\int_0^T e_{CA}i_{OA}\,dt$ is read by wattmeter P_A and wattmeter P_B reads the value $\frac{1}{T}\int_0^T e_{CB}i_{OB}\,dt$.

* *Measurement of the Energy of Polyphase Currents*, by A. Blondel, Proceedings of the International Congress held at Chicago, 1893, p. 112.

Attention is called to the fact that, in setting up equation (15–23), the voltage e_{CA} from line C (in which there is no current coil of a wattmeter) to line A and the voltage e_{CB} from line C again to line B were used; and such combinations as e_{CA} and e_{BC} or e_{AC} and e_{CB} were not used. The correct result may be obtained by using the voltages e_{AC} and e_{BC} with the currents i_{AO} and i_{BO}. Hence, the total power consumed by a balanced or an unbalanced Y-connected load in a three-phase, three-wire system is

$$P_Y = P_A + P_B$$

or

$$P_Y = E_{CA} I_{OA} \cos \theta_{I_{OA}}^{E_{CA}} + E_{CB} I_{OB} \cos \theta_{I_{OB}}^{E_{CB}} \tag{15–25}$$

Also,

$$P_Y = E_{AC} I_{AO} \cos \theta_{I_{AO}}^{E_{AC}} + E_{BC} I_{BO} \cos \theta_{I_{BO}}^{E_{BC}} \tag{15–26}$$

Since $\dot{I}_{AA'} = \dot{I}_{OA}$ and $\dot{I}_{BB'} = I_{OB}$, then

$$P_Y = E_{CA} I_{AA'} \cos \theta_{I_{AA'}}^{E_{CA}} + E_{CB} I_{BB'} \cos \theta_{I_{BB'}}^{E_{CB}} \tag{15–25a}$$

or

$$P_Y = E_{AC} I_{A'A} \cos \theta_{I_{A'A}}^{E_{AC}} + E_{BC}\, I_{B'B} \cos \theta_{I_{B'B}}^{E_{BC}} \tag{15–26a}$$

The following example illustrates the use of these equations.

Example 15–1.—Three equal single-phase loads $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = 8 + j6$ are connected in wye to a three-phase, three-wire, 60-cycle, 100-volt system, as shown in Fig. 15–3(a). By using the two-wattmeter method, find the total power taken.

Solution.—If the sequence ABC is used and $\dot{E}_{CB}$ is taken as reference, then the expressions for the voltage drops are:

$$\dot{E}_{CB} = 100\underline{/0^\circ} \qquad \dot{E}_{OA} = \frac{100}{\sqrt{3}}\underline{/90^\circ}$$

$$\dot{E}_{BA} = 100\underline{/120^\circ} \qquad \dot{E}_{OB} = 57.7\underline{/-30^\circ}$$

$$\dot{E}_{AC} = 100\underline{/240^\circ} \qquad \dot{E}_{OC} = 57.7\underline{/210^\circ}$$

The line currents are:

$$\dot{I}_{AA'} = \dot{I}_{OA} = \frac{\dot{E}_{OA}}{\dot{Z}_a} = \frac{57.7\underline{/90^\circ}}{10\underline{/36.8^\circ}} = 5.77\underline{/53.2^\circ}$$

$$\dot{I}_{BB'} = \dot{I}_{OB} = \frac{\dot{E}_{OB}}{\dot{Z}_b} = \frac{57.7\underline{/-30^\circ}}{10\underline{/36.8^\circ}} = 5.77\underline{/-66.8^\circ}$$

$$\dot{I}_{CC'} = \dot{I}_{OC} = \frac{\dot{E}_{OC}}{\dot{Z}_c} = \frac{57.7\underline{/210^\circ}}{10\underline{/36.8^\circ}} = 5.77\underline{/173.2^\circ}$$

The vector diagram is shown in Fig. 15–4. It can be seen from this vector diagram that the phase angle between $\dot{E}_{CA}$ and $\dot{I}_{OA} = \dot{I}_{AA'}$ is equal to that between $\dot{E}_{AC}$ and $\dot{I}_{AO} = \dot{I}_{A'A}$ and the phase angle between $\dot{E}_{CB}$ and $\dot{I}_{OB} = \dot{I}_{BB'}$ is equal to that between $\dot{E}_{BC}$ and $\dot{I}_{BO} = \dot{I}_{B'B}$. That is,

$$\theta_{I_{OA}}^{E_{CA}} = 6.8° \quad \text{and} \quad \theta_{I_{OB}}^{E_{CB}} = 66.8°$$

$$\theta_{I_{AO}}^{E_{AC}} = 6.8° \quad \text{and} \quad \theta_{I_{BO}}^{E_{BC}} = 66.8°$$

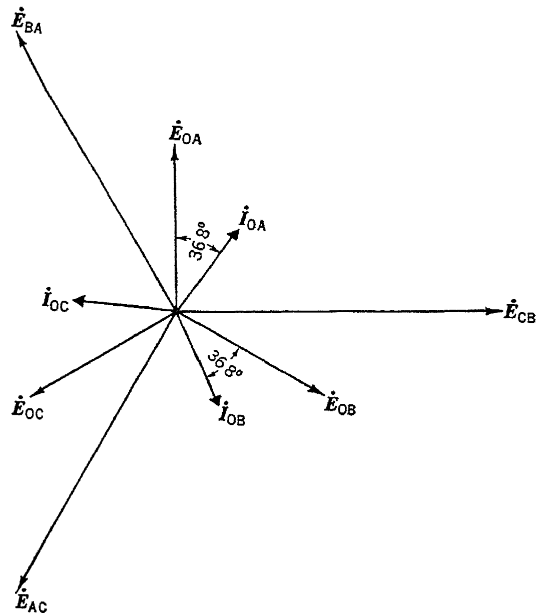


FIG. 15–4

Then either equation (15–25) or equation (15–26) will give the total power taken by the load. Thus,

$$P_Y = 100 \times 5.77 \cos 6.8° + 100 \times 5.77 \cos 66.8° = 798.9 \text{ watts}$$

As a check,

$$P_Y = I_A^2 R_a + I_B^2 R_b + I_C^2 R_c = 3 I_A^2 R_a = 3 \times 5.77^2 \times 8 = 798.9 \text{ watts}$$

15–6. Proof of the Two-Wattmeter Method for Measuring Power Taken by a Δ-Connected Load.—The instantaneous power taken by the three impedances in Fig. 15–3(*b*) is

$$p = e_{BA} i_{BA} + e_{AC} i_{AC} + e_{CB} i_{CB} \tag{15–27}$$

When Kirchhoff's Current Law is applied at points A and B, equation (15–27) becomes

$$p = e_{BA}i_{BA} + e_{AC}(i_{BA} - i_{AA'}) + e_{CB}(i_{BA} + i_{BB'})$$
$$= -e_{AC}i_{AA'} + e_{CB}i_{BB'} + i_{BA}(e_{BA} + e_{AC} + e_{CB})$$

or
$$p = e_{CA}i_{AA'} + e_{CB}i_{BB'} + i_{BA}(e_{BA} + e_{AC} + e_{CB}) \qquad (15\text{–}28)$$

Since $e_{BA} + e_{AC} + e_{CB} = 0$, equation (15–28) reduces to

$$p = e_{CA}i_{AA'} + e_{CB}i_{BB'} \qquad (15\text{–}29)$$

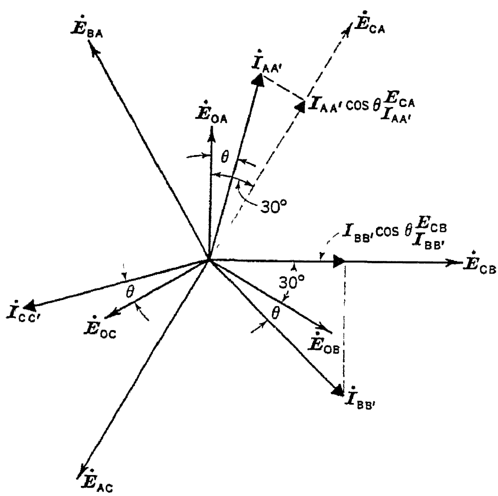


FIG. 15–5

Integrating equation (15–29) over one period gives the average power. Thus,

$$P = \frac{1}{T}\int_0^T e_{CA}i_{AA'}\,dt + \frac{1}{T}\int_0^T e_{CB}i_{BB'}\,dt \qquad (15\text{–}30)$$

where $\frac{1}{T}\int_0^T e_{CA}i_{AA'}\,dt$ is given by the reading of wattmeter P_A and wattmeter P_B reads $\frac{1}{T}\int_0^T e_{CB}i_{BB'}\,dt$. That is,

$$P_\Delta = E_{CA}I_{AA'} \cos \theta_{I_{AA'}}^{E_{CA}} + E_{CB}I_{BB'} \cos \theta_{I_{BB'}}^{E_{CB}} \qquad (15\text{–}30a)$$

or
$$P_\Delta = E_{AC}I_{A'A} \cos \theta_{I_{A'A}}^{E_{AC}} + E_{BC}I_{B'B} \cos \theta_{I_{B'B}}^{E_{BC}} \qquad (15\text{–}30b)$$

It should be noted that equations (15–30a) and (15–25a) are similar expressions and equations (15–30b) and (15–26a) are also similar expressions, because the currents are line currents.

15–7. Methods of Determining Whether the Two-Wattmeter Readings Should Be Added or Subtracted in Balanced Three-Phase Loads.—If the impedances $\dot{Z}_a$, $\dot{Z}_b$, and $\dot{Z}_c$ in Fig. 15–3(*a*) are inductive and equal, the Y-connected load is a balanced three-phase load with lagging power factor. A vector diagram for the *ABC* sequence of voltages is shown in Fig. 15–5. It can be seen from the vector diagram that, for a lagging power factor and positive values for θ, the angle between $\dot{E}_{CA}$ and $\dot{I}_{AA'}$ is $\theta_{I_{AA'}}^{E_{CA}} = (30° - \theta)$ and the angle between $\dot{E}_{CB}$ and $\dot{I}_{BB'}$ is $\theta_{I_{BB'}}^{E_{CB}} = (30° + \theta)$. Then, from equation (15–25a), in which it is assumed that P_A is in line *A* and P_B is in line *B*,

$$P_A = E_{CA} I_{AA'} \cos \theta_{I_{AA'}}^{E_{CA}} = E_{CA} I_{AA'} \cos (30° - \theta) \qquad \textbf{(15–31)}$$

and

$$P_B = E_{CB} I_{BB'} \cos \theta_{I_{BB'}}^{E_{CB}} = E_{CB} I_{BB'} \cos (30° + \theta) \qquad \textbf{(15–32)}$$

For a balanced three-phase load with leading power factor, θ assumes negative values in equations (15–31) and (15–32). Thus, for a balanced Y-connected or Δ-connected load, the relations between the power factor and the power are as follows:

(*a*) When P.F. $= 1$ or $\theta = 0°$, the readings of the two wattmeters will be alike, or $P_A = P_B$. Both readings are positive for $\theta = 0°$ to $\theta = \pm 60°$.

(*b*) When P.F. $= 0.5$ lagging or $\theta = 60°$, P_B will be zero and P_A will represent the total power.

(*c*) When P.F. $= 0.5$ leading or $\theta = -60°$, P_A will be zero and P_B will represent the total power.

(*d*) When P.F. < 0.5 or $\theta > 60°$, either positive or negative, the smaller wattmeter reading is negative and it must be subtracted from the larger wattmeter reading to obtain the correct power taken by the load (see the vector diagram in Fig. 15–6).

(*e*) When P.F. $=0$ or $\theta = \pm 90°$, the wattmeter readings will be alike in value but opposite in sign. However, these conditions are never attainable in practical circuits.

When a test is performed *on a balanced three-phase circuit* in the laboratory and the power factor is not known, the sign of the smaller reading of the two wattmeters may be determined in the following ways:

Method (*a*): If the wattmeters are connected alike (and both wattmeters read up-scale), that is, if the current is led from the load to corresponding current terminals and the corresponding ends of the potential coils are connected to the common line, in which there is no current coil, the readings of both wattmeters are positive. But, if the larger-reading wattmeter reads up-scale and either the potential or current leads of the smaller-reading wattmeter must be reversed to make this second wattmeter read up-scale, then the sign of the smaller reading is negative.

Method (*b*): Let the potential lead of the low-reading wattmeter (both wattmeters being connected to read up-scale without regard to symmetry), which is connected to the common line with no current coil, be disconnected and transferred to the line in which the current coil of the other wattmeter is connected. If the deflection of the wattmeter is now in the same direction as before, then its reading, as originally connected, is positive and should be added to the larger reading to obtain the total power received by the load. However, if the deflection is reversed, then its reading, as originally connected, is negative and should be subtracted from the larger reading to determine the total power supplied to the load.

Example 15–2.—Three single-phase loads $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = 10\underline{/70°}$ are connected in wye to a three-phase, three-wire, 173.2-volt system, as shown in Fig. 15–3(*a*). Determine the sign of the smaller wattmeter reading by Method (*b*).

Solution.—If sequence *ABC* is used and $\dot{E}_{CB}$ is taken as reference, then the expressions for the voltage drops are:

$$\dot{E}_{CB} = 173.2\underline{/0°} \qquad \dot{E}_{OA} = 100\underline{/90°}$$

$$\dot{E}_{BA} = 173.2\underline{/120°} \qquad \dot{E}_{OB} = 100\underline{/-30°}$$

$$\dot{E}_{AC} = 173.2\underline{/240°} \qquad \dot{E}_{OC} = 100\underline{/210°}$$

The line currents are:

$$\dot{I}_{AA'} = \dot{I}_{OA} = \frac{\dot{E}_{OA}}{\dot{Z}_a} = \frac{100\underline{/90^\circ}}{10\underline{/70^\circ}} = 10\underline{/20^\circ}$$

$$\dot{I}_{BB'} = \dot{I}_{OB} = \frac{\dot{E}_{OB}}{\dot{Z}_b} = \frac{100\underline{/-30^\circ}}{10\underline{/70^\circ}} = 10\underline{/-100^\circ}$$

$$\dot{I}_{CC'} = \dot{I}_{OC} = \frac{\dot{E}_{OC}}{\dot{Z}_c} = \frac{100\underline{/210^\circ}}{10\underline{/70^\circ}} = 10\underline{/140^\circ}$$

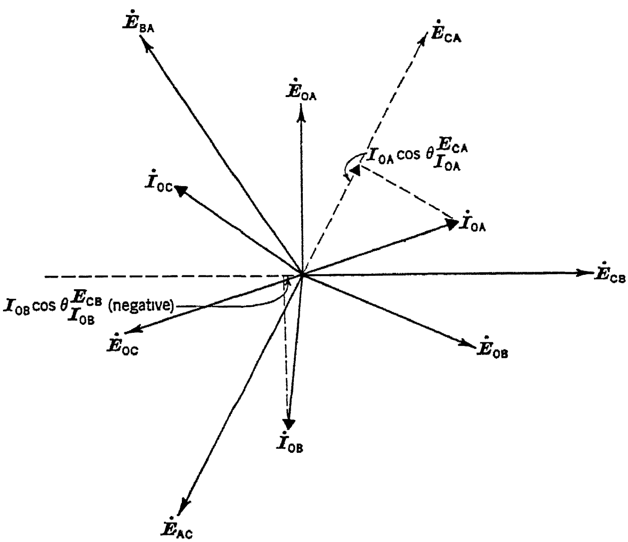


FIG. 15–6

The vector diagram is given in Fig. 15–6. The actual wattmeter readings (in absolute values), as obtained in the laboratory, are:

$$P_A = E_{CA} I_{OA} \cos \theta_{I_{OA}}^{E_{CA}} = 173.2 \times 10 \times \cos 40^\circ = 1328 \text{ watts}$$

$$P_B = E_{CB} I_{OB} \cos \theta_{I_{OB}}^{E_{CB}} = \pm 173.2 \times 10 \times \cos 100^\circ = \pm 271 \text{ watts}$$

To determine whether the smaller reading P_B is positive or negative, move potential lead 4 from line C to line A, Fig. 15–3(a). If the meter still reads up-scale, the reading is positive; if the deflection is reversed, the reading is negative. When the potential lead 4 is moved from line C to line A, it can be seen from Fig. 15–3(a) that wattmeter P_B now reads

$$P_B = E_{AB} I_{OB} \cos \theta_{I_{OB}}^{E_{AB}}$$

From Fig. 15–6 the deflection should reverse because the angle between E_{CB} and I_{OB} is greater than 90° and the angle between E_{AB} and I_{OB} is less than 90°. The voltage E_{AB} is used, and not E_{BA}, because E_{CB} is the voltage used in finding P_B and E_{CB} is from potential lead 4 to potential lead 3.

15–8. Two-Wattmeter Method for Determining the Power Factor of a Balanced Three-Phase Load.—When a balanced three-phase load is connected to a three-phase, three-wire system, its power factor may be determined from the readings of two wattmeters which are connected to read the total power supplied to the load. For simplicity, let E be the line-to-line voltage and I be the line current. Expanding equations (15–31) and (15–32), which are based on sequence ABC for the line voltage drops and on the fact that θ assumes positive values for lagging power factor, gives:

$$P_A = EI\ (\cos 30° \cos\theta + \sin 30° \sin\theta) \qquad (15\text{–}33)$$

$$P_B = EI\ (\cos 30° \cos\theta - \sin 30° \sin\theta) \qquad (15\text{–}34)$$

From these equations,

$$P_A + P_B = 2EI \cos 30° \cos\theta = \sqrt{3}\ EI \cos\theta \qquad (15\text{–}35)$$

$$P_A - P_B = 2EI \sin 30° \sin\theta = EI \sin\theta \qquad (15\text{–}36)$$

Dividing equation (15–36) by equation (15–35), we obtain:

$$\frac{P_A - P_B}{P_A + P_B} = \frac{EI \sin\theta}{\sqrt{3}\ EI \cos\theta}$$

or

$$\tan\theta = \sqrt{3}\left[\frac{P_A - P_B}{P_A + P_B}\right] \qquad (15\text{–}37)$$

When the power factor angle θ is known, the power factor can be found by looking up the cosine of the angle.

It must be remembered that, if wattmeter P_A is connected in the line marked A and wattmeter P_B is connected in the line marked B, then equation (15–37) requires that the phase sequence be ABC. Also, equation (15–37) gives positive values for θ when the power factor is lagging (P_B = smaller wattmeter reading) and gives negative values when the power factor is leading (P_B = larger wattmeter reading). In the laboratory, it is convenient to proceed in the following manner:

Step 1: Determine the phase sequence of the three line wires with the aid of a phase-sequence indicator.

Step 2: Mark the line wires A, B, and C for sequence ABC.

Step 3: Connect wattmeter P_A in line marked A and wattmeter P_B in line marked B.

When, for example, P_A reads 1500 watts and P_B reads -200 watts, then

$$\tan\theta = \sqrt{3}\left[\frac{1500+200}{1500-200}\right]$$

and

$$\theta = +66.2°$$

or

$$\text{P.F.} = \cos 66.2° = 0.404 \text{ (lagging)}$$

When P_A reads -200 watts and P_B reads 1500 watts, then

$$\tan\theta = \sqrt{3}\left[\frac{-200-1500}{-200+1500}\right]$$

$$\theta = -66.2°$$

$$\text{P.F.} = \cos(-66.2°) = 0.404 \text{ (leading)}$$

PROBLEMS

15–1. Three similar single-phase loads $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = 3 + j4$ are connected in wye to a three-phase, four-wire, 60-cycle, 110-volt system, as shown in Fig. 15–7. The three wattmeters are alike. What are the readings of the wattmeters and what is their algebraic sum which will give the total power taken by the balanced polyphase load? Draw the vector diagram to approximate scale. Check the results by finding the I^2R of each branch.

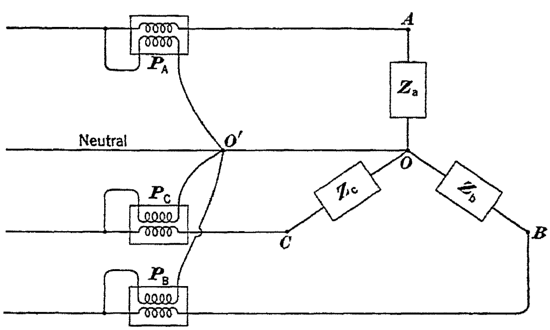


FIG. 15–7

15–2. If the neutral wire of the circuit in Problem 15–1 is disconnected from the points O and O', what would be the readings of the wattmeters?

15–3. Three single-phase loads $\dot{Z}_{BA} = \dot{Z}_{AC} = \dot{Z}_{CB} = 3 - j4$ are connected in delta to a three-phase, three-wire, 60-cycle, 110-volt system, as shown in Fig. 15–8. What are the six wattmeter readings and what is the algebraic sum of the readings of P_A, P_B, and P_C which will give the total power taken by the polyphase load? Check the total power by the use of the formula $P = \sqrt{3}\, EI \cos\theta$.

15-4. Three similar single-phase loads $\dot{Z} = 3 + j4$ are connected in wye to a three-phase, three-wire, 60-cycle, 110-volt system. The two-wattmeter method is used to measure the power taken by the load. What are the readings of the wattmeters and what is their algebraic sum which will give the total power taken by the balanced polyphase load? Draw the vector diagram to scale and check the analytical data.

15-5. Repeat Problem 15-4 with the single-phase loads connected in delta.

15-6. If $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = 10 + j0$ in Fig. 15-3(*a*) and the Y-load is connected to a three-phase, three-wire, 110-volt system, demonstrate that $P_A = P_B$. Draw the vector diagram.

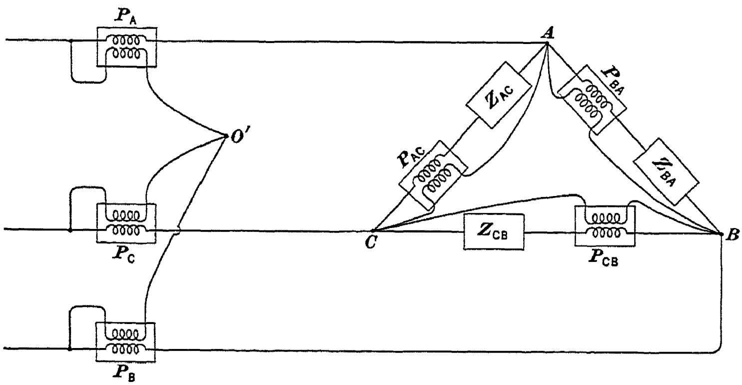


FIG. 15-8

15-7. If the impedances of Problem 15-6 are changed to $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = 10\underline{/60°}$, demonstrate that one wattmeter reading is zero. Draw the vector diagram to scale and check the data. Repeat also for $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = 10\underline{/-60°}$.

15-8. The impedances in Problem 15-6 are changed to $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = 10\underline{/80°}$. Show that the smaller wattmeter reading is negative. Draw the vector diagram to scale and check the data.

15-9. The impedances in Problem 15-6 are changed to $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = 10\underline{/90°}$. Show that the wattmeter readings are equal and opposite in sign. Draw the vector diagram. Repeat for $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = 10\underline{/-90°}$.

15-10. Three single-phase loads are connected to a three-phase, four-wire, 220-volt source, as shown in Fig. 15-9. The sequence is such that E_{BA} leads E_{CB} by 120°. Connect correctly the minimum number of wattmeters with current coils in the lines to measure the total power taken from the three-phase source. What is their algebraic sum which will give the total power? Draw the vector diagram.

15–11. Repeat Problem 15–10 with the direction of rotation of the generator reversed. Note the magnitudes of the line and neutral currents and the readings of the wattmeters as connected in Problem 15–10.

15–12. A balanced three-phase load operating at 0.8 power factor lagging takes 100 amp per line wire. The voltage from line to line is 2300 volts. Determine the reading of each of two wattmeters connected to read correctly the total power. Check the total power by the use of the formula $P = \sqrt{3}\, EI \cos \theta$.

15–13. Three identical impedance coils are connected in wye to a three-phase, three-wire, 60-cycle system and consume 4000 watts at 0.6 power factor. How many watts will they take when connected in delta to the same system and what will be the power factor?

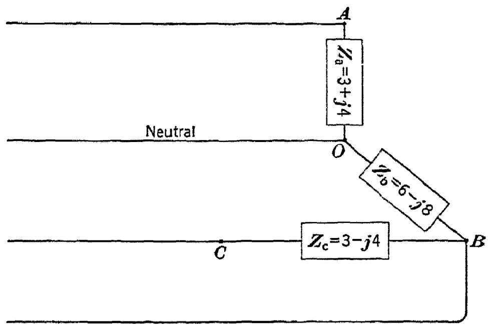


FIG. 15–9

15–14. Two wattmeters are connected correctly to measure the total power taken by a balanced three-phase load. One of the meters indicates negatively an amount equal to one-half of the positive indication of the other wattmeter. What must be the power factor of the load? Illustrate your solution by a vector diagram.

15–15. A balanced three-phase load takes 15 kw from a three-phase, three-wire, 220-volt system. The line current is 50 amp. If two wattmeters were connected correctly to measure the total power, what would be the reading of each meter? Draw the vector diagram.

15–16. Two wattmeters connected correctly to measure the total power of a balanced three-phase load read 1000 kw and −374 kw, respectively. Determine the power factor of the load.

15–17. A polyphase wattmeter is used to measure the power taken from a three-phase, 440-volt system. The current coils are in lines *A* and *B*. Ammeters in lines *A*, *B*, and *C* read 100, 180, and 180 amp, respectively. The wattmeter indicates 20 kw but it is afterwards found that the potential circuit to line *B* is broken. Determine the power, assuming the sequence *ABC* for both the line voltages and the line currents. (Attention is called to the fact that there are several possible answers.)

15–18. Show that the sign of the smaller wattmeter reading may also be determined by disconnecting the potential lead of the high-reading wattmeter which is connected to the line with the no-current coil and transferring it to the line which contains the current coil of the other wattmeter. If the deflection is still up-scale, the two wattmeter readings should be added. If the deflection is reversed, their difference gives the total power.

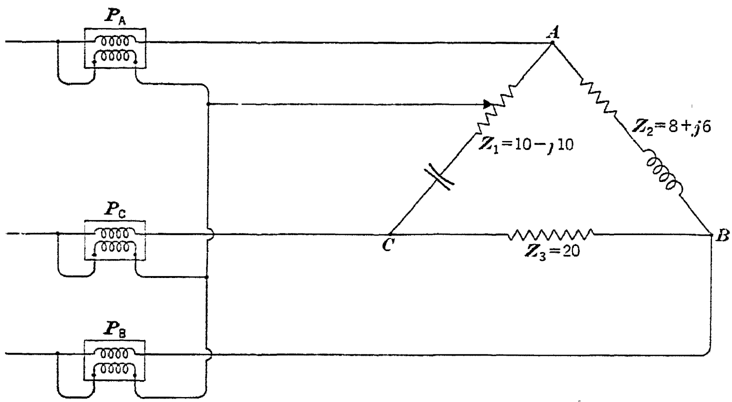


FIG. 15–10

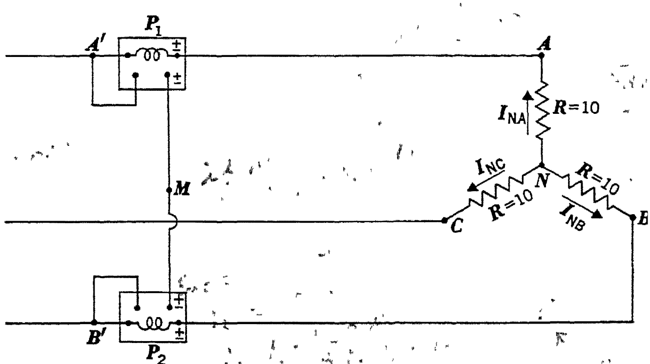


FIG. 15–11

15–19. Three single-phase loads are connected in delta to a three-phase, three-wire, 100-volt system, as shown in Fig. 15–10. E_{BA} leads E_{CB} by 120°. With $\dot{E}_{CB}$ as a reference vector, plot curves of the readings of wattmeters P_A, P_B, and P_C as functions of the resistance included between the slider and the point A.

15–20. Prove in detail that $(n-1)$ wattmeters are necessary to read correctly the total power in an n-wire system.

15–21. A balanced Y-load is connected to a three-phase, three-wire, 173.2-volt system, as shown in Fig. 15–11. The two identical wattmeters are connected "alike" with point M floating (that is, the wattmeters are connected properly to read the total power delivered to the load if M is connected to line C). For uniformity, use sequence ABC for the voltages and $\dot{E}_{CB}$ as reference. Find the readings of wattmeters P_1 and P_2.

15–22. A balanced three-phase, Y-connected load is connected to a three-phase, three-wire system, as in Fig. 15–12. The wattmeter P_A reads 1000 watts, wattmeter P_B reads 100 watts, and each ammeter reads 10 amp. When the potential coil lead of P_B is disconnected from line C and touched to line A, the reading of P_B reverses. In the analysis, assume that the sequence is ABC and take $\dot{E}_{CB}$ as reference. (*a*) What is the resistance of each branch of the Y-connected load? (*b*) What is the power factor of the load? (*c*) What is the line voltage of the three-phase system?

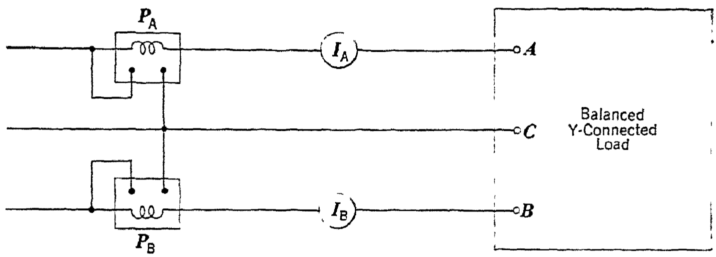


FIG. 15–12

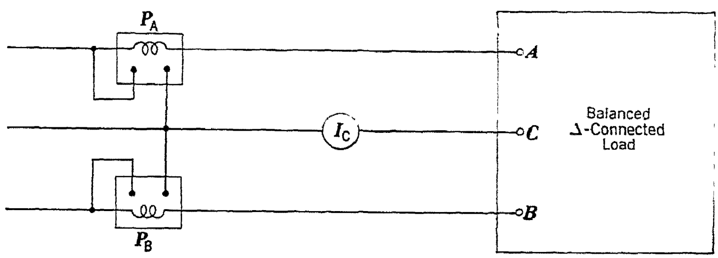


FIG. 15–13

15–23. A balanced three-phase, Δ-connected load is connected to a three-phase, three-wire system, as shown in Fig. 15–13. The wattmeter P_A reads 100 watts, wattmeter P_B reads 1000 watts, and the ammeter I_C reads 17.32 amp. When the potential coil lead of P_A is disconnected from line C and is touched to line B, the reading of P_A reverses. In the analysis, assume that the sequence is ABC and take $\dot{E}_{CB}$ as reference. (*a*) What is the resistance of each branch of the Δ-connected load? (*b*) What is the power factor of the load? (*c*) What is the line voltage of the three-phase system?

15–24. A balanced inductive Y-load is connected to a three-phase, three-wire system, similar to that shown in Fig. 15–12. The wattmeters are connected properly to read the total power. The total power taken is 2598 watts. The wattmeter P_A reads 1732 watts, and the wattmeter P_B reads 866 watts.

If the potential lead of P_A is disconnected from line C and touched to line B, what will its reading be? In the analysis, assume that the sequence is ABC and take $\dot{E}_{CB}$ as reference.

15–25. A balanced Y-load $\dot{Z}_a = \dot{Z}_b = \dot{Z}_c = R \pm jX = 10\underline{/\pm\theta}$ is connected to a three-phase, 173.2-volt system, as shown in Fig. 15–3(*a*). The wattmeters are connected properly to read the total average power consumed by the load. The reading of wattmeter P_A is 1500 watts. When the potential lead 2 of P_A is transferred from line C to line B, the wattmeter P_A reads zero. If the sequence is ABC and $\dot{E}_{CB}$ is used as reference, find: (*a*) the R and X of the load, stating whether X is inductive or capacitive; (*b*) the reading of P_B.

CHAPTER 16

SYMMETRICAL COMPONENTS

In the operation of polyphase systems, it is usually attempted to keep all loads, and hence the voltages and currents, as nearly balanced as possible. In spite of this fact, there are systems where quite severe unbalance may exist. This may be particularly the case where heavy single-phase loads, such as electric furnaces, are connected. Unbalance is also caused by unsymmetrical short circuits, such as line-to-line shorts.

The solution of a problem involving unbalanced impedances, voltages, and currents is frequently quite laborious. When sufficient data are available, such a problem can be solved by the application of Ohm's Law and Kirchhoff's Laws. This method, however, is often inconvenient and may even lead to considerable difficulty when rotating machines are involved. The problems involved in the operation of balanced three-phase circuits—or, in general, of balanced polyphase circuits—are comparatively simple and easy of solution. This condition suggests that *the solution of unbalanced polyphase systems may be accomplished conveniently by the combination of solutions of balanced systems.* A very convenient method based upon the fact that any unsymmetrical system of vectors may be resolved into a definite number of symmetrical sets of vectors was developed by C. L. Fortescue.* This procedure is known as the *Method of Symmetrical Components.*

As most polyphase circuits are three-phase, only unbalanced three-phase systems will be considered in this chapter. The voltages and currents are assumed to be sinusoidal. It will be shown first that two balanced, three-phase systems of currents (or voltages) of opposite phase sequence will add up to an unbalanced system of currents (or voltages) whose vector sum is zero. Then it will be indicated that any unbalanced three-phase system of currents (or voltages) whose vector sum is not zero can be resolved into two symmetrical three-phase systems and one single-phase system. The determination of power of unbalanced three-

* Fortescue, C. L., *Methods of Symmetrical Coordinates Applied to the Solution of Polyphase Networks*, Trans. AIEE, 1918, page 1027.

phase circuits in terms of the different system components will also be considered.

The advantage of this method lies in the analysis of problems involving the unbalanced loading of three-phase transformer banks, line-to-line or line-to-ground shorts, the operation of a three-phase induction motor on an unbalanced system of voltages, etc. The justification for including this chapter is that a student should have a reading knowledge of the Method of Symmetrical Components since an increasing number of articles appearing in the technical press assume some knowledge of this method. Only a brief explanation of the method will be given. The method will be applied to the solution of simple and unbalanced three-phase circuits which are familiar to the reader.

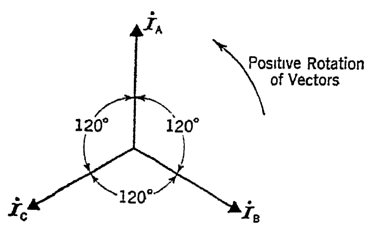


FIG. 16–1

16–1. Vector Components.—A balanced symmetrical three-phase system of vectors, by definition, is made up of three vectors of equal magnitudes, each displaced 120° in phase from the others as shown in Fig. 16–1. These revolving time vectors represent sinusoidal quantities.

The conception of an operator a is also introduced. This operator a is a quantity which, when it is attached to a vector, rotates the vector through a positive angle of 120° in the conventional counter-clockwise direction without change in magnitude. Mathematically it is defined as follows:

$$a = 1\underline{/120°} = -0.5 + j0.866 \qquad \textbf{(16–1)}$$

It is analogous to the well-known operator j, which is

$$j = 1\underline{/90°} = \sqrt{-1}$$

Similar to j^2, which rotates a vector through a positive angle of $2 \times 90°$ or 180°, the operator a^2 has the property of rotating a vector through a positive angle of $2 \times 120°$ or $240°$ without change of magnitude. Thus,

$$a^2 = 1\underline{/240°} = -0.5 - j0.866 \qquad (16\text{–}2)$$

Likewise, the operator a^3 rotates a vector through 360° without change of magnitude, or

$$a^3 = 1\underline{/360°} = 1 \qquad (16\text{–}3)$$

Also,

$$a^4 = a \qquad (16\text{–}4)$$

The currents $\dot{I}_B$ and $\dot{I}_C$ in Fig. 16–1 may be expressed in terms of $\dot{I}_A$. Thus,

$$\left.\begin{aligned} \dot{I}_B &= a^2\dot{I}_A \\ \dot{I}_C &= a\dot{I}_A \end{aligned}\right\} \qquad (16\text{–}5)$$

When expressing $\dot{I}_B$ in terms of $\dot{I}_A$, it should be noted that $\dot{I}_B$ cannot be obtained by operating on $\dot{I}_A$ by $-a$ to indicate a clockwise rotation of 120°. The vector $-a\dot{I}_A$ is equal to $-(a\dot{I}_A)$ or $-\dot{I}_C$.

A property of the operator a, of which use will be made later, is that

$$1 + a + a^2 = 1 + (-0.5 + j0.866) + (-0.5 - j0.866) = 0 \quad (16\text{–}6)$$

16–2. Degrees of Freedom for a System of Vectors.—By degrees of freedom in any system of vectors is meant the number of independent ways in which a quantity may vary or the number of independent parameters required to specify the quantity completely. A scalar has but one degree of freedom, since it can vary only in magnitude. A vector within a plane has two degrees of freedom, since it can vary in magnitude and direction. A vector in space has three degrees of freedom.

A system of three vectors in a plane can be described by specifying each vector separately, and this requires six parameters. The system has, in the general case, six degrees of freedom. If the system is symmetrical, however, the six parameters are not independent. The relation between the different parameters is expressed by the nature of the symmetry. A symmetrical system of three vectors in a plane has but two degrees of freedom. This condition suggests the possibility of defining any system of three coplanar, unsymmetrical vectors, having six degrees of freedom, in terms of three systems of components, each having two degrees of freedom.

16-3. Synthesis of Three-Phase Vector Systems.—The currents $\dot{I}_A$, $\dot{I}_B$, and $\dot{I}_C$ of Fig. 16-2 constitute a balanced symmetrical three-phase system of currents having the phase order or sequence ABC. The currents $\dot{I}_{A'}$, $\dot{I}_{B'}$, $\dot{I}_{C'}$ constitute a second balanced symmetrical system of currents of the same phase order or sequence as the first. If the two systems are added, each phase being considered separately, the results are:

$$\dot{I}_a = \dot{I}_A + \dot{I}_{A'} \tag{16-7}$$

$$\dot{I}_b = \dot{I}_B + \dot{I}_{B'} \tag{16-8}$$

$$\dot{I}_c = \dot{I}_C + \dot{I}_{C'} \tag{16-9}$$

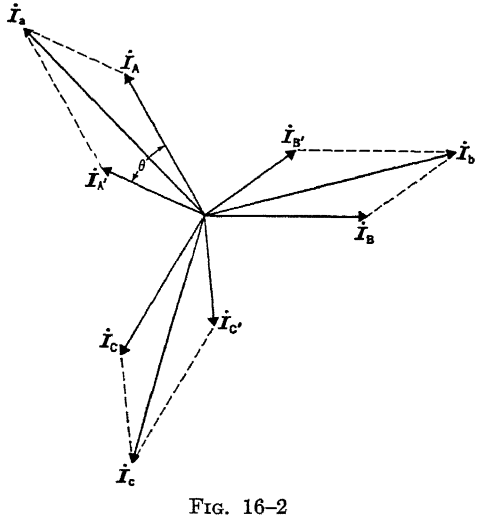


FIG. 16-2

It can be seen from Fig. 16-2 that the resultant system $\dot{I}_a$, $\dot{I}_b$, and $\dot{I}_c$ is also a balanced symmetrical system and has the same phase order or sequence as the first system. This will be found to be true regardless of the magnitudes of the vectors in the two original systems and the phase angle θ between $\dot{I}_A$ and $\dot{I}_{A'}$.

In Fig. 16-3(a), the phase sequence of the second system is reversed from that considered for the same vectors in Fig. 16-2, so that the sequence now becomes $\dot{I}_{A'}$, $\dot{I}_{C'}$, $\dot{I}_{B'}$, that is, sequence ACB. The reader should note that the phase order, and not the direction of vector rotation, is negative (see the definition of phase sequence in Art. 14-8). If we make the same additions as those indicated in equations (16–7) to (16–9), the resultant currents will now form an unsymmetrical system of vectors $\dot{I}_a$, $\dot{I}_b$, and $\dot{I}_c$. How badly this resultant vector system is unbalanced depends on the relative magnitudes of the two balanced symmetrical systems and on the phase angle θ between $\dot{I}_A$ and $\dot{I}_{A'}$.

Thus, it is seen that two symmetrical systems, when added together, produce a symmetrical resultant system if their phase sequences are the same, and produce an unsymmetrical resultant system if their phase sequences are opposite.

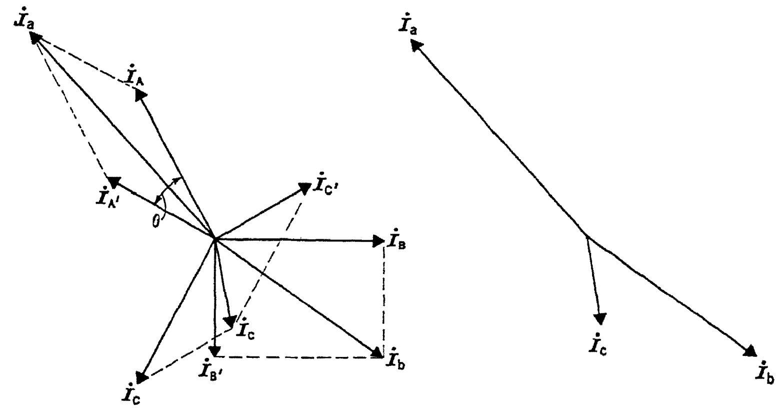


FIG. 16–3

In Fig. 16–3, the following conditions hold:

$$\dot{I}_A+\dot{I}_B+\dot{I}_C=0 \text{ (being symmetrical)} \qquad (16\text{–}10)$$

$$\dot{I}_{A'}+\dot{I}_{B'}+\dot{I}_{C'}=0 \text{ (being symmetrical)} \qquad (16\text{–}11)$$

When equations (16–10) and (16–11) are added, the resultant must become zero. Thus,

$$\dot{I}_a+\dot{I}_b+\dot{I}_c=0 \qquad (16\text{–}12)$$

Hence, it may be concluded that *any unsymmetrical system of three vectors whose vector sum is zero may be represented by two symmetrical systems having opposite phase sequence.*

16–4. Meaning of Positive-Sequence and Negative-Sequence Systems.—The concept of sequence systems of vectors will now be considered. It has been shown that any unbalanced system of three vectors whose vector sum is zero, as $\dot{I}_a$, $\dot{I}_b$, and $\dot{I}_c$, may be resolved into two systems of components. A *positive-sequence* system of vectors will be considered as a balanced system in which the phase order is ABC. A *negative-sequence* system will be considered as a balanced system whose phase order is ACB. In the double subscript notation, the small letters a, b, and c will be used to indicate the phases, while the numerals 1 and 2 will be employed to designate the positive-sequence and negative-sequence systems, respectively. Thus, if $\dot{I}_a$, $\dot{I}_b$, and $\dot{I}_c$ form an unbalanced system of vectors whose vector sum is zero, then the positive-sequence components are designated by $\dot{I}_{a1}$, $\dot{I}_{b1}$, and $\dot{I}_{c1}$, as shown in Fig. 16–4(*a*), and the negative-sequence components by $\dot{I}_{a2}$, $\dot{I}_{c2}$, and $\dot{I}_{b2}$, as shown in Fig. 16–4(*b*). In general, $\dot{I}_{a1}$ and $\dot{I}_{a2}$ will have different magnitudes and different phase angles, as shown in Fig. 16–8.

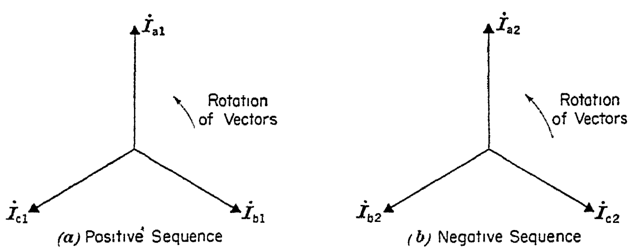


FIG. 16–4

The vectors in the positive-sequence system are related to $\dot{I}_{a1}$ by the following relations:

$$\left.\begin{aligned}\dot{I}_{a1}&=\dot{I}_{a1}\\ \dot{I}_{b1}&=a^2\dot{I}_{a1}\\ \dot{I}_{c1}&=a\dot{I}_{a1}\end{aligned}\right\}\qquad(16\text{–}13)$$

In the negative-sequence system, the vectors are related as follows:

$$\left.\begin{aligned}\dot{I}_{a2}&=\dot{I}_{a2}\\ \dot{I}_{b2}&=a\dot{I}_{a2}\\ \dot{I}_{c2}&=a^2\dot{I}_{a2}\end{aligned}\right\}\qquad(16\text{–}14)$$

16–5. Voltage and Current Relations in Balanced or Unbalanced Three-Phase Circuits.—The following relations hold for balanced or unbalanced three-phase circuits:

(*a*) In any three-phase system, the vector sum of the line voltages is zero.

(*b*) In any three-phase, three-wire system, whether the load is connected in wye or in delta, the vector sum of the line currents is zero.

(*c*) If an unbalanced Y-load is connected to a three-phase, three-wire system, the vector sum of the voltage drops across the three branches is, in general, not zero.

(*d*) In a Δ-connected load, the vector sum of the branch currents is, in general, not zero.

(*e*) In a three-phase, four-wire system with Y-connected loads, the vector sum of the branch currents is, in general, not zero.

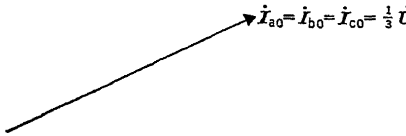


FIG. 16–5

16–6. Zero-Sequence Components.—As indicated in (*c*), (*d*), and (*e*) of the preceding article, the vector sum of currents or voltages in a three-phase system will not always be zero and cannot be completely defined by the positive-sequence and negative-sequence systems. There is, in general, a residual current or voltage due to the unbalancing. In the case of an unbalanced load on a three-phase, four-wire system, the vector sum of the line currents is a residual current flowing in the neutral wire. Thus,

$$\dot{I}_a + \dot{I}_b + \dot{I}_c = \dot{U} \tag{16–15}$$

This residual current may be represented by three equal vectors having zero phase; that is, they are not separated in time phase and will vary simultaneously in each branch, as shown in Fig. 16–5. Each has the value of $\frac{1}{3}\dot{U}$. This system is called the *zero-sequence system,* and the zero-sequence components are designated by $\dot{I}_{a0}$, $\dot{I}_{b0}$, and $\dot{I}_{c0}$. In the figure it may appear as if only a single vector were shown; but actually there are three, one on top of another. Then, by combining this idea with the concept developed in Art. 16–3, it will be seen that *any system of unbalanced, three-phase currents or voltages whose vector sum is not zero may be represented by three symmetrical component systems, namely, a positive-sequence* *system, a negative-sequence system, and a zero-sequence system* (see Example 16–3).

16–7. Determination of the Phase-Sequence Components.—It has been indicated that: (*a*) any unbalanced unsymmetrical three-phase system of currents or voltages whose vector sum is zero can be resolved into two systems of symmetrical components; and (*b*) any unbalanced, unsymmetrical three-phase system of currents or voltages whose vector sum is not zero can be resolved into three systems of symmetrical components. It now remains to determine the components in terms of the three known unbalanced vectors.

(*a*) *With No Neutral Connection:* In a three-phase, three-wire system the vector sum of the line currents is zero, or

$$\dot{I}_a + \dot{I}_b + \dot{I}_c = 0 \tag{16–16}$$

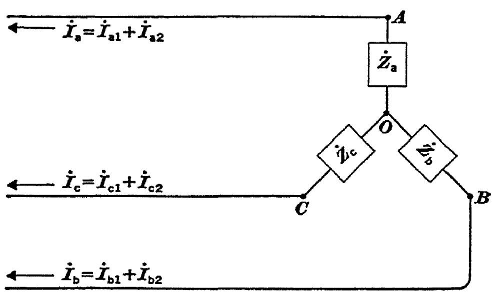


FIG. 16–6

There will be no zero-sequence currents since there is no residual current. The line currents $\dot{I}_a$, $\dot{I}_b$, and $\dot{I}_c$ may be resolved into their positive-sequence and negative-sequence components. Thus,

$$\left.\begin{aligned} \dot{I}_a &= \dot{I}_{a1} + \dot{I}_{a2} \\ \dot{I}_b &= \dot{I}_{b1} + \dot{I}_{b2} \\ \dot{I}_c &= \dot{I}_{c1} + \dot{I}_{c2} \end{aligned}\right\} \tag{16–17}$$

The positive-sequence and negative-sequence components of each current flow in the same line, as shown in Fig. 16–6. The components $\dot{I}_{a1}$, $\dot{I}_{b1}$, and $\dot{I}_{c1}$, being equal in magnitude and 120° apart, add up to zero at point *O*; and, likewise, the negative-sequence components add up to zero at point *O*.

To simplify the mathematics in the determination of the sequence components, the components of $\dot{I}_b$ and $\dot{I}_c$ are expressed in terms of the components of $\dot{I}_a$. Substitution of values from equations (16–13) and (16–14) in equation (16–17) gives:

$$\dot{I}_a = \dot{I}_{a1} + \dot{I}_{a2} \tag{16–18}$$

$$\dot{I}_b = a^2\dot{I}_{a1} + a\dot{I}_{a2} \tag{16–19}$$

$$\dot{I}_c = a\dot{I}_{a1} + a^2\dot{I}_{a2} \tag{16–20}$$

In the determination of $\dot{I}_{a1}$, multiply equation (16–19) by a and multiply equation (16–20) by a^2. The results are:

$$a\dot{I}_b = a^3\dot{I}_{a1} + a^2\dot{I}_{a2} = \dot{I}_{a1} + a^2\dot{I}_{a2} \tag{16–21}$$

$$a^2\dot{I}_c = a^3\dot{I}_{a1} + a^4\dot{I}_{a2} = \dot{I}_{a1} + a\dot{I}_{a2} \tag{16–22}$$

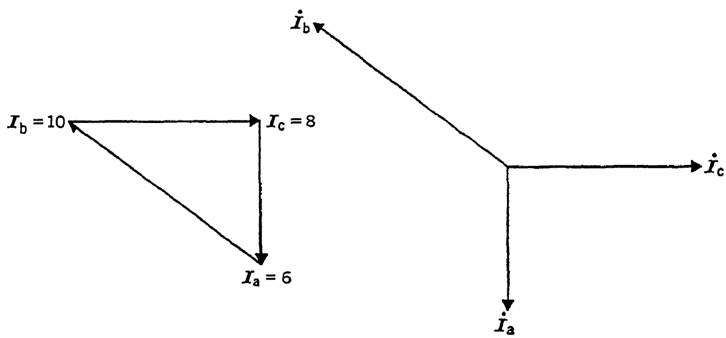


FIG. 16–7

Adding equations (16–18), (16–21), and (16–22) gives

$$\dot{I}_a + a\dot{I}_b + a^2\dot{I}_c = 3\dot{I}_{a1} + (1 + a + a^2)\dot{I}_{a2} = 3\dot{I}_{a1}$$

or

$$\dot{I}_{a1} = \frac{\dot{I}_a + a\dot{I}_b + a^2\dot{I}_c}{3} \tag{16–23}$$

In a similar manner, multiplying equation (16–19) by a^2, multiplying equation (16–20) by a, and then adding the results to equation (16–18) gives

$$\dot{I}_{a2} = \frac{\dot{I}_a + a^2\dot{I}_b + a\dot{I}_c}{3} \tag{16–24}$$

When $\dot{I}_{a1}$ and $\dot{I}_{a2}$ are known, the other components can be found easily by the relations given in equations (16–13) and (16–14).

Example 16–1.—The three line currents in a three-phase, three-wire system are $\dot{I}_a = -j6$, $\dot{I}_b = -8 + j6$, and $\dot{I}_c = 8$. Find the positive-sequence and negative-sequence components and draw the vector diagram.

Solution.—The vector sum of the currents is zero, as shown in Fig. 16–7. There will be no zero-sequence components.

When values are substituted in equations (16–23) and (16–24), the results are:

$$\dot{I}_{a1} = \tfrac{1}{3}(6\underline{/-90^\circ} + 1\underline{/120^\circ} \times 10\underline{/143.2^\circ} + 1\underline{/240^\circ} \times 8) = 7.82\underline{/257.2^\circ}$$

$$\dot{I}_{a2} = \tfrac{1}{3}(6\underline{/-90^\circ} + 1\underline{/240^\circ} \times 10\underline{/143.2^\circ} + 1\underline{/120^\circ} \times 8) = 2.37\underline{/43.1^\circ}$$

From equations (16–13) and (16–14),

$$\dot{I}_{b1} = a^2\dot{I}_{a1} = 7.82\underline{/137.2^\circ}$$

$$\dot{I}_{c1} = a\dot{I}_{a1} = 7.82\underline{/17.2^\circ}$$

$$\dot{I}_{b2} = a\dot{I}_{a2} = 2.37\underline{/163.1^\circ}$$

$$\dot{I}_{c2} = a^2\dot{I}_{a2} = 2.37\underline{/283.1^\circ}$$

The vector diagram is shown in Fig. 16–8.

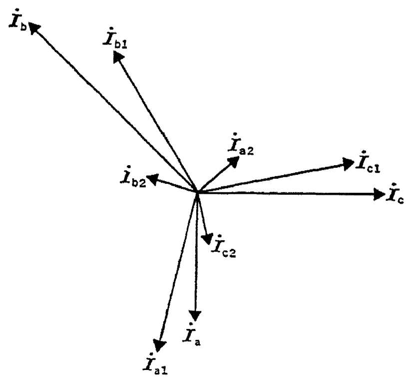


FIG. 16–8

Example 16–2.—The load is so adjusted that the three line currents in a three-phase, three-wire system are $\dot{I}_a = -j6$, $\dot{I}_c = -8+j6$, and $\dot{I}_b = 8$. Find the positive-sequence and negative-sequence components.

Solution.—Although the sequence of the line currents is ACB, equation (16–23) still gives the positive-sequence component because of the definitions adopted in Art. 16–4. Thus, from equations (16–23) and (16–24),

$$\dot{I}_{a1} = \tfrac{1}{3}(6\underline{/-90^\circ} + 1\underline{/120^\circ} \times 8 + 1\underline{/240^\circ} \times 10\underline{/143.2^\circ}) = 2.37\underline{/43.1^\circ}$$

$$\dot{I}_{a2} = \tfrac{1}{3}(6\underline{/-90^\circ} + 1\underline{/240^\circ} \times 8 + 1\underline{/120^\circ} \times 10\underline{/143.2^\circ}) = 7.82\underline{/257.2^\circ}$$

Also, from equations (16–13) and (16–14),

$$\dot{I}_{b1} = a^2\dot{I}_{a1} = 2.37\underline{/283.1^\circ} = 2.37\underline{/-76.9^\circ}$$

$$\dot{I}_{c1} = a\dot{I}_{a1} = 2.37\underline{/163.1^\circ} = 2.37\underline{/163.1^\circ}$$

$$\dot{I}_{b2} = a\dot{I}_{a2} = 7.82\underline{/377.2^\circ} = 7.82\underline{/17.2^\circ}$$

$$\dot{I}_{c2} = a^2\dot{I}_{a2} = 7.82\underline{/497.2^\circ} = 7.82\underline{/137.2^\circ}$$

(b) *With Neutral Connection:* With a neutral connection, as in the case of a three-phase, four-wire system, it is necessary to consider not only the positive-sequence and negative-sequence components but also the zero-sequence components. The currents, in terms of their sequence components, are:

$$\dot{I}_a = \dot{I}_{a0} + \dot{I}_{a1} + \dot{I}_{a2} \tag{16-25}$$

$$\dot{I}_b = \dot{I}_{b0} + \dot{I}_{b1} + \dot{I}_{b2} \tag{16-26}$$

$$\dot{I}_c = \dot{I}_{c0} + \dot{I}_{c1} + \dot{I}_{c2} \tag{16-27}$$

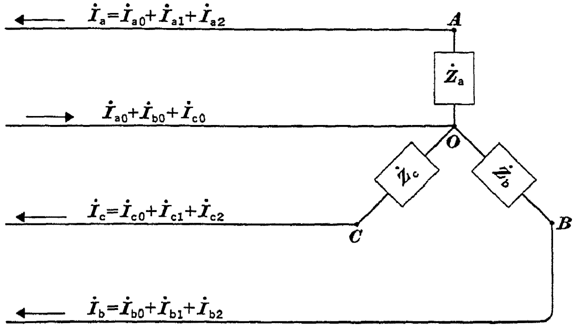


FIG. 16–9

The positive-sequence, negative-sequence, and zero-sequence components of each current flow in the same line wire, as shown in Fig. 16–9. The positive-sequence system and the negative-sequence system each add up to zero at point O. The zero-sequence components are in phase with each other, and, hence, they add up to form the current in the neutral wire.

With $\dot{I}_{a0} = \dot{I}_{b0} = \dot{I}_{c0}$, and the relations in equations (16–13) and (16–14), all the sequence components of $\dot{I}_a$, $\dot{I}_b$, and $\dot{I}_c$ can be expressed in terms of those of $\dot{I}_a$. Thus,

$$\dot{I}_a = \dot{I}_{a0} + \dot{I}_{a1} + \dot{I}_{a2} \tag{16-28}$$

$$\dot{I}_b = \dot{I}_{a0} + a^2\dot{I}_{a1} + a\dot{I}_{a2} \tag{16-29}$$

$$\dot{I}_c = \dot{I}_{a0} + a\dot{I}_{a1} + a^2\dot{I}_{a2} \tag{16-30}$$

To determine the zero-sequence component $\dot{I}_{a0}$, add equations (16–28), (16–29), and (16–30). The result is:

$$\dot{I}_a + \dot{I}_b + \dot{I}_c = 3\dot{I}_{a0} + (1 + a^2 + a)\dot{I}_{a1} + (1 + a + a^2)\dot{I}_{a2}$$

from which

$$\dot{I}_{a0} = \tfrac{1}{3}(\dot{I}_a + \dot{I}_b + \dot{I}_c) \tag{16-31}$$

To find $\dot{I}_{a1}$ multiply equation (16–29) by a, multiply equation (16–30) by a^2, and add these results to equation (16–28). This procedure gives:

$$\dot{I}_a + a\dot{I}_b + a^2\dot{I}_c = (1 + a + a^2)\dot{I}_{a0} + 3\dot{I}_{a1} + (1 + a^2 + a)\dot{I}_{a2}$$

from which

$$\dot{I}_{a1} = \tfrac{1}{3}(\dot{I}_a + a\dot{I}_b + a^2\dot{I}_c) \qquad (16\text{–}32)$$

In a similar manner, $\dot{I}_{a2}$ may be found by multiplying equation (16–29) by a^2, multiplying equation (16–30) by a, and adding the results to equation (16–28). The result is

$$\dot{I}_a + a^2\dot{I}_b + a\dot{I}_c = (1 + a^2 + a)\dot{I}_{a0} + (1 + a + a^2)\dot{I}_{a1} + 3\dot{I}_{a2}$$

from which

$$\dot{I}_{a2} = \tfrac{1}{3}(\dot{I}_a + a^2\dot{I}_b + a\dot{I}_c) \qquad (16\text{–}33)$$

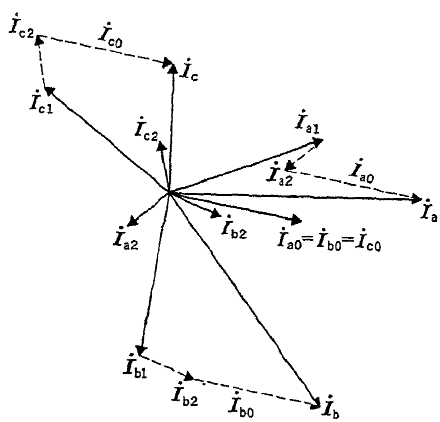


Fig. 16–10

Example 16–3.—In a three-phase, four-wire system, like that in Fig. 16–9, the line currents are $\dot{I}_a = 100$, $\dot{I}_b = 60 - j80$, and $\dot{I}_c = j50$. Determine the positive-sequence, negative-sequence and zero-sequence components of the line currents.

Solution.—Since the vector sum of the line currents is not equal to zero, there will exist in the system zero-sequence components. Substitution of values in equations (16–31) to (16–33) gives:

$$\dot{I}_{a0} = \tfrac{1}{3}(100\underline{/0^\circ} + 100\underline{/-53.2} + 50\underline{/90^\circ}) = 54.2\underline{/-10.6^\circ}$$

$$\dot{I}_{a1} = \tfrac{1}{3}(100\underline{/0^\circ} + 1\underline{/120^\circ} \times 100\underline{/-53.2^\circ} + 1\underline{/240^\circ} \times 50\underline{/90^\circ}) = 64.9\underline{/20.1^\circ}$$

$$\dot{I}_{a2} = \tfrac{1}{3}(100\underline{/0^\circ} + 1\underline{/240^\circ} \times 100\underline{/-53.2^\circ} + 1\underline{/120^\circ} \times 50\underline{/90^\circ}) = 18.75\underline{/221^\circ}$$

The components of $\dot{I}_b$ and $\dot{I}_c$ are:

$$\dot{I}_{b0} = \dot{I}_{c0} = \dot{I}_{a0} = 54.2\underline{/-10.6°}$$

$$\dot{I}_{b1} = a^2\dot{I}_{a1} = 64.9\underline{/260.1°}$$

$$\dot{I}_{c1} = a\dot{I}_{a1} = 64.9\underline{/140.1°}$$

$$\dot{I}_{b2} = a\dot{I}_{a2} = 18\ 75\underline{/341°}$$

$$\dot{I}_{c2} = a^2\dot{I}_{a2} = 18.75\underline{/101°}$$

The vector diagram is shown in Fig. 16–10.

The positive-sequence components $\dot{I}_{a1}$, $\dot{I}_{b1}$, and $\dot{I}_{c1}$ add up to zero at the point O. This is also true for the negative-sequence components $\dot{I}_{a2}$, $\dot{I}_{c2}$, and $\dot{I}_{b2}$. The zero-sequence components $\dot{I}_{a0}$, $\dot{I}_{b0}$, and $\dot{I}_{c0}$ being in phase, all flow away from or toward the point O at the same time. The neutral wire carries the sum of the three, or the entire residual current $162.6\underline{/-10.6°}$. If the neutral consists of a ground return, as is sometimes the case in transmission lines with grounded neutrals at the transformers, the zero-sequence components flow over the line wires in parallel and return through the grounded connection.

Up to this point, only systems of current vectors have been considered. It is obvious that formulas for systems of voltage vectors will be similar to those for the current vectors. Thus, for $\dot{E}_a+\dot{E}_b+\dot{E}_c\neq 0$,

$$\left.\begin{aligned}\dot{E}_a &= \dot{E}_{a0}+\dot{E}_{a1}+\dot{E}_{a2}\\ \dot{E}_b &= \dot{E}_{a0}+a^2\dot{E}_{a1}+a\dot{E}_{a2}\\ \dot{E}_c &= \dot{E}_{a0}+a\dot{E}_{a1}+a^2\dot{E}_{a2}\end{aligned}\right\} \quad (16\text{–}34)$$

$$\left.\begin{aligned}\dot{E}_{a0} &= \tfrac{1}{3}(\dot{E}_a+\dot{E}_b+\dot{E}_c)\\ \dot{E}_{a1} &= \tfrac{1}{3}(\dot{E}_a+a\dot{E}_b+a^2\dot{E}_c)\\ \dot{E}_{a2} &= \tfrac{1}{3}(\dot{E}_a+a^2\dot{E}_b+a\dot{E}_c)\end{aligned}\right\} \quad (16\text{–}35)$$

$$\left.\begin{aligned}\dot{E}_{b1} &= a^2\dot{E}_{a1}\\ \dot{E}_{c1} &= a\dot{E}_{a1}\\ \dot{E}_{b2} &= a\dot{E}_{a2}\\ \dot{E}_{c2} &= a^2\dot{E}_{a2}\end{aligned}\right\} \quad (16\text{–}36)$$

16–8. Balanced Voltages Impressed on Unbalanced Δ-Connected Loads.—Consider the case where three unequal load impedances $\dot{Z}_{CB}$, $\dot{Z}_{BA}$, and $\dot{Z}_{AC}$ are connected in delta to a balanced, three-phase system, as shown in Fig. 16–11. It is assumed that the line voltage drops of sequence ABC are $\dot{E}_{CB}$, $\dot{E}_{BA}$, and $\dot{E}_{AC}$. This sequence can be determined by examining the complex expressions for the voltages in a numerical problem. Also, the branch currents of sequence ABC are $\dot{I}_{CB}$, $\dot{I}_{BA}$, and $\dot{I}_{AC}$. This sequence is similar to that indicated in Fig. 16–12(*a*) without the subscripts 1. The purpose of this article is to give an example for establishing a notation for the sequence components in a Δ-system and, also, for indicating how to determine the following quantities in such a network: (*a*) the sequence components of the voltage drops across the branch impedances; (*b*) the sequence components of the branch currents; and (*c*) the sequence components of the line currents.

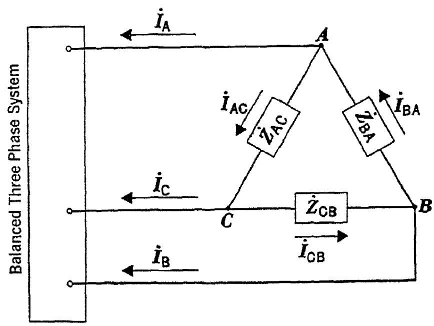


FIG. 16–11

Since the impressed voltages are balanced and symmetrical, the line voltage drops will have no negative-sequence and no zero-sequence components. The positive-sequence components are

$$\left.\begin{aligned}\dot{E}_{CB1} &= \dot{E}_{CB}\\ \dot{E}_{BA1} &= \dot{E}_{BA}\\ \dot{E}_{AC1} &= \dot{E}_{AC}\end{aligned}\right\} \tag{16–37}$$

Next, the sequence components of the branch currents will be considered. With $\dot{E}_{CB}$, $\dot{E}_{BA}$, and $\dot{E}_{AC}$ as the voltage drops across the load impedances in the direction of the current arrows, the branch currents are:

$$\dot{I}_{CB} = \frac{\dot{E}_{CB}}{\dot{Z}_{CB}} = \dot{E}_{CB}\dot{Y}_{CB} \tag{16–38}$$

$$\dot{I}_{BA} = \frac{\dot{E}_{BA}}{\dot{Z}_{BA}} = \dot{E}_{BA}\dot{Y}_{BA} \tag{16–39}$$

$$\dot{I}_{AC} = \frac{\dot{E}_{AC}}{\dot{Z}_{AC}} = \dot{E}_{AC}\dot{Y}_{AC} \tag{16–40}$$

In general, the vector sum of the branch currents in a Δ-connected load is not zero. This sum will contain all three sequence components. It will be seen that the sequence components may be expressed in terms of $\dot{E}_{CB}$. Relating components to those of branch CB, we have, with the aid of Fig. 16–12:

$$\dot{I}_{CB} = \dot{I}_{CB0} + \dot{I}_{CB1} + \dot{I}_{CB2} \tag{16–41}$$

$$\dot{I}_{BA} = \dot{I}_{BA0} + \dot{I}_{BA1} + \dot{I}_{BA2} = \dot{I}_{CB0} + a\dot{I}_{CB1} + a^2\dot{I}_{CB2} \tag{16–42}$$

$$\dot{I}_{AC} = \dot{I}_{AC0} + \dot{I}_{AC1} + \dot{I}_{AC2} = \dot{I}_{CB0} + a^2\dot{I}_{CB1} + a\dot{I}_{CB2} \tag{16–43}$$

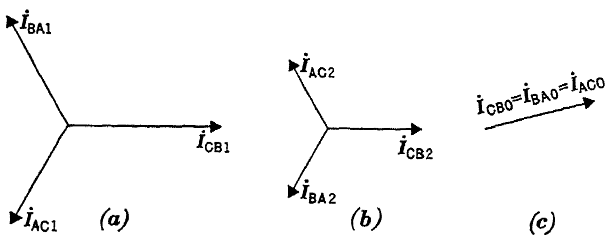


FIG. 16–12

In general, the magnitudes and phase angles of $\dot{I}_{CB1}$, $\dot{I}_{CB2}$, and $\dot{I}_{CB0}$ are different.

Solving these equations simultaneously and substituting values from equations (16–38) to (16–40), we find that:

$$\dot{I}_{CB0} = \tfrac{1}{3}(\dot{I}_{CB} + \dot{I}_{BA} + \dot{I}_{AC}) = \tfrac{1}{3}(\dot{E}_{CB}\dot{Y}_{CB} + \dot{E}_{BA}\dot{Y}_{BA} + \dot{E}_{AC}\dot{Y}_{AC}) \tag{16–44}$$

$$\dot{I}_{CB1} = \tfrac{1}{3}(\dot{I}_{CB} + a^2\dot{I}_{BA} + a\dot{I}_{AC}) = \tfrac{1}{3}(\dot{E}_{CB}\dot{Y}_{CB} + a^2\dot{E}_{BA}\dot{Y}_{BA} + a\dot{E}_{AC}\dot{Y}_{AC}) \tag{16–45}$$

$$\dot{I}_{CB2} = \tfrac{1}{3}(\dot{I}_{CB} + a\dot{I}_{BA} + a^2\dot{I}_{AC}) = \tfrac{1}{3}(\dot{E}_{CB}\dot{Y}_{CB} + a\dot{E}_{BA}\dot{Y}_{BA} + a^2\dot{E}_{AC}\dot{Y}_{AC}) \tag{16–46}$$

Expressing $\dot{E}_{BA}$ and $\dot{E}_{AC}$ in terms of $\dot{E}_{CB}$ and rearranging, we obtain:

$$\dot{I}_{CB0} = \frac{\dot{E}_{CB}}{3}(\dot{Y}_{CB} + a\dot{Y}_{BA} + a^2\dot{Y}_{AC}) = \dot{E}_{CB}\dot{Y}_{CB0} \tag{16–47}$$

$$\dot{I}_{CB1} = \frac{\dot{E}_{CB}}{3}(\dot{Y}_{CB} + \dot{Y}_{BA} + \dot{Y}_{AC}) = \dot{E}_{CB}\dot{Y}_{CB1} \tag{16–48}$$

$$\dot{I}_{CB2} = \frac{\dot{E}_{CB}}{3}(\dot{Y}_{CB} + a^2\dot{Y}_{BA} + a\dot{Y}_{AC}) = \dot{E}_{CB}\dot{Y}_{CB2} \tag{16–49}$$

where

$$\left.\begin{aligned} \dot{Y}_{CB0} &= \tfrac{1}{3}(\dot{Y}_{CB} + a\dot{Y}_{BA} + a^2\dot{Y}_{AC}) \\ \dot{Y}_{CB1} &= \tfrac{1}{3}(\dot{Y}_{CB} + \dot{Y}_{BA} + \dot{Y}_{AC}) \\ \dot{Y}_{CB2} &= \tfrac{1}{3}(\dot{Y}_{CB} + a^2\dot{Y}_{BA} + a\dot{Y}_{AC}) \end{aligned}\right\} \qquad (16\text{–}50)$$

Thus, when an unbalanced Δ-load is connected to a three-phase system of balanced voltages, the sequence components of the branch currents may be obtained by first finding the values of $\dot{Y}$ indicated in equation (16–50) and then multiplying each by the line voltage drop $\dot{E}_{CB}$ as given in equations (16–47) to (16–49).

The sequence components of $\dot{I}_{BA}$ and $\dot{I}_{AC}$ are determined with the aid of Fig. 16–12.

The sequence components of the line currents may be determined in the following way. With the positive senses as indicated in Fig. 16–11, we have:

$$\left.\begin{aligned} \dot{I}_A &= \dot{I}_{BA} - \dot{I}_{AC} \\ \dot{I}_B &= \dot{I}_{CB} - \dot{I}_{BA} \\ \dot{I}_C &= \dot{I}_{AC} - \dot{I}_{CB} \end{aligned}\right\} \qquad (16\text{–}51)$$

Since the vector sum of the line currents is zero, there will be no zero-sequence components for the line currents. Then,

$$\dot{I}_{A1} + \dot{I}_{A2} = (\dot{I}_{BA0} + \dot{I}_{BA1} + \dot{I}_{BA2}) - (\dot{I}_{AC0} + \dot{I}_{AC1} + \dot{I}_{AC2})$$

or

$$\dot{I}_{A1} + \dot{I}_{A2} = (\dot{I}_{BA1} - \dot{I}_{AC1}) + (\dot{I}_{BA2} - \dot{I}_{AC2}) \qquad (16\text{–}52)$$

where

$$\left.\begin{aligned} \dot{I}_{A1} &= \dot{I}_{BA1} - \dot{I}_{AC1} = a\dot{I}_{CB1} - a^2\dot{I}_{CB1} = (a - a^2)\dot{I}_{CB1} \\ \dot{I}_{A2} &= \dot{I}_{BA2} - \dot{I}_{AC2} = a^2\dot{I}_{CB2} - a\dot{I}_{CB2} = (a^2 - a)\dot{I}_{CB2} \end{aligned}\right\} \qquad (16\text{–}53a)$$

It can be shown that

$$(a - a^2) = \sqrt{3}\underline{/90^\circ}$$

$$(a^2 - a) = \sqrt{3}\underline{/-90^\circ}$$

Then

$$\left.\begin{aligned} \dot{I}_{A1} &= \sqrt{3}\underline{/90^\circ}\ \dot{I}_{CB1} = \sqrt{3}\underline{/90^\circ}\ \dot{E}_{CB}\dot{Y}_{CB1} \\ \dot{I}_{A2} &= \sqrt{3}\underline{/-90^\circ}\ \dot{I}_{CB2} = \sqrt{3}\underline{/-90^\circ}\ \dot{E}_{CB}\dot{Y}_{CB2} \end{aligned}\right\} \qquad (16\text{–}53b)$$

Also,

$$\left.\begin{aligned} \dot{I}_{B1} &= a^2\dot{I}_{A1} \quad \text{and} \quad \dot{I}_{B2} = a\dot{I}_{A2} \\ \dot{I}_{C1} &= a\dot{I}_{A1} \quad \text{and} \quad \dot{I}_{C2} = a^2\dot{I}_{A2} \end{aligned}\right\} \qquad (16\text{–}53c)$$

Example 16–4.—Three unequal load impedances $\dot{Z}_{CB} = 10\underline{/0^\circ}$, $\dot{Z}_{BA} = 10\underline{/-90^\circ}$, and $\dot{Z}_{AC} = 10\underline{/90^\circ}$ are connected in delta to a three-phase system with balanced voltages, as in Fig. 16–11. The line-to-line voltage is 100 volts. Assuming that the sequence of the line voltage drops across the load impedances is ABC, determine the sequence components of the branch currents by the straightforward method, that is, by using equations (16–44) to (16–46). The student should check the answers by the use of equations (16–47) to (16–49).

Solution.—The expressions for the line voltage drops across the load impedances of sequence ABC are:

$$\dot{E}_{CB} = 100\underline{/0°} = \dot{E}_{CB1}$$

$$\dot{E}_{BA} = 100\underline{/120°} = \dot{E}_{BA1}$$

$$\dot{E}_{AC} = 100\underline{/240°} = \dot{E}_{AC1}$$

The branch currents are:

$$\dot{I}_{CB} = \frac{\dot{E}_{CB}}{\dot{Z}_{CB}} = \frac{100\underline{/0°}}{10\underline{/0°}} = 10\underline{/0°}$$

$$\dot{I}_{BA} = \frac{\dot{E}_{BA}}{\dot{Z}_{BA}} = \frac{100\underline{/120°}}{10\underline{/-90°}} = 10\underline{/210°}$$

$$\dot{I}_{AC} = \frac{\dot{E}_{AC}}{\dot{Z}_{AC}} = \frac{100\underline{/240°}}{10\underline{/90°}} = 10\underline{/150°}$$

Their sequence components are:

$$\begin{aligned}\dot{I}_{CB0} &= \tfrac{1}{3}(\dot{I}_{CB} + \dot{I}_{BA} + \dot{I}_{AC}) \\ &= \tfrac{1}{3}(10\underline{/0°} + 10\underline{/210°} + 10\underline{/150°}) = 2.44\underline{/180°}\end{aligned}$$

$$\begin{aligned}\dot{I}_{CB1} &= \tfrac{1}{3}(\dot{I}_{CB} + a^2\dot{I}_{BA} + a\dot{I}_{AC}) \\ &= \tfrac{1}{3}(10\underline{/0°} + 1\underline{/240°} \times 10\underline{/210°} + 1\underline{/120°} \times 10\underline{/150°}) = 3.33\underline{/0°}\end{aligned}$$

$$\begin{aligned}\dot{I}_{CB2} &= \tfrac{1}{3}(\dot{I}_{CB} + a\dot{I}_{BA} + a^2\dot{I}_{AC}) \\ &= \tfrac{1}{3}(10\underline{/0°} + 1\underline{/120°} \times 10\underline{/210°} + 1\underline{/240°} \times 10\underline{/150°}) = 9.11\underline{/0°}\end{aligned}$$

$$\dot{I}_{BA0} = \dot{I}_{AC0} = \dot{I}_{CB0} = 2.44\underline{/180°}$$

$$\dot{I}_{BA1} = a\dot{I}_{CB1} = 3.33\underline{/120°}$$

$$\dot{I}_{AC1} = a^2\dot{I}_{CB1} = 3.33\underline{/240°}$$

$$\dot{I}_{BA2} = a^2\dot{I}_{CB2} = 9.11\underline{/240°}$$

$$\dot{I}_{AC2} = a\dot{I}_{CB2} = 9\ 11\underline{/120°}$$

16–9. Unbalanced Voltages Impressed on Balanced Y-Connected Loads.—In a three-phase system, the line voltages form a closed triangle and their vector sum is zero. There is no zero-sequence component of the line voltages, and

$$\left.\begin{aligned}\dot{E}_{CB} &= \dot{E}_{CB1} + \dot{E}_{CB2} \\ \dot{E}_{BA} &= a\dot{E}_{CB1} + a^2\dot{E}_{CB2} \\ \dot{E}_{AC} &= a^2\dot{E}_{CB1} + a\dot{E}_{CB2}\end{aligned}\right\} \tag{16-54}$$

The sequence components of the line voltage $\dot{E}_{CB}$ are:

$$\dot{E}_{CB1} = \tfrac{1}{3}(\dot{E}_{CB} + a^2\dot{E}_{BA} + a\dot{E}_{AC}) \tag{16-55}$$

$$\dot{E}_{CB2} = \tfrac{1}{3}(\dot{E}_{CB} + a\dot{E}_{BA} + a^2\dot{E}_{AC}) = \dot{E}_{CB} - \dot{E}_{CB1} \tag{16-56}$$

One procedure in solving a three-phase problem with unbalanced impressed voltages is to find the results for each sequence of the line voltages (each is a balanced three-phase system) and then add the results vectorially when the Superposition Theorem can be used. For example, the terminal voltages of a three-phase induction motor may be unbalanced intentionally for the purpose of varying the motor speed over a wider range than normally possible. The performance of the induction motor under such a condition may be analyzed by determining the performance for each sequence, which is equivalent to determining the performance under the condition of balanced three-phase impressed voltages, and then combining the results.

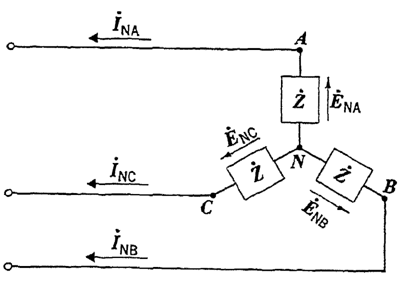


FIG. 16–13

Consider the case where unbalanced voltages are impressed on a balanced Y-connected load, as in Fig. 16–13. It is assumed that the voltage drop from C to B is $\dot{E}_{CB}$, the drop from B to A is $\dot{E}_{BA}$, the drop from A to C is $\dot{E}_{AC}$, and their sequence is ABC. The vector diagram for the positive sequence is given in Fig. 16–14(a), and that for the negative sequence is given in Fig. 16–14(b). Expressions for the line currents may be derived by the method of symmetrical components. The sequence components of the line currents will be determined first.

With $\dot{E}_{NA}$, $\dot{E}_{NB}$, and $\dot{E}_{NC}$ as voltage drops in the direction of the current arrows, it can be seen from Figs. 16–13 and 16–14 that:

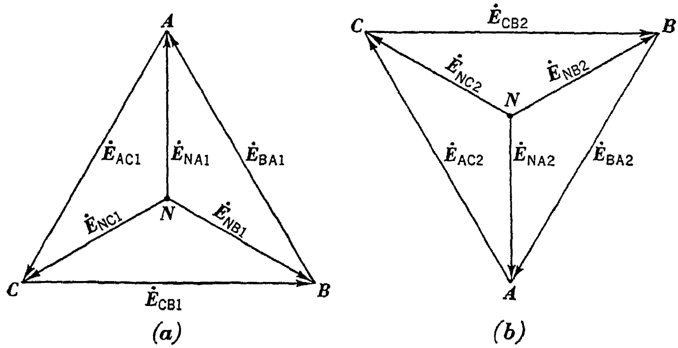


FIG. 16–14

$$\dot{E}_{NA1}=\frac{\dot{E}_{CB1}}{\sqrt{3}}\underline{/90^\circ} \quad \text{and} \quad \dot{I}_{NA1}=\frac{\dot{E}_{CB1}}{\sqrt{3}\dot{Z}}\underline{/90^\circ} \tag{16–57a}$$

$$\dot{E}_{NB1}=\frac{\dot{E}_{CB1}}{\sqrt{3}}\underline{/-30^\circ} \quad \text{and} \quad \dot{I}_{NB1}=\frac{\dot{E}_{CB1}}{\sqrt{3}\dot{Z}}\underline{/-30^\circ} \tag{16–57b}$$

$$\dot{E}_{NC1}=\frac{\dot{E}_{CB1}}{\sqrt{3}}\underline{/-150^\circ} \quad \text{and} \quad \dot{I}_{NC1}=\frac{\dot{E}_{CB1}}{\sqrt{3}\dot{Z}}\underline{/-150^\circ} \tag{16–57c}$$

$$\dot{E}_{NA2}=\frac{\dot{E}_{CB2}}{\sqrt{3}}\underline{/-90^\circ} \quad \text{and} \quad \dot{I}_{NA2}=\frac{\dot{E}_{CB2}}{\sqrt{3}\dot{Z}}\underline{/-90^\circ} \tag{16–58a}$$

$$\dot{E}_{NB2}=\frac{\dot{E}_{CB2}}{\sqrt{3}}\underline{/30^\circ} \quad \text{and} \quad \dot{I}_{NB2}=\frac{\dot{E}_{CB2}}{\sqrt{3}\dot{Z}}\underline{/30^\circ} \tag{16–58b}$$

$$\dot{E}_{NC2}=\frac{\dot{E}_{CB2}}{\sqrt{3}}\underline{/150^\circ} \quad \text{and} \quad \dot{I}_{NC2}=\frac{\dot{E}_{CB2}}{\sqrt{3}\dot{Z}}\underline{/150^\circ} \tag{16–58c}$$

The line current $\dot{I}_{NA}$ is

$$\dot{I}_{NA}=\dot{I}_{NA1}+\dot{I}_{NA2}=\frac{1}{\sqrt{3}\dot{Z}}\left[\dot{E}_{CB1}\underline{/90^\circ}+\dot{E}_{CB2}\underline{/-90^\circ}\right] \tag{16–59}$$

Also,

$$\dot{I}_{NA}=\frac{(\dot{E}_{CB}+a^2\dot{E}_{BA}+a\dot{E}_{AC})\underline{/90^\circ}}{3\sqrt{3}\dot{Z}}+\frac{(\dot{E}_{CB}+a\dot{E}_{BA}+a^2\dot{E}_{AC})\underline{/-90^\circ}}{3\sqrt{3}\dot{Z}}$$

or

$$\dot{I}_{NA}=\frac{(\dot{E}_{CB}+a^2\dot{E}_{BA}+a\dot{E}_{AC})\sqrt{3}\underline{/90^\circ}}{9\dot{Z}}+\frac{(\dot{E}_{CB}+a\dot{E}_{BA}+a^2\dot{E}_{AC})\sqrt{3}\underline{/-90^\circ}}{9\dot{Z}} \qquad (16\text{–}60)$$

It can be shown that

$$\left.\begin{aligned}(a-a^2)&=\sqrt{3}\underline{/90^\circ}\\(a^2-a)&=\sqrt{3}\underline{/-90^\circ}\end{aligned}\right\} \qquad (16\text{–}61)$$

Substituting these values in equation (16–60), expanding, and rearranging, we obtain:

$$\dot{I}_{NA}=\frac{1}{9\dot{Z}}[\dot{E}_{BA}(2-a-a^2)-\dot{E}_{AC}(2-a-a^2)] \qquad (16\text{–}62)$$

Since $2-a-a^2=3-(1+a+a^2)=3$, then

$$\dot{I}_{NA}=\frac{1}{3\dot{Z}}(\dot{E}_{BA}-\dot{E}_{AC})=\frac{1}{3\dot{Z}}(\dot{E}_{BA}+\dot{E}_{CA}) \qquad (16\text{–}63)$$

By following a similar procedure, we get:

$$\dot{I}_{NB}=\frac{1}{3\dot{Z}}(\dot{E}_{CB}-\dot{E}_{BA})=\frac{1}{3\dot{Z}}(\dot{E}_{CB}+\dot{E}_{AB}) \qquad (16\text{–}64)$$

$$\dot{I}_{NC}=\frac{1}{3\dot{Z}}(\dot{E}_{AC}-\dot{E}_{CB})=\frac{1}{3\dot{Z}}(\dot{E}_{AC}+\dot{E}_{BC}) \qquad (16\text{–}65)$$

Equations (16–63) to (16–65) state that the current in any line is equal to $\frac{1}{3\dot{Z}}$ times the vector sum of the voltages from the other two lines to the line considered.

Example 16–5.—In Fig. 16–13, assume that $\dot{Z}=R=10\underline{/0^\circ}$ and the line voltage drops are:

$$\dot{E}_{CB}=80+j0=80\underline{/0^\circ}$$
$$\dot{E}_{BA}=-80+j60=100\underline{/143.2^\circ}$$
$$\dot{E}_{AC}=0-j60=60\underline{/-90^\circ}$$

Find: (*a*) the sequence components of the line currents and (*b*) the line currents by the use of equations (16–63) to (16–65).

Solution.—From equations (16–55) and (16–56),

$$\dot{E}_{CB1} = \tfrac{1}{3}(\dot{E}_{CB} + a^2\dot{E}_{BA} + a\dot{E}_{AC})$$

$$= \tfrac{1}{3}(80 + 1\underline{/240°} \times 100\underline{/143.2°} + 1\underline{/120°} \times 60\underline{/-90°})$$

$$= 74.62 + j23.1 = 78.1\underline{/17.2°}$$

$$\dot{E}_{CB2} = \dot{E}_{CB} - \dot{E}_{CB1} = 5.38 - j23.1 = 23.7\underline{/-77°}$$

From equations (16–57) to (16–62), the sequence components of the line currents are:

$$\dot{E}_{NA1} = \frac{78.1}{\sqrt{3}}\underline{/107.2°} \quad \text{and} \quad \dot{I}_{NA1} = \frac{7.81}{\sqrt{3}}\underline{/107.2°}$$

$$\dot{E}_{NB1} = \frac{78.1}{\sqrt{3}}\underline{/-12.8°} \quad \text{and} \quad \dot{I}_{NB1} = \frac{7.81}{\sqrt{3}}\underline{/-12.8°}$$

$$\dot{E}_{NC1} = \frac{78.1}{\sqrt{3}}\underline{/-132.8°} \quad \text{and} \quad \dot{I}_{NC1} = \frac{7.81}{\sqrt{3}}\underline{/-132.8°}$$

$$\dot{E}_{NA2} = \frac{23.7}{\sqrt{3}}\underline{/-167°} \quad \text{and} \quad \dot{I}_{NA2} = \frac{2.37}{\sqrt{3}}\underline{/-167°}$$

$$\dot{E}_{NB2} = \frac{23.7}{\sqrt{3}}\underline{/-47°} \quad \text{and} \quad \dot{I}_{NB2} = \frac{2.37}{\sqrt{3}}\underline{/-47°}$$

$$\dot{E}_{NC2} = \frac{23.7}{\sqrt{3}}\underline{/73°} \quad \text{and} \quad \dot{I}_{NC2} = \frac{2.37}{\sqrt{3}}\underline{/73°}$$

From equations (16–63) to (16–65), the line currents are:

$$\dot{I}_{NA} = \frac{1}{3 \times 10\underline{/0°}}(-80 + j60 + j60) = 4.8\underline{/123.7°}$$

$$\dot{I}_{NB} = \frac{1}{3 \times 10\underline{/0°}}(80 + 80 - j60) = 5.67\underline{/-20.6°}$$

$$\dot{I}_{NC} = \frac{1}{3 \times 10\underline{/0°}}(-j60 - 80) = 3.33\underline{/216.9°}$$

16–10. Single-Phase Loads on Three-Phase Systems.—In a three-phase, three-wire system single-phase loads can be connected between line and line; while in a three-phase, four-wire system single-phase loads may be connected between line and line and between line and neutral. The load $\dot{Z}_{CB}$ connected between line wires C and B in Fig. 16–15 may be considered as an unbalanced Δ-connected load with infinite impedances between lines B and A and between lines A and C. It will be seen in the next article on *Power in Unbalanced Three-Phase Circuits* that it is important to distinguish between the sequence components of the line currents $\dot{I}_a$, $\dot{I}_b$, and $\dot{I}_c$ and those of the branch currents $\dot{I}_{CB}$, $\dot{I}_{BA}$, and $\dot{I}_{AC}$. For example, if the sequence components of the branch voltages (line voltages) are considered in the expression for finding power, then the sequence components of the branch currents must be used in the same expression when equation (16–74) is to be used.

The sequence components of the line currents or of the branch currents can be found by following the straightforward method; that is, by first finding the line currents or the branch currents by the simplest method and then substituting values in the proper equations for the sequence components. The line currents will be determined first.

By inspection of Fig. 16–15, it can be seen that:

$$\dot{I}_a = 0$$

$$\dot{I}_b = \dot{I}_{CB} = \frac{\dot{E}_{CB}}{\dot{Z}_{CB}} = \dot{E}_{CB}\dot{Y}_{CB} = -\dot{I}_c$$

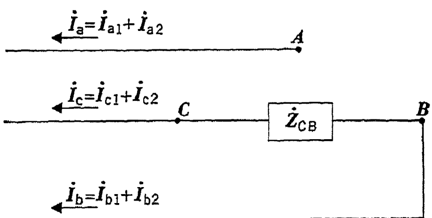


FIG. 16–15

Also, the line currents contain no zero-sequence components since $\dot{I}_a+\dot{I}_b+\dot{I}_c=0$. Although $\dot{I}_a=0$, its sequence components are not necessarily zero; only the vector sum of $\dot{I}_{a1}$ and $\dot{I}_{a2}$ is zero. By substituting values in equations (16–23) and (16–24), we obtain:

$$\dot{I}_{a1} = \frac{1}{3}(a\dot{I}_b + a^2\dot{I}_c) = \frac{a-a^2}{3}\dot{E}_{CB}\dot{Y}_{CB} = \frac{\dot{E}_{CB}\dot{Y}_{CB}}{\sqrt{3}} \angle 90^\circ \quad \textbf{(16–66)}$$

$$\dot{I}_{a2} = \frac{1}{3}(a^2\dot{I}_b + a\dot{I}_c) = \frac{a^2-a}{3}\dot{E}_{CB}\dot{Y}_{CB} = \frac{\dot{E}_{CB}\dot{Y}_{CB}}{\sqrt{3}} \angle -90^\circ \quad \textbf{(16–67)}$$

It should be noted that $\dot{I}_{a1}$ and $\dot{I}_{a2}$ are 180° out of phase and, thus, $\dot{I}_{a1}+\dot{I}_{a2}=\dot{I}_a=0$.

The branch currents are:

$$\dot{I}_{CB} = \frac{\dot{E}_{CB}}{\dot{Z}_{CB}} = \dot{E}_{CB}\dot{Y}_{CB}$$

$$\dot{I}_{BA} = 0$$

$$\dot{I}_{AC} = 0$$

Their sequence components can be determined by substituting values in equations (16–44) to (16–46). Thus,

$$\dot{I}_{CB0} = \dot{I}_{CB1} = \dot{I}_{CB2} = \tfrac{1}{3}\dot{I}_{CB} = \tfrac{1}{3}\dot{E}_{CB}\dot{Y}_{CB} \tag{16–68}$$

16–11. Power in Unbalanced Three-Phase Circuits.—The power per phase of an unbalanced three-phase circuit depends on the voltage, the current, and the power factor of that phase. Let the voltage and current be expressed in the polar form:

$$\dot{E} = E\underline{/\alpha}$$

$$\dot{I} = I\underline{/\beta}$$

Then, the power expression is

$$P = EI \cos (\beta - \alpha) \tag{16–69}$$

The total power in a three-phase circuit is

$$P_0 = P_a + P_b + P_c$$

where

$$\left.\begin{aligned} P_a &= E_a I_a \cos (\beta_a - \alpha_a) \\ P_b &= E_b I_b \cos (\beta_b - \alpha_b) \\ P_c &= E_c I_c \cos (\beta_c - \alpha_c) \end{aligned}\right\} \tag{16–70}$$

These are the powers in the phases a, b, and c, respectively.

To find the power in terms of the sequence components of the voltages and currents, let:

$$\dot{E}_{a0} = E_{a0}\underline{/\alpha_0},\ \dot{E}_{b0} = \dot{E}_{a0} = E_{a0}\underline{/\alpha_0},\ \dot{E}_{c0} = \dot{E}_{a0} = E_{a0}\underline{/\alpha_0}$$

$$\dot{E}_{a1} = E_{a1}\underline{/\alpha_1},\ \dot{E}_{b1} = a^2\dot{E}_{a1} = E_{a1}\underline{/\alpha_1 - 120^\circ},\ \dot{E}_{c1} = a\dot{E}_{a1} = E_{a1}\underline{/\alpha_1 + 120^\circ}$$

$$\dot{E}_{a2} = E_{a2}\underline{/\alpha_2},\ \dot{E}_{b2} = a\dot{E}_{a2} = E_{a2}\underline{/\alpha_2 + 120^\circ},\ \dot{E}_{c2} = a^2\dot{E}_{a2} = E_{a2}\underline{/\alpha_2 - 120^\circ}$$

$$\dot{I}_{a0} = I_{a0}\underline{/\beta_0},\ \dot{I}_{b0} = \dot{I}_{a0} = I_{a0}\underline{/\beta_0},\ \dot{I}_{c0} = \dot{I}_{a0} = I_{a0}\underline{/\beta_0}$$

$$\dot{I}_{a1} = I_{a1}\underline{/\beta_1},\ \dot{I}_{b1} = a^2\dot{I}_{a1} = I_{a1}\underline{/\beta_1 - 120^\circ},\ \dot{I}_{c1} = a\dot{I}_{a1} = I_{a1}\underline{/\beta_1 + 120^\circ}$$

$$\dot{I}_{a2} = I_{a2}\underline{/\beta_2},\ \dot{I}_{b2} = a\dot{I}_{a2} = I_{a2}\underline{/\beta_2 + 120^\circ},\ \dot{I}_{c2} = a^2\dot{I}_{a2} = I_{a2}\underline{/\beta_2 - 120^\circ}$$

Then, since $\dot{E}_a = \dot{E}_{a0} + \dot{E}_{a1} + \dot{E}_{a2}$, $\dot{I}_a = \dot{I}_{a0} + \dot{I}_{a1} + \dot{I}_{a2}$, etc., equations (16–70) become:

$$\begin{aligned} P_a = {} & E_{a0}I_{a0}\cos(\beta_0-\alpha_0) + E_{a1}I_{a1}\cos(\beta_1-\alpha_1) + E_{a2}I_{a2}\cos(\beta_2-\alpha_2) \\ & + E_{a0}I_{a1}\cos(\beta_1-\alpha_0) + E_{a1}I_{a2}\cos(\beta_2-\alpha_1) + E_{a2}I_{a0}\cos(\beta_0-\alpha_2) \\ & + E_{a0}I_{a2}\cos(\beta_2-\alpha_0) + E_{a1}I_{a0}\cos(\beta_0-\alpha_1) + E_{a2}I_{a1}\cos(\beta_1-\alpha_2) \end{aligned} \qquad (16\text{–}71)$$

$$\begin{aligned} P_b = {} & E_{a0}I_{a0}\cos(\beta_0-\alpha_0) + E_{a1}I_{a1}\cos(\beta_1-\alpha_1) + E_{a2}I_{a2}\cos(\beta_2-\alpha_2) \\ & + E_{a0}I_{a1}\cos(\beta_1-\alpha_0-120^\circ) + E_{a1}I_{a2}\cos(\beta_2-\alpha_1-120^\circ) \\ & + E_{a2}I_{a0}\cos(\beta_0-\alpha_2-120^\circ) + E_{a0}I_{a2}\cos(\beta_2-\alpha_0+120^\circ) \\ & + E_{a1}I_{a0}\cos(\beta_2-\alpha_1+120^\circ) + E_{a2}I_{a1}\cos(\beta_1-\alpha_2+120^\circ) \end{aligned} \qquad (16\text{–}72)$$

$$\begin{aligned} P_c = {} & E_{a0}I_{a0}\cos(\beta_0-\alpha_0) + E_{a1}I_{a1}\cos(\beta_1-\alpha_1) + E_{a2}I_{a2}\cos(\beta_2-\alpha_2) \\ & + E_{a0}I_{a1}\cos(\beta_1-\alpha_0+120^\circ) + E_{a1}I_{a2}\cos(\beta_2-\alpha_1+120^\circ) \\ & + E_{a2}I_{a0}\cos(\beta_0-\alpha_2+120^\circ) + E_{a0}I_{a2}\cos(\beta_1-\alpha_0-120^\circ) \\ & + E_{a1}I_{a0}\cos(\beta_0-\alpha_1-120^\circ) + E_{a2}I_{a1}\cos(\beta_1-\alpha_2-120^\circ) \end{aligned} \qquad (16\text{–}73)$$

But, combinations of the form $\cos(\beta_1-\alpha_0) + \cos(\beta_1-\alpha_0-120^\circ) + \cos(\beta_1-\alpha_0+120^\circ)$ become 0. Hence, when the addition is performed to obtain the total power, all terms in equations (16–71) to (16–73) add to zero except the first three in each equation. Thus,

$$P_0 = 3E_{a0}I_{a0}\cos(\beta_0-\alpha_0) + 3E_{a1}I_{a1}\cos(\beta_1-\alpha_1) + 3E_{a2}I_{a2}\cos(\beta_2-\alpha_2)$$

or $$P_0 = 3E_{a0}I_{a0}\cos\theta_0 + 3E_{a1}I_{a1}\cos\theta_1 + 3E_{a2}I_{a2}\cos\theta_2 \qquad (16\text{–}74)$$

where θ_0, θ_1, and θ_2 are the phase angles between the corresponding sequence components of current and voltage.

Hence, the total power in an unbalanced three-phase circuit is the sum of the powers due to the separate phase-sequence components (that is, the zero-sequence voltage E_{a0} and the zero-sequence current I_{a0} will give average power, etc.).

Example 16–6.—The impedance $\dot{Z}_{CB}$ in Fig. 16–15 is a pure resistance of 10 ohms, and the line voltage of the balanced three-phase system is 100 volts. Determine the total power delivered to the resistor by using the sequence components of the voltages and currents.

Solution.—The power will be determined by the use of equation (16–74). If the sequence components of the branch voltage drops (equal to the line voltage drops) are to be used, then those of the branch currents must be used.

Since the branch voltage drops form a balanced symmetrical system, only the positive-sequence system exists. For sequence ABC and with the voltage drop $\dot{E}_{CB}$ from C to B as the reference vector, then:

$$\dot{E}_{CB1} = \dot{E}_{CB} = 100\underline{/0^\circ}$$

$$\dot{E}_{BA1} = \dot{E}_{BA} = 100\underline{/120^\circ}$$

$$\dot{E}_{AC1} = \dot{E}_{AC} = 100\underline{/240^\circ}$$

The branch currents are:

$$\dot{I}_{CB} = \frac{\dot{E}_{CB}}{\dot{Z}_{CB}} = \frac{100\underline{/0^\circ}}{10\underline{/0^\circ}} = 10\underline{/0^\circ}$$

$$\dot{I}_{BA} = 0$$

$$\dot{I}_{AC} = 0$$

Only the positive-sequence component is needed in this case, since $\dot{E}_{CB0}$ and $\dot{E}_{CB2}$ are zero, and

$$\dot{I}_{CB1} = \frac{1}{3}\,\dot{I}_{CB} = \frac{10}{3}\,\underline{/0^\circ}$$

The power is

$$P = 3E_{CB1}I_{CB1}\cos\theta_{I_{CB1}}^{E_{CB1}}$$

$$= 3 \times 100 \times \frac{10}{3}\cos 0^\circ = 1000 \text{ watts}$$

Example 16–7.—Find the total power delivered to the resistor in Example 16–6 by using the sequence components of the line currents.

Solution.—Again, the power will be determined by the use of equation (16–74) and for voltages of sequence ABC. If the sequence components of the line currents are to be used, then those of the line-to-neutral voltages must be used.

Since the line voltages are balanced, then the line-to-neutral voltages are balanced. If the voltage drop $\dot{E}_{CB}$ is the reference vector and if the voltage drop from N to line A be $\dot{E}_{NA}$, the drop from N to B be $\dot{E}_{NB}$, and the drop from N to C be $\dot{E}_{NC}$, then, with the aid of Fig. 16–14(a), we have:

$$\dot{E}_{NA1} = \dot{E}_{NA} = \frac{100}{\sqrt{3}}\underline{/90^\circ}$$

$$\dot{E}_{NB1} = \dot{E}_{NB} = \frac{100}{\sqrt{3}}\underline{/-30^\circ}$$

$$\dot{E}_{NC1} = \dot{E}_{NC} = \frac{100}{\sqrt{3}}\underline{/-150^\circ}$$

The line currents are:

$$\dot{I}_a = 0$$

$$\dot{I}_b = \dot{I}_{CB} = 10\underline{/0^\circ} = -\dot{I}_c$$

Only the positive-sequence component is needed in this case, since $\dot{E}_{NA0}$ and $\dot{E}_{NA2}$ are zero. With the aid of equations (16–23) and (16–61),

$$\dot{I}_{a1} = \frac{\dot{I}_{CB}}{\sqrt{3}}\underline{/90^\circ} = \frac{10}{\sqrt{3}}\underline{/90^\circ}$$

The power is

$$P = 3E_{NA1}I_{a1}\cos\theta_{I_{a1}}^{E_{NA1}}$$

$$= 3\times\frac{100}{\sqrt{3}}\times\frac{10}{\sqrt{3}}\cos 0° = 1000 \text{ watts}$$

PROBLEMS

16–1. Evaluate, in the rectangular and polar forms, the following: $1-a$' $1-a^2$, a^2-a, $a-a^2$, $a-1$, a^2-1, $2a-a^2+3$, $1+a^2+a^5$.

16–2. Evaluate, in the rectangular and polar forms, the following: $\frac{1-a^2}{1+a}$; $\frac{1-a}{a^2-a}$; $\frac{1+a}{1-a^2}$.

16–3. Evaluate, in both the rectangular and polar forms, the following: $(1-a)(a-1)$; $(a^2-a)(1+a)$; $(1+a^2+a^5)(a^2-1)$.

16–4. Two balanced and symmetrical systems of opposite phase sequence are:

$$\dot{I}_{a1} = K\underline{/0°}, \qquad \dot{I}_{b1} = K\underline{/240°}, \qquad \dot{I}_{c1} = K\underline{/120°}$$

$$\dot{I}_{a2} = B\underline{/\theta}, \qquad \dot{I}_{c2} = B\underline{/\theta+240°}, \qquad \dot{I}_{b2} = B\underline{/\theta+120°}$$

These two systems are added together as indicated in Fig. 16–3. Determine the resultant system: (*a*) when $K = B$ and $\theta = 0°$; (*b*) when $K = 2B$ and $\theta = 0°$; (*c*) when $K = B$ and $\theta = 30°$; (*d*) when $K = 2B$ and $\theta = 60°$; (*e*) when $K = \frac{1}{2}B$ and $\theta = -30°$.

16–5. The line currents in a three-phase, three-wire system are $\dot{I}_a = 2+j8$, $\dot{I}_b = 3-j5$, and $\dot{I}_c = -5-j3$. Find the sequence components of the line currents. Draw the separate diagrams of the sequence systems and a composite diagram showing the phase relation between the systems.

16–6. Repeat Problem 16–5 for the following line currents: $\dot{I}_a = 2+j8$, $\dot{I}_b = -5-j3$, and $\dot{I}_c = 3-j5$..

16–7. Solve Problems 16–5 and 16–6 graphically. That is, rotate $\dot{I}_b$ and $\dot{I}_c$ through angles of 120° or 240°, as indicated in the equations for the sequence components, and then take one-third of the resultant vector which is obtained by performing the proper addition.

16–8. In a three-phase, three-wire system, the voltage drops across the three impedances connected in wye are $\dot{E}_a = 100\underline{/30°}$, $\dot{E}_b = 100\underline{/60°}$, and $\dot{E}_c = 50\underline{/-120°}$. Determine the sequence components of the load voltage drops. Draw the vector diagram of each sequence component and, also, a composite vector diagram showing the phase relations of the different vectors.

16–9. Solve Problem 16–8 graphically.

16–10. The line currents in a three-phase system are $\dot{I}_a = 60+j80$, $\dot{I}_b = 80-j60$, and $\dot{I}_c = 100+j0$. What are the symmetrical components of the three currents? Draw the vector diagram for the original currents and the vector diagrams for the positive-sequence, negative-sequence, and zero-sequence systems.

16–11. The line currents in a three-phase, four-wire system are $\dot{I}_a = 3+j4$, $\dot{I}_b = 5+j0$, and $\dot{I}_c = 0-j5$. What are the symmetrical components of the currents? What is the current flowing in the neutral wire? How does the magnitude of the neutral current compare with the magnitude of the zero-sequence component? Draw a complete vector diagram, showing the original line currents and their sequence components.

16–12. Three single-phase loads $\dot{Z}_{CB} = 10\underline{/0°}$, $\dot{Z}_{BA} = 5\underline{/-36.8°}$, and $\dot{Z}_{AC} = 5\underline{/53.2°}$ are connected in delta to a balanced, three-phase, 100-volt system. Determine the positive-sequence, negative-sequence, and zero-sequence components of the branch currents. Also, determine the sequence components of the line currents.

16–13. Repeat Example 16–5 for $\dot{Z} = 10\underline{/90°}$.

16–14. Repeat Example 16–5 for $\dot{Z} = 10\underline{/-90°}$.

16–15. Repeat Example 16–5 for $\dot{Z} = 10\underline{/53.2°}$.

16–16. Repeat Example 16–5 for $\dot{Z} = 10\underline{/-36.8°}$.

16–17. Three equal impedances $\dot{Z} = R = 30$ ohms are connected in delta to a three-phase system, in which the line voltage drops are:

$$\dot{E}_{CB} = 80 + j0 = 80\underline{/0°}$$

$$\dot{E}_{BA} = -80 + j60 = 100\underline{/143.2°}$$

$$\dot{E}_{AC} = 0 - j60 = 60\underline{/-90°}$$

Find the sequence components of the branch currents.

16–18. Repeat Problem 16–17 for $\dot{Z} = 30\underline{/90°}$.

16–19. Repeat Problem 16–17 for $\dot{Z} = 30\underline{/53.2°}$.

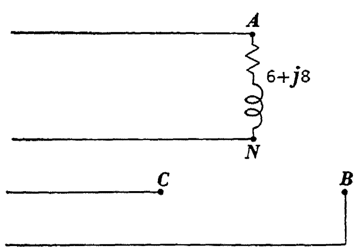


FIG. 16–16

16–20. Three impedances $\dot{Z}_{CB} = 8 - j6$, $\dot{Z}_{BA} = 6 + j8$, and $\dot{Z}_{AC} = 10 + j0$ are connected in delta to a three-phase, three-wire system. If the line voltages are $\dot{E}_{CB} = 34.57 + j19.35$, $\dot{E}_{BA} = 90\underline{/100°}$, and $\dot{E}_{AC} = 110\underline{/260°}$, find the sequence components of the line currents and draw the complete vector diagram.

16–21. A single-phase load $\dot{Z}_{NA} = 6 + j8$ is connected between line A and the neutral in a three-phase, four-wire, 173.2-volt system, as shown in Fig. 16–16. Find the sequence components of the line currents. What is the neutral current?

16–22. The impedance $\dot{Z} = 8 - j6$ is connected between lines A and B of a three-phase, four-wire, 100-volt supply, as shown in Fig. 16–17. Find the sequence components of the line currents. Calculate the power taken by the load.

16–23. Two impedances $\dot{Z}_{BA} = 8 - j6$ and $\dot{Z}_{NA} = 6 + j8$ are connected to a three-phase, four-wire, 173.2-volt supply, as shown in Fig. 16–18. Find the line and neutral currents and their sequence components. Calculate the power and draw the vector diagram.

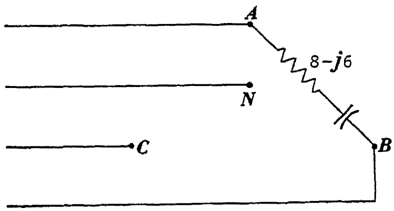


FIG. 16–17

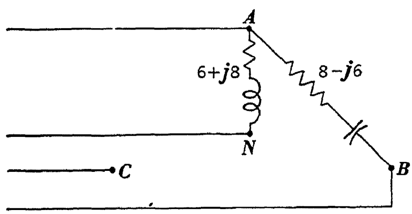


FIG. 16–18

16–24. A balanced, Y-connected, pure resistive load, for which $R = 10$ for each branch, is connected to a three-phase, three-wire system. If the line voltages are $\dot{E}_{CB} = 100 + j0$, $\dot{E}_{BA} = -45 + j78$, and $\dot{E}_{AC} = -55 - j78$, find the sequence components of the line currents and the power taken. Draw the vector diagram.

16–25. Each wire of a three-phase, three-wire, 100-volt system has an impedance $\dot{Z} = 1 + j2$. There is no load connected across the load terminals. If line wires A and C are accidentally shorted at the load terminals, calculate the sequence components of the line currents.

16–26. A three-phase, 100-volt system has a ground return. A ground return is equivalent to a neutral wire connected to the neutral of the generator. Each of the three conductors has an impedance $\dot{Z} = 2 + j3$, and the ground return is assumed to have negligible impedance. If line wire A is shorted to ground, determine the sequence components of the line currents.

16–27. If there is a double line-to-ground short in Problem 16–26, calculate the sequence components of the line currents.

16–28. A balanced delta load $\dot{Z} = 3 + j4$ is connected to a three-phase system with unbalanced voltages $\dot{E}_{CB} = 110\underline{/0°}$, $\dot{E}_{BA} = 100\underline{/120°}$, and $\dot{E}_{AC} = -60 - j86.6$. Find the sequence components of the line currents and draw the vector diagram.

16–29. A Y-load is connected to a three-phase, 173.2-volt system, in which the sequence components of the line currents are:

$$\dot{I}_{a0} = \frac{10}{3}\underline{/240°}$$

$$\dot{I}_{b1} = \frac{10}{3}\underline{/0°}$$

$$\dot{I}_{c2} = \frac{20}{3}\underline{/60°}$$

Find the impedances of the Y-load.

CHAPTER 17

NON-SINUSOIDAL WAVES

The wave forms of commercial alternators are seldom exactly sinusoidal and under certain conditions may depart considerably from the ideal form. Even if the poles of an alternator could be shaped to give a flux distribution that would produce a sinusoidal voltage at no load, the wave form of the alternator under load might differ from a sinusoid because of the effect of armature reaction in distorting the flux distribution. A sinusoidal voltage impressed on a circuit with iron core will produce a non-sinusoidal current. An inductive load will tend to smooth out the irregularities or suppress in the current wave the higher harmonics that existed in the voltage wave. A capacitive load will tend to amplify in the current wave the higher harmonics in the voltage wave.

The purposes of this chapter are: (*a*) to study the effect of even and odd harmonics on the resultant wave form; (*b*) to develop equations for the average and effective values of non-sinusoidal waves; (*c*) to derive an equation for the power taken by a circuit when a non-sinusoidal voltage is applied to the circuit; and (*d*) to analyze complex or distorted waves and to evaluate their component harmonics.

17–1. Representation of a Non-Sinusoidal Wave by a Fourier Series.—Joseph Fourier, a French physicist and mathematician, stated in his *Analytical Theory of Heat* (1822) that any function which is periodic, single-valued, finite, and continuous (or at least with a finite number of discontinuities in an interval of length 2π) may be represented by a series of the form

$$y=f(\alpha)=A_1 \sin \alpha+A_2 \sin 2\alpha+A_3 \sin 3\alpha+\dots+ A_n \sin n\alpha+B_0+B_1 \cos \alpha+B_2 \cos 2\alpha+ B_3 \cos 3\alpha+\dots+B_n \cos n\alpha \qquad (17\text{–}1)$$

or

$$y=B_0+\sum_{n=1}^{n=\infty} A_n \sin n\alpha+\sum_{n=1}^{n=\infty} B_n \cos n\alpha \qquad (17\text{–}2)$$

where B_0 is a constant;

$A_1 \ldots A_n$ and $B_1 \ldots B_n$ are amplitudes of the different harmonics;

α is the independent variable—in our work, either *time* or *angle*;

n is any integer, as 1, 2, 3, 4, ...

The sine and cosine terms are called harmonics. It is convenient to use n for designating the order of the harmonics. If $n=1$, the harmonics (lowest frequency) are called the fundamentals; if $n=2$, the terms are known as the second harmonics; and if $n=\beta$, the harmonics are called the βth harmonics.

The expression in equation (17–1) is, in general, an infinite series. Theoretically, it is necessary to have an infinite number of terms to represent an arbitrary function with absolute exactness by a series for every value of α in the chosen interval. If the number of terms selected, say n, is finite, equation (17–1) will be a finite series and it will be exactly equal to the function at n points. The deviation between the function and the series will be smaller as n becomes greater. Fortunately, in the case of waves obtained from alternators, the first few terms are sufficient. Of course, it should be noted that, in certain cases, the function may be represented exactly by a series containing a finite number of terms.

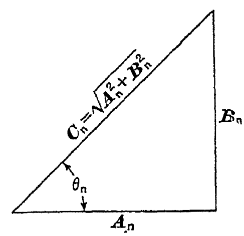


FIG. 17–1

For convenience, the Fourier series in equation (17–1) may be reduced to a series involving sine terms alone by combining $A_1 \sin \alpha$ and $B_1 \cos \alpha$ to give $C_1 \sin(\alpha+\theta_1)$, etc. Consider any two terms such as $A_n \sin n\alpha$ and $B_n \cos n\alpha$ and construct a triangle as shown in Fig. 17–1. It is obvious from the figure that:

$$\left.\begin{aligned} \cos\theta_n &= \frac{A_n}{\sqrt{A_n^2+B_n^2}} \\ \sin\theta_n &= \frac{B_n}{\sqrt{A_n^2+B_n^2}} \\ \tan\theta_n &= \frac{B_n}{A_n} \end{aligned}\right\} \qquad (17\text{–}3)$$

Multiply the expression $(A_n \sin n\alpha + B_n \cos n\alpha)$ by $\frac{\sqrt{A_n^2+B_n^2}}{\sqrt{A_n^2+B_n^2}}$, and rearrange the result as follows:

$$\sqrt{A_n^2+B_n^2}\left(\sin n\alpha \frac{A_n}{\sqrt{A_n^2+B_n^2}} + \cos n\alpha \frac{B_n}{\sqrt{A_n^2+B_n^2}}\right)$$

When we substitute values from equation (17–3), we get:

$$\sqrt{A_n^2+B_n^2}[\sin n\alpha \cos \theta_n + \cos n\alpha \sin \theta_n] = C_n \sin (n\alpha + \theta_n)$$

or

$$A_n \sin n\alpha + B_n \cos n\alpha = C_n \sin (n\alpha + \theta_n) \qquad \textbf{(17–4)}$$

Thus,

$$C_1 \sin (\alpha + \theta_1) = A_1 \sin \alpha + B_1 \cos \alpha$$
$$C_2 \sin (2\alpha + \theta_2) = A_2 \sin 2\alpha + B_2 \cos 2\alpha$$
$$C_n \sin (n\alpha + \theta_n) = A_n \sin n\alpha + B_n \cos n\alpha$$

Hence, equation (17–1) may be written as follows:

$$y = f(\alpha) = B_0 + C_1 \sin (\alpha + \theta_1) + C_2 \sin (2\alpha + \theta_2)$$
$$+ C_3 \sin (3\alpha + \theta_3) + \ldots + C_n \sin (n\alpha + \theta_n) \qquad \textbf{(17–5)}$$

where n = any integer and

$$\left.\begin{aligned} C_1 &= \sqrt{A_1^2+B_1^2} \\ C_n &= \sqrt{A_n^2+B_n^2} \end{aligned}\right\} \qquad \textbf{(17–6)}$$

$$\left.\begin{aligned} \tan \theta_1 &= \frac{B_1}{A_1} \\ \tan \theta_n &= \frac{B_n}{A_n} \end{aligned}\right\} \qquad \textbf{(17–7)}$$

The constants $C_1, C_2, C_3, \ldots . C_n$ are the amplitudes of the component waves. The angles $\theta_1, \theta_2, \theta_3, \ldots . \theta_n$ are the angles measured from the origin to the points of crossing from negative to positive values of the component waves.

The signs of A_n and B_n must be considered in determining the value for θ_n. For example, if A_n and B_n are both negative, the angle is in the third quadrant.

17–2. Symmetry.—A wave is said to have a "half-wave" symmetry when any ordinate in the positive lobe will have an ordinate of equal value but of opposite sign at a distance of 180 degrees along the time axis, as shown in Fig. 17–2. Such a wave shall be called a *symmetrical wave.* Mathematically,

$$f(\alpha) = -f(\alpha+\pi) \tag{17–8}$$

A wave is said to have a "quarter-wave" symmetry when the curve is symmetrical with respect to the ordinate at $\frac{\pi}{2}$, as shown in Fig. 17–2; or, mathematically,

$$f(\alpha) = f(\pi-\alpha) \tag{17–9}$$

A wave having "quarter-wave" symmetry does not necessarily have "half-wave" symmetry. For clearness, the term *symmetrical wave* shall be restricted hereafter to waves having "half-wave" symmetry.

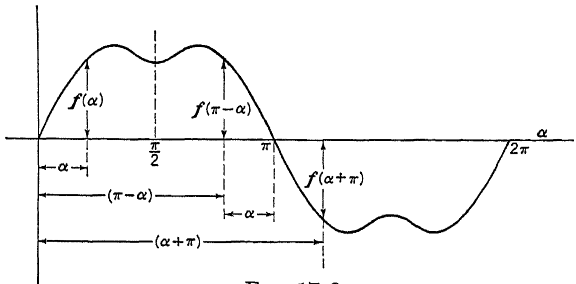


Fig. 17–2

It will be shown in Art. 17–5 that it is always possible to change the reference point from which angles are reckoned to that shown in Fig. 17–2.

17–3. Synthesis of Waves.—To have a better conception of complex or distorted waves being composed of harmonics and a constant term, a study of the effect of a constant term, of an even harmonic, and of an odd harmonic on wave form will be made.

(*a*) *The Constant Term:* To study the effect of the constant term on wave form, plot the wave

$$e = e_0 + e_1 = 10 + 20 \sin \omega t \tag{17–10}$$

It is obvious that $e_0 = 10$ is a straight line parallel to and displaced 10 units above the ωt-axis, Fig. 17–3. The sine wave 20 sin ωt is superimposed upon this constant term. The constant term does not distort the sine wave. It simply shifts the horizontal axis of the wave B_0 units above or below the ωt-axis, the direction depending on whether B_0 is positive or negative. The resultant wave is unsymmetrical according to the definition of a symmetrical wave given by equation (17–8). Thus, it may be said that a wave containing a constant term is unsymmetrical.

(*b*) *Effect of Even Harmonics on Wave Form:* To study the effect of even harmonics on the resultant wave shape, add a second harmonic to a fundamental at various phase angles. In Figs. 17–4, 17–5, and 17–6 are shown plots, for several values of θ_2, of the wave

$$e = 10 \sin \omega t + 5 \sin (2\omega t + \theta_2) \qquad \textbf{(17–11)}$$

In Fig. 17–4, $\theta_2 = 0°$. In Fig. 17–5, $\theta_2 = -90°$. In Fig. 17–6, $\theta_2 = +90°$.

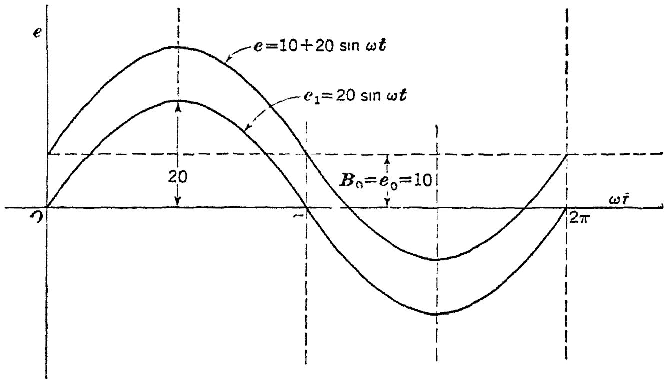


FIG. 17–3

In plotting the wave $e = 10 \sin \omega t + 5 \sin (2\omega t - 90°)$ in Fig. 17–5, it may be helpful to write the expression as follows:

$$e = 10 \sin \omega t + 5 \sin 2(\omega t - 45°) \qquad \textbf{(17–12)}$$

The angle 45° in equation (17–12) is in fundamental degrees or to the same scale to which ωt is plotted; whereas, the angle θ_2 or $-90°$ in equation (17–11) is in "second harmonic" degrees. That is, the interval from 0 to π along the ωt-axis for the fundamental is equal to the interval from 0 to 2π for the second harmonic.

In Fig. 17–4, the lobes may at first appear symmetrical; but, upon application of the rule of "half-wave" symmetry, it is found that an ordinate in the positive lobe is not equal to the negative of the corresponding ordinate in the negative lobe at a distance of 180 degrees along the ωt-axis. Inspection of Fig. 17–4, 17–5, or 17–6

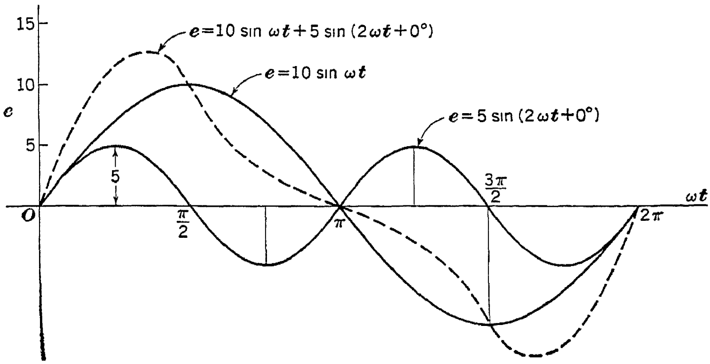


FIG. 17–4

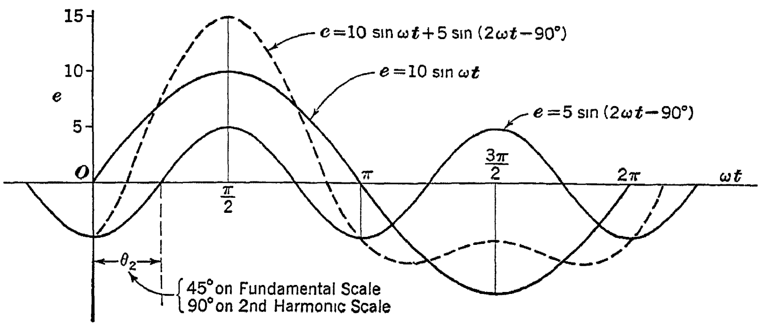


FIG. 17–5

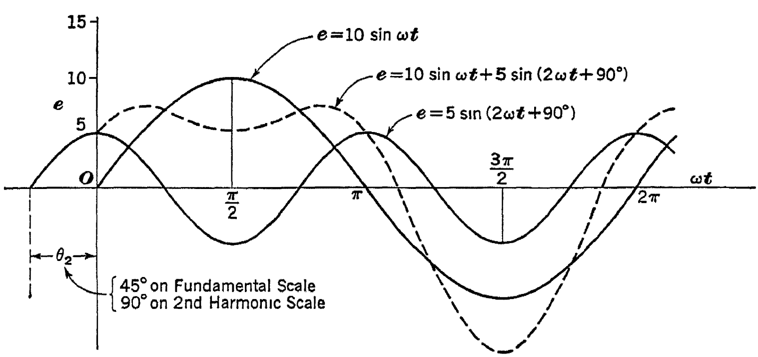


FIG. 17–6

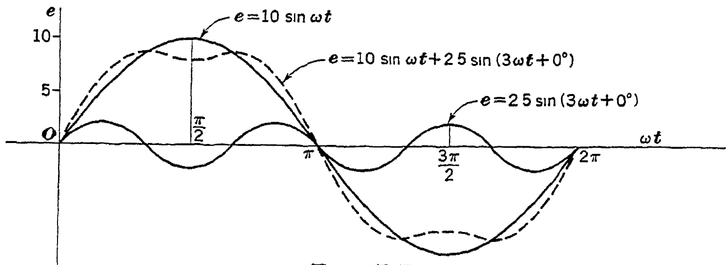


FIG. 17–7

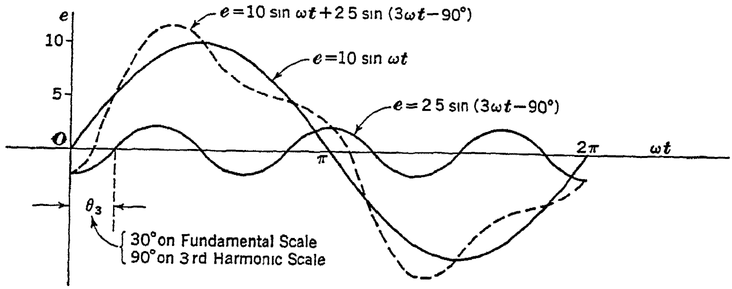


FIG. 17–8

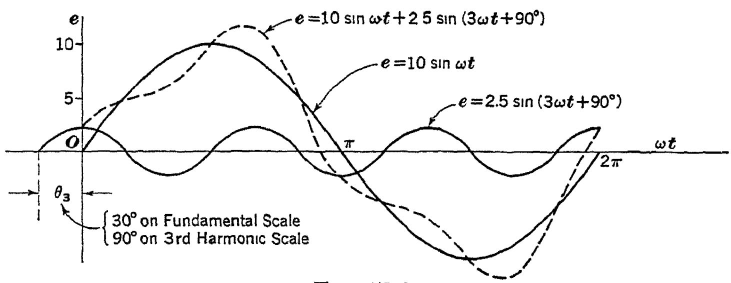


FIG. 17–9

shows that the positive and negative lobes of the resultant waves are not symmetrical. Hence, it may be said that *an even harmonic causes a resultant wave to be unsymmetrical.*

(*c*) *Effect of Odd Harmonics on Wave Form:* In a similar manner, the effect of odd harmonics on the resultant wave form may be studied by adding a third harmonic to a fundamental at various angles. In Figs. 17–7, 17–8, and 17–9 are shown plots, for several values of θ_3, of the wave

$$e = 10 \sin \omega t + 2.5 \sin (3\omega t + \theta_3) \qquad \textbf{(17–13)}$$

In Fig. 17–7, $\theta_3=0°$. In Fig. 17–8, $\theta_3=-90°$. In Fig. 17–9, $\theta_3=+90°$.

By inspection, it is obvious that, in each of these cases, an odd harmonic produces a symmetrical resultant wave. Note that the combined wave has the same period, 0 to 2π, as that for the fundamental component.

It should be noted from this study that, in plotting the nth harmonic component of the wave $e_n=E_{mn}\sin(n\alpha+\theta_n)$, the angle θ_n is measured on the scale of the nth harmonic. That is, the interval from 0 to π along the ωt-axis for the fundamental is equal to the interval from 0 to $n\pi$ for the nth harmonic. In Fig. 17–8, where θ_3 is negative, the third harmonic crosses the axis at a distance of $\frac{\theta_3}{3}=\frac{90}{3}$ "fundamental" degrees to the right of the origin. When θ_n is positive, as in Fig. 17–9, the nth harmonic crosses the axis at a distance of $\frac{\theta_n}{n}$ "fundamental" degrees to the left of the origin.

17–4. Symmetrical Waves Cannot Contain Even Harmonics. Let a wave be represented by the following Fourier Series:

$$e=C_1\sin(\omega t+\theta_1)+C_2\sin(2\omega t+\theta_2)+C_3\sin(3\omega t+\theta_3)+\ldots+C_n\sin(n\omega t+\theta_n) \quad \textbf{(17–14)}$$

If this wave is symmetrical (half-wave symmetry), its instantaneous values at two instants of time which are separated by one-half period, such as t and $\left(t+\frac{T}{2}\right)=\left(t+\frac{1}{2f}\right)$ where $f=$ the fundamental frequency, must be equal in magnitude but opposite in sign. This follows from the definition given by equation (17–8). Equation (17–14) gives the instantaneous value for e for a time $t=t$; and for $t=t+\frac{1}{2f}$ the instantaneous value of e is

$$e'=C_1\sin\left(\omega t+\frac{\omega}{2f}+\theta_1\right)+C_2\sin\left(2\omega t+\frac{\omega}{f}+\theta_2\right)+C_3\sin\left(3\omega t+\frac{3\omega}{2f}+\theta_3\right)+\ldots+C_n\sin\left(n\omega t+\frac{n\omega}{2f}+\theta_n\right) \quad \textbf{(17–15)}$$

It is known that $\omega=2\pi f$ and also that phase displacement of any multiple of 2π radians is equal to zero displacement as far as phase relations are concerned. Equation (17–15) becomes

$$
\begin{aligned}
e' &= C_1 \sin(\omega t + \pi + \theta_1) + C_2 \sin(2\omega t + \theta_2) \\
&\quad + C_3 \sin(3\omega t + \pi + \theta_3) + \ldots + C_n \sin(n\omega t + n\pi + \theta_n) \\
&= -C_1 \sin(\omega t + \theta_1) + C_2 \sin(2\omega t + \theta_2) \\
&\quad - C_3 \sin(3\omega t + \theta_3) + \ldots + (-1)^n C_n \sin(n\omega t + \theta_n) \qquad (17\text{–}16)
\end{aligned}
$$

It will be seen by comparing equations (17–14) and (17–16) that, while the fundamentals and the corresponding odd harmonics for points one-half a period apart are opposite in sign, the even harmonics are of the same sign. Therefore, *the two halves of a wave containing even harmonics cannot satisfy the definition of half-wave symmetry.*

17–5. Changing the Reference Point From Which Angles and Time Are Measured in a Complex Wave.—It is frequently convenient and often necessary, when considering complex waves, to change the position of the point from which angles and time are measured. In some cases, the Fourier expression for a non-sinusoidal wave can be simplified by a proper choice of the origin, as indicated in Art. 17–14 and in Problems 17–23 and 17–24. When two waves are compared for similarity, it may be necessary to shift one wave in phase to determine whether or not the phase relations of the different harmonics are alike in the two waves. This feature will now be considered.

For two waves to be similar they not only must contain like harmonics, but the relative magnitudes of the harmonics and fundamental and their phase relations must be alike in the two waves. It is a simple matter to tell from the equations whether or not the waves contain the same harmonics. The relative magnitudes of the harmonics and the fundamental may be determined from the coefficients of the Fourier series. But, unless the points from which angles are measured occupy the same positions with respect to the fundamentals, the relative phases of the harmonics in the waves will not be obvious.

Consider the following waves:

$$e_1 = 10 \sin(\omega t + 30^\circ) + 4 \sin(3\omega t + 75^\circ) + 2 \sin(5\omega t - 60^\circ) \qquad (17\text{–}17)$$

$$e_2 = 15 \sin(\omega t - 10^\circ) + 6 \sin(3\omega t - 45^\circ) - 3 \sin(5\omega t - 80^\circ) \qquad (17\text{–}18)$$

The waves contain like harmonics, and the relative magnitudes of the harmonics and fundamentals are the same in both waves.

However, to compare the phase relations of the harmonics and the fundamentals, one wave must be shifted in phase by an amount equal to the difference between the fundamental phase angles of the two waves. In this case the shift is $\pm 40°$, the sign depending on whether e_1 or e_2 is shifted.

Shifting e_1 in the positive direction of ωt would mean subtracting 40°, that is, substituting $(\omega t - 40°)$ for ωt in equation (17–17). Thus,

$$e_1' = 10 \sin (\omega t + 30° - 40°) + 4 \sin (3\omega t + 75° - 3 \times 40°) + 2 \sin (5\omega t - 60° - 5 \times 40°)$$

$$= 10 \sin (\omega t - 10°) + 4 \sin (3\omega t - 45°) + 2 \sin (5\omega t - 260°)$$

or

$$e_1' = 10 \sin (\omega t - 10°) + 4 \sin (3\omega t - 45°) - 2 \sin (5\omega t - 80°) \quad \textbf{(17–19)}$$

It is obvious from equations (17–18) and (17–19) that the phase differences are also alike, and the waves are actually of the same shape.

17–6. Average Value of Non-Sinusoidal Emf or Current. Consider the alternating emf wave

$$e = E_0 + E_{m1} \sin (\alpha + \theta_1) + E_{m2} \sin (2\alpha + \theta_2) + E_{m3} \sin (3\alpha + \theta_3) + \ldots + E_{mn} \sin (n\alpha + \theta_n) \quad \textbf{(17–20)}$$

It is obvious that the average value of any sine wave during one complete cycle or any number n of complete cycles is zero. Therefore, the average value of every term in equation (17–20) except E_0 is zero; and the average value of an unsymmetrical wave, between the limits 0 and 2π measured on the fundamental scale, is

$$E_{\text{avg}} = E_0 \quad \textbf{(17–21)}$$

This value is commonly called by communication engineers the d-c value or the d-c effect. Of course, the average value of a symmetrical wave over one complete cycle is zero.

Similarly, if i is of the form of equation (17–20), then the average value between 0 and 2π measured on the fundamental scale is

$$I_{\text{avg}} = I_0 \quad \textbf{(17–22)}$$

17–7. Effective or RMS Value of a Non-Sinusoidal Emf or Current.—Again, let the emf wave be as given by equation (17–20). The effective value is given by the expression

$$E = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} e^2\, d\alpha}$$

or

$$E = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} [E_0 + E_{m1}\sin(\alpha+\theta_1) + E_{m2}\sin(2\alpha+\theta_2) + \ . \ . \ . \ ]^2 d\alpha} \tag{17–23}$$

Equation (17–23) contains, under the radical sign, the following terms:

(*a*) The E_0^2 term, which is

$$\frac{1}{2\pi}\int_0^{2\pi} E_0^2\, d\alpha$$

(*b*) The product terms containing E_0 as a factor, which are of the general form

$$\frac{1}{2\pi}\int_0^{2\pi} E_0 E_{mn} \sin(n\alpha+\theta_n)\, d\alpha$$

(*c*) The square terms, which are of the general form

$$\frac{1}{2\pi}\int_0^{2\pi} E_{mn}^2 \sin^2(n\alpha+\theta_n)\, d\alpha$$

(*d*) The product terms not containing E_0, which are of the general form

$$\frac{1}{2\pi}\int_0^{2\pi} E_{mn}\sin(n\alpha+\theta_n) \times E_{mk}\sin(k\alpha+\theta_k)\, d\alpha$$

where k is any integer, as 1, 2, 3, . . . , that is different from n. When $k = n$, the term is a square term, as in (*c*).

The evaluation of each of the terms of types (*a*), (*b*), (*c*), and (*d*) will now be considered.

(*a*) It is obvious that

$$\frac{1}{2\pi}\int_0^{2\pi} E_0^2\, d\alpha = E_0^2 \tag{17–24}$$

(*b*) The value of the expression $\int_0^{2\pi} \sin (n\alpha + \theta_n)\, d\alpha$ will be zero for any value of n, since the integral of a sine wave between the limits 0 and 2π is zero. Thus, the product terms containing E_0 as a factor are zero.

(*c*) The value of any square term may be determined by considering the product of two sine terms of like frequency (indicated by $n\alpha$ in the two sine terms). This method of analysis is used here because it will be useful in the study of power, as shown in Art. 17–9. Let

$$G(\alpha) = A \sin (n\alpha + \theta) \times B \sin (n\alpha + \theta')$$

or

$$G(\alpha) = AB \sin (n\alpha + \theta) \sin (n\alpha + \theta') \qquad (17\text{–}25)$$

Since $\sin x \sin y = \frac{1}{2}[\cos (x - y) - \cos (x + y)]$, then

$$G(\alpha) = \frac{AB}{2}[\cos (\theta - \theta') - \cos (2n\alpha + \theta + \theta')] \qquad (17\text{–}26)$$

When α is the only variable, the integral of the second term in the brackets between the limits 0 and 2π is zero; and

$$\frac{1}{2\pi}\int_0^{2\pi} G(\alpha)\, d\alpha = \frac{AB}{2} \cos (\theta - \theta') \qquad (17\text{–}27)$$

When $A = B$ and $\theta = \theta'$, equation (17–27) gives a square term. Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} G(\alpha)\, d\alpha = \frac{A^2}{2} \qquad (17\text{–}28)$$

Hence, the sum of the square terms in equation (17–23) is

$$\frac{E_{m1}^2}{2} + \frac{E_{m2}^2}{2} + \frac{E_{m3}^2}{2} + \dots = \left(\frac{E_{m1}}{\sqrt{2}}\right)^2 + \left(\frac{E_{m2}}{\sqrt{2}}\right)^2 + \left(\frac{E_{m3}}{\sqrt{2}}\right)^2 + \dots$$
$$= E_1^2 + E_2^2 + E_3^2 + \dots \qquad (17\text{–}29)$$

where $E_1, E_2, E_3, \dots E_n$ are the effective values of the harmonic components of the wave.

(*d*) The values of the product terms not containing E_0 may be obtained by studying the product of two sine terms of unlike frequency. Let

$$H(\alpha) = A \sin (n\alpha + \theta_n) \times B \sin (k\alpha + \theta_k)$$

or

$$H(\alpha) = AB \sin (n\alpha + \theta_n) \sin (k\alpha + \theta_k) \qquad (17\text{–}30)$$

where $n \neq k$; that is, the frequencies are different.

Since $\sin x \sin y = \frac{1}{2}[\cos (x-y) - \cos (x+y)]$, then

$$H(\alpha) = \frac{AB}{2}\left\{\cos [(n-k)\alpha + \theta_n - \theta_k] - \cos [(n+k)\alpha + \theta_n + \theta_k]\right\} \quad (17\text{–}31)$$

It follows that

$$\frac{1}{2\pi}\int_0^{2\pi} H(\alpha)\, d\alpha = 0 \quad (17\text{–}32)$$

because integrating a cosine function of α between the limits 0 and 2π gives zero as a result.

Therefore, the effective value of any non-sinusoidal wave is the square root of the sum of the square of the constant term and the squares of the effective values of the component waves; or

$$E = \sqrt{E_0^2 + E_1^2 + E_2^2 + E_3^2 + \ldots + E_n^2} \quad (17\text{–}33)$$

Equation (17–33) is independent of the phase relation between the component waves and of the starting point of the wave.

Similarly, let an alternating current of period $T = \frac{2\pi}{\omega}$ be of the form

$$i = I_0 + I_{m1} \sin (\omega t + \theta_1 + \beta_1) + I_{m2} \sin (2\omega t + \theta_2 + \beta_2) + I_{m3} \sin (3\omega t + \theta_3 + \beta_3) + \ldots + I_{mn} \sin (n\omega t + \theta_n + \beta_n) \quad (17\text{–}34)$$

Then its effective value is

$$I = \sqrt{I_0^2 + I_1^2 + I_2^2 + I_3^2 + \ldots + I_n^2} \quad (17\text{-}35)$$

where $I_1, I_2, I_3, \ldots I_n$ are the effective values of the component waves. The effective values of the emf and current for a non-sinusoidal wave are automatically indicated by an ordinary a-c voltmeter and ammeter.

Example 17–1.—Find the effective value of the following wave:

$$i = 10 + 100\sqrt{2} \sin \omega t + 50\sqrt{2} \sin (3\omega t + 60°) + 10\sqrt{2} \sin (7\omega t - 210°)$$

Solution.—Substituting effective values in equation (17–35) gives

$$I = \sqrt{10^2 + 100^2 + 50^2 + 10^2} = 112.7 \text{ amp}$$

Note that the I_0^2 term is not $\left(\frac{10}{\sqrt{2}}\right)^2$.

17–8. Effective Value of a Symmetrical Emf or Current Wave. When a wave is symmetrical, the constant term E_0 or I_0 is zero and the even harmonics do not appear in the expressions. Equations (17–33) and (17–35) then become, respectively,

$$E = \sqrt{E_1^2 + E_3^2 + E_5^2 + E_7^2 + \dots} \qquad (17\text{–}36)$$

$$I = \sqrt{I_1^2 + I_3^2 + I_5^2 + I_7^2 + \dots} \qquad (17\text{–}37)$$

17–9. Power When the Voltage and Current Are Non-Sinusoidal Waves.—The power delivered to a circuit at any instant is defined by the relation

$$p = ei$$

Also, the average power, or wattmeter reading, is given by the expression

$$P = \frac{1}{2\pi}\int_0^{2\pi} p \, d\alpha = \frac{1}{2\pi}\int_0^{2\pi} ei \, d\alpha$$

Let the emf and the current waves be expressed, respectively, by:

$$e = E_0 + E_{m1} \sin(\alpha + \theta_1) + E_{m2} \sin(2\alpha + \theta_2) + E_{m3} \sin(3\alpha + \theta_3) + \dots$$

or

$$e = E_0 + \sum_{n=1}^{n=n} E_{mn} \sin(n\alpha + \theta_n) \qquad (17\text{–}38)$$

and

$$i = I_0 + I_{m1} \sin(\alpha + \theta_1 + \beta_1) + I_{m2} \sin(2\alpha + \theta_2 + \beta_2) + I_{m3} \sin(3\alpha + \theta_3 + \beta_3) + \dots$$

$$= I_0 + I_{m1} \sin(\alpha + \theta_1') + I_{m2} \sin(2\alpha + \theta_2') + I_{m3} \sin(3\alpha + \theta_3') + \dots$$

or

$$i = I_0 + \sum_{n=1}^{n=n} I_{mn} \sin(n\alpha + \theta_n') \qquad (17\text{–}39)$$

Here, β_n may be considered as the phase angle between the nth harmonic voltage and the nth harmonic current, and $\theta_n' = (\theta_n + \beta_n)$.

The average power is

$$P = \frac{1}{2\pi}\int_0^{2\pi} \left[E_0 + \sum_{n=1}^{n=n} E_{mn} \sin(n\alpha + \theta_n)\right] \times \left[I_0 + \sum_{n=1}^{n=n} I_{mn} \sin(n\alpha + \theta_n')\right] d\alpha \qquad (17\text{–}40)$$

When the emf is multiplied by the current, equation (17–40) will give a product term of the two constants, product terms of constant and harmonic components, product terms of like frequencies, and product terms of unlike frequencies. On integration, the product term of the two constants will give E_0I_0; each product term of a constant and a harmonic component will be zero; all product terms of unlike frequencies will be zero [from equation (17–32)]; and product terms of like frequencies will give [from equation (17–27)]

$$\sum_{n=1}^{n=n} \frac{E_{mn}I_{mn}}{2} \cos\,(\theta_n - \theta_n') = \sum_{n=1}^{n=n} \frac{E_{mn}I_{mn}}{2} \cos\,\beta_n \qquad (17\text{–}41)$$

Thus, the average power is

$$P = E_0I_0 + \frac{E_{m1}I_{m1}}{2}\cos\,\beta_1 + \frac{E_{m2}I_{m2}}{2}\cos\,\beta_2 + \frac{E_{m3}I_{m3}}{2}\cos\,\beta_3 + \ldots$$
$$= E_0I_0 + E_1I_1\cos\,\beta_1 + E_2I_2\cos\,\beta_2 + E_3I_3\cos\,\beta_3 + \ldots$$

or

$$P = E_0I_0 + E_1I_1\cos(\theta_1 - \theta_1') + E_2I_2\cos(\theta_2 - \theta_2') + E_3I_3\cos(\theta_3 - \theta_3') + \ldots \qquad (17\text{–}42)$$

If the emf and current waves are symmetrical, then equation (17–42) becomes

$$P = E_1I_1\cos\beta_1 + E_3I_3\cos\beta_3 + E_5I_5\cos\beta_5 + \ldots$$

or

$$P = E_1I_1\cos\,(\theta_1 - \theta_1') + E_3I_3\cos\,(\theta_3 - \theta_3') + E_5I_5\cos\,(\theta_5 - \theta_5') + \ldots \qquad (17\text{–}43)$$

It should be noted that the average power produced by the voltage and current waves of different frequencies is zero. *Each individual harmonic produces its part of the power, independent of the existence of the other harmonics.* For example, the fundamental component of the voltage and the fundamental component of the current will produce average power, but the fundamental component of the voltage and the third harmonic of the current will produce no average power. The true power of non-sinusoidal waves is automatically indicated by an ordinary wattmeter.

Example 17–2.—The voltage applied across the terminals of a circuit is

$$e = 141.4\sin\,\omega t + 70.7\sin\,(3\omega t + 90°) - 14.14\sin\,(5\omega t - 150°)$$

and the current flowing in the circuit is

$$i = 14.14\sin\,(\omega t - 30°) + 7.07\sin\,(5\omega t + 270°)$$

Find the average power delivered to the circuit.

Solution.—Substituting numerical values in equation (17–43) gives

$$P = 100 \times 10 \cos (0^\circ + 30^\circ) + 0 + (-10) \times 5 \cos (-150^\circ - 270^\circ) = 841 \text{ watts}$$

The term $E_3 I_3 \cos (\theta_3 - \theta_3')$ is equal to zero because there is no third harmonic in the current wave and $I_3 = 0$.

17–10. Power Factor When the Voltage and Current Are Non-Sinusoidal Waves.—The general definition of power factor, which holds for both sinusoidal and non-sinusoidal waves, is

$$\text{Power factor} = \frac{\text{True power}}{\text{Effective voltage} \times \text{Effective current}} \tag{17–44}$$

where the true power is given by equation (17–42), the effective voltage by equation (17–33), and the effective current by equation (17–35). That is,

$$\text{Power factor} = \frac{E_0 I_0 + \sum_{n=1}^{n=n} E_n I_n \cos (\theta_n - \theta_n')}{\sqrt{E_0^2 + \sum_{n=1}^{n=n} E_n^2} \times \sqrt{I_0^2 + \sum_{n=1}^{n=n} I_n^2}} \tag{17–45}$$

A harmonic which occurs in the current of a circuit and does not exist in the voltage does not contribute to the average power. It does increase the effective value of the current required to produce the power. Therefore, the power factor of a circuit containing in its current a harmonic which is not present in its voltage cannot be unity. Similarly, the power factor of a circuit containing in its voltage a harmonic which is not present in its current cannot be unity. It will be seen that the conditions for the power factor to be unity are:

$$\left.\begin{aligned} \cos (\theta_1 - \theta_1') = \cos (\theta_2 - \theta_2') = \cos (\theta_3 - \theta_3') = \text{etc.} = 1 \\ \frac{E_0}{I_0} = \frac{E_1}{I_1} = \frac{E_2}{I_2} = \frac{E_3}{I_3} = \text{etc.} = k \end{aligned}\right\} \tag{17–46}$$

That is, unity power factor can occur only when the voltage and current waves are exactly similar in form and have no phase displacement with respect to each other. That the conditions in equation (17–46) are necessary for unity power factor will now be demonstrated.

Since the input power can never be greater than the input volt-amperes, it can be seen from equation (17–45) that to attain unity power factor the numerator must be as large as possible; that is,

$$\cos(\theta_1 - \theta_1') = \cos(\theta_2 - \theta_2') = \cos(\theta_3 - \theta_3') = \text{etc.} = 1$$

When the second condition in equation (17–46) is also satisfied, then equation (17–45) can be written as follows:

$$\text{Power factor} = \frac{E_0 I_0 + E_1 I_1 + E_2 I_2 + \ldots}{\sqrt{E_0^2 + E_1^2 + E_2^2 + \ldots} \times \sqrt{I_0^2 + I_1^2 + I_2^2 + \ldots}}$$

$$= \frac{k[I_0^2 + I_1^2 + I_2^2 + \ldots]}{k\sqrt{I_0^2 + I_1^2 + I_2^2 + \ldots} \times \sqrt{I_0^2 + I_1^2 + I_2^2 + \ldots}} = 1$$

Physically, the first condition in equation (17–46) means that every component in the voltage wave must be in phase with the corresponding component in the current wave to give the maximum possible power. The second condition states that the circuit should be purely resistive since k, which has the dimension of impedance, must be constant for the different ratios of voltage to current and, hence, must be independent of frequency.

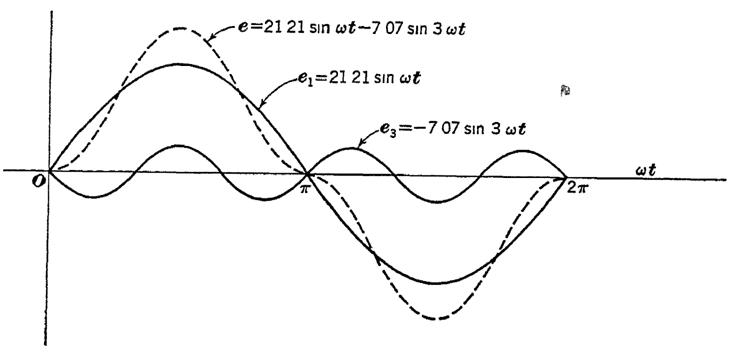


FIG. 17–10

17–11. Definitions.—(*a*) *Crest, or Peak, Factor:* The crest factor or peak factor of a wave is the ratio of its maximum value to its effective value. The crest factor of a sine wave is $\sqrt{2}$. A knowledge of the crest factor is of importance in testing insulation, since the stress to which an insulation is subjected by a given impressed voltage depends on the maximum value and not on its effective value.

(*b*) *Equivalent Sine Wave:* When a wave form of voltage or current does not differ greatly from a sinusoid, it is sufficiently accurate to replace the non-sinusoidal wave by an equivalent sine wave. This equivalent sine wave has the same *effective* value as the actual wave it replaces and has the same frequency as the fundamental wave or the wave of the lowest order. An example of the use of an equivalent sine wave is in the case of the no-load current of a tranfsormer. The non-sinusoidal no-load current is replaced by an equivalent sine wave in order to represent the no-load current by a vector in the vector diagram.

Example 17–3.—Find the maximum value E_m of the equivalent sine wave and the crest or peak factor of $e = 21.21 \sin \omega t - 7.07 \sin 3\omega t$

Solution.—The graph of the wave is shown in Fig. 17–10. It is obvious from the figure that the maximum value of the complex wave is

$$E'_m = 21.21 + 7.07 = 15\sqrt{2} + 5\sqrt{2} = 28.28 \text{ volts}$$

The effective value is

$$E = \sqrt{15^2 + 5^2} = 15.8 \text{ volts}$$

The equivalent sine wave has a maximum value of

$$E_m = \sqrt{2} \times 15.8 = 22.35 \text{ volts}$$

Also, the crest factor of the complex wave is

$$\text{Crest factor} = \frac{E'_m}{E} = \frac{28.28}{15.8} = 1.79$$

17–12. Analysis of Waves by Fourier's Series.—It is often necessary to analyze a recurring wave, obtained by an oscillograph, a Rosa Curve Tracer, or other means, into the components of its Fourier Series in order to determine the harmonics present and their magnitudes. Equation (17–1) is

$$f(\alpha) = A_1 \sin \alpha + A_2 \sin 2\alpha + \dots + A_n \sin n\alpha$$
$$+ B_0 + B_1 \cos \alpha + B_2 \cos 2\alpha + \dots + B_n \cos n\alpha \qquad (17\text{–}1)$$

This may be written also as shown in equation (17–5), which is

$$f(\alpha) = B_0 + C_1 \sin (\alpha + \theta_1) + C_2 \sin (2\alpha + \theta_2) + C_3 \sin (3\alpha + \theta_3)$$
$$+ C_4 \sin (4\alpha + \theta_4) + \dots + C_n \sin (n\alpha + \theta_n) \qquad (17\text{–}5)$$

where $C_n = \sqrt{A_n^2 + B_n^2}$ and $\theta_n = \tan^{-1} \frac{B_n}{A_n}$.

Equation (17–5) is a convenient form to have. But, in order to determine the values of C_n and θ_n, it is necessary to obtain the values of A_n and B_n. Two methods of accomplishing this, the better one depending on the individual problem, will be given here. They are the analytical method and the graphical method.

17–13. Analytical Method.—If portions of the wave can be represented by simple mathematical equations, as is possible with rectified sinusoidal waves, the analytical method has preference over the graphical method.

(*a*) *Determination of the Constant Term:* To obtain the value of the constant term B_0, multiply equation (17–1) by $d\alpha$ and integrate between the limits 0 and 2π. Thus,

$$\int_0^{2\pi} f(\alpha)\, d\alpha = \int_0^{2\pi} B_0\, d\alpha + A_1\int_0^{2\pi}\sin\alpha\, d\alpha + A_2\int_0^{2\pi}\sin 2\alpha\, d\alpha + \ldots$$
$$+ A_n\int_0^{2\pi}\sin n\alpha\, d\alpha + B_1\int_0^{2\pi}\cos\alpha\, d\alpha$$
$$+ B_2\int_0^{2\pi}\cos 2\alpha\, d\alpha + \ldots + B_n\int_0^{2\pi}\cos n\alpha\, d\alpha \qquad (17\text{–}47)$$

All terms in the right-hand side of the equation, except the first, become zero, leaving

$$\int_0^{2\pi} f(\alpha)\, d\alpha = 2\pi B_0$$

or

$$B_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\, d\alpha \qquad (17\text{–}48)$$

Thus, B_0 is the average height of the wave taken over one complete cycle, since $\int_0^{2\pi} f(\alpha)\, d\alpha$ is the net area under the distorted curve $f(\alpha)$ from 0 to 2π and 2π is the base. The value of B_0 would be zero if the net area were zero, irrespective of the shape of the complex wave.

(*b*) *Determination of the Coefficient of Any Sine Term:* To find the value of the coefficient A_n of the term $A_n \sin n\alpha$, multiply both sides of equation (17–1) by $\sin n\alpha\, d\alpha$ and integrate between the limits 0 and 2π. Thus,

$$\int_0^{2\pi} f(\alpha)\sin n\alpha\,d\alpha = \int_0^{2\pi} B_0 \sin n\alpha\,d\alpha + \int_0^{2\pi} A_1 \sin\alpha \sin n\alpha\,d\alpha$$
$$+\int_0^{2\pi} A_2 \sin 2\alpha \sin n\alpha\,d\alpha + \dots + \int_0^{2\pi} A_n \sin^2 n\alpha\,d\alpha$$
$$+\int_0^{2\pi} B_1 \cos\alpha \sin n\alpha\,d\alpha + \dots + \int_0^{2\pi} B_n \cos n\alpha \sin n\alpha\,d\alpha \quad \textbf{(17–49)}$$

After terms in the right-hand side of the equation are integrated and the limits are introduced, equation (17–49) becomes

$$\int_0^{2\pi} f(\alpha)\sin n\alpha\,d\alpha = \int_0^{2\pi} A_n \sin^2 n\alpha\,d\alpha = \frac{2\pi A_n}{2}$$

from which

$$A_n = 2\left[\frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\sin n\alpha\,d\alpha\right] \quad \textbf{(17–50)}$$

Thus, the value of A_n is twice the average height of the curve $F_1(\alpha) = f(\alpha)\sin n\alpha$ plotted with values of α between the limits 0 and 2π as abscissas.

(c) *Determination of the Coefficient of Any Cosine Term:* To determine the value of the coefficient B_n of the term $B_n \cos n\alpha$, multiply both sides of equation (17–1) by $\cos n\alpha\,d\alpha$ and integrate between the limits of 0 and 2π. Thus,

$$\int_0^{2\pi} f(\alpha)\cos n\alpha\,d\alpha = \int_0^{2\pi} B_0 \cos n\alpha\,d\alpha + \int_0^{2\pi} A_1 \sin\alpha \cos n\alpha\,d\alpha$$
$$+\int_0^{2\pi} A_2 \sin 2\alpha \cos n\alpha\,d\alpha + \dots + \int_0^{2\pi} A_n \sin n\alpha \cos n\alpha\,d\alpha$$
$$+\int_0^{2\pi} B_1 \cos\alpha \cos n\alpha\,d\alpha + \dots + \int_0^{2\pi} B_n \cos^2 n\alpha\,d\alpha \quad \textbf{(17–51)}$$

After integration of the terms in the right-hand side of the equation, it will be found that equation (17–51) reduces to

$$\int_0^{2\pi} f(\alpha)\cos n\alpha\,d\alpha = \int_0^{2\pi} B_n \cos^2 n\alpha\,d\alpha = \frac{2\pi B_n}{2}$$

from which

$$B_n = 2\left[\frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\cos n\alpha\,d\alpha\right] \quad \textbf{(17–52)}$$

Equation (17–52) states that B_n is twice the average height of the curve $F_2(\alpha) = f(\alpha)\cos n\alpha$ when plotted with values of α between the limits 0 and 2π as abscissas.

Example 17–4.—A rectangular wave is shown in Fig. 17–11. Since portions of the curve can be expressed by mathematical expressions, find its equation in terms of harmonics, neglecting those beyond the second, by the analytical method.

Solution.—(*a*) The constant term is

$$B_0=\frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\,d\alpha=\frac{1}{2\pi}\left\{\int_0^{\theta} f_1(\alpha)\,d\alpha+\int_{\theta}^{2\pi} f_2(\alpha)\,d\alpha\right\}$$

$$=\frac{1}{2\pi}\left\{\int_0^{\frac{3}{2}\pi} 20\,d\alpha+\int_{\frac{3}{2}\pi}^{2\pi}(-10)\,d\alpha\right\}=\frac{1}{2\pi}[30\pi-5\pi]=12.5$$

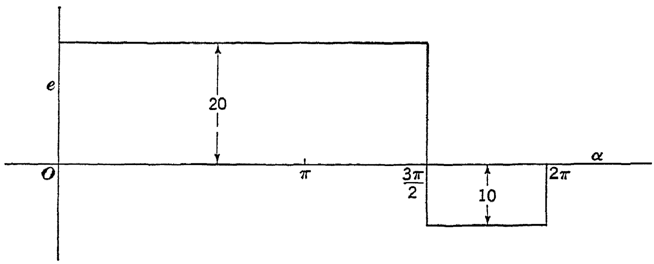


FIG. 17–11

In other words, the value of B_0 or the average ordinate is determined by dividing the net area by 2π. Thus,

$$B_0=\frac{1}{2\pi}\left[\left(20\times\frac{3\pi}{2}\right)-\left(10\times\frac{\pi}{2}\right)\right]=12.5$$

(*b*) The values of A_1, B_1, C_1, and θ_1 are found as follows:

$$A_1=2\left[\frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\sin\alpha\,d\alpha\right]=\frac{1}{\pi}\left\{\int_0^{\theta} f_1(\alpha)\sin\alpha\,d\alpha+\int_{\theta}^{2\pi} f_2(\alpha)\sin\alpha\,d\alpha\right\}$$

$$=\frac{1}{\pi}\left\{\int_0^{\frac{3}{2}\pi} 20\sin\alpha\,d\alpha+\int_{\frac{3}{2}\pi}^{2\pi}(-10)\sin\alpha\,d\alpha\right\}=\frac{1}{\pi}[20+10]=9.53$$

$$B_1=2\left[\frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\cos\alpha\,d\alpha\right]=\frac{1}{\pi}\left\{\int_0^{\frac{3}{2}\pi} f_1(\alpha)\cos\alpha\,d\alpha+\int_{\frac{3}{2}\pi}^{2\pi} f_2(\alpha)\cos\alpha\,d\alpha\right\}$$

$$=\frac{1}{\pi}\left\{\int_0^{\frac{3}{2}\pi} 20\cos\alpha\,d\alpha+\int_{\frac{3}{2}\pi}^{2\pi}(-10)\cos\alpha\,d\alpha\right\}=\frac{1}{\pi}[-20-10]=-9.53$$

$$C_1=\sqrt{A_1^2+B_1^2}=\sqrt{9.53^2+(-9.53)^2}=13.47$$

$$\theta_1=\tan^{-1}\frac{B_1}{A_1}=\tan^{-1}\frac{-9.53}{9.53}=-45°$$

The fundamental term of the series is

$$C_1 \sin(\alpha+\theta_1) = 13.47 \sin(\alpha - 45°)$$

(c) The values of A_2, B_2, C_2, and θ_2 are found in the following manner:

$$A_2 = 2\left[\frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\sin 2\alpha\, d\alpha\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\frac{3}{2}\pi} f_1(\alpha)\sin 2\alpha\, d\alpha + \int_{\frac{3}{2}\pi}^{2\pi} f_2(\alpha)\sin 2\alpha\, d\alpha\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\frac{3}{2}\pi} 20\sin 2\alpha\, d\alpha + \int_{\frac{3}{2}\pi}^{2\pi} (-10)\sin 2\alpha\, d\alpha\right\} = 9.53$$

$$B_2 = 2\left[\frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\cos 2\alpha\, d\alpha\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\frac{3}{2}\pi} f_1(\alpha)\cos 2\alpha\, d\alpha + \int_{\frac{3}{2}\pi}^{2\pi} f_2(\alpha)\cos 2\alpha\, d\alpha\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\frac{3}{2}\pi} 20\cos 2\alpha\, d\alpha + \int_{\frac{3}{2}\pi}^{2\pi} (-10)\cos 2\alpha\, d\alpha\right\} = 0$$

$$C_2 = \sqrt{A_2^2 + B_2^2} = A_2 = 9.53$$

$$\theta_2 = \tan^{-1}\frac{B_2}{A_2} = \tan^{-1}\frac{0}{9.53} = 0$$

The second harmonic is

$$C_2 \sin(2\alpha+\theta_2) = 9.53 \sin 2\alpha$$

Thus, the equation for the wave, analyzed for harmonics including the second, is

$$e = 12.5 + 13.47\sin(\alpha - 45°) + 9.53\sin 2\alpha + \ldots$$

The student should plot this wave and compare the result with the original wave in Fig. 17–11.

17–14. Effect of Symmetry on the Fourier Expression.—In making the analysis of a non-sinusoidal wave, it is helpful to employ the principles given in this article for the following reasons:

(a) To avoid spending unnecessary time and labor in finding the amplitudes of harmonics that are non-existent.

(b) To know that no important harmonics have been omitted.

The few rules that follow are useful in determining which terms vanish and which terms exist in a series because of the symmetry of the wave and the choice of the origin.

(1) If a wave has equal area above and below the horizontal axis in a complete cycle, $B_0 = 0$ (Figs. 17–4, 17–5, and 17–6).

(2) If a wave is symmetrical so that $f(\alpha) = -f(\alpha+\pi)$, B_0 and all even harmonics are zero (Figs. 17–8 and 17–9). That is, the wave can be represented by a series of odd harmonics only. Integration for all the coefficients of the harmonics may be made for 180° instead of 360°; that is, equations (17–50) and (17–52) become:

$$A_n = 2\left[\frac{1}{\pi}\int_0^{\pi} f(\alpha)\sin n\alpha\, d\alpha\right] \tag{17–50a}$$

$$B_n = 2\left[\frac{1}{\pi}\int_0^{\pi} f(\alpha)\cos n\alpha\, d\alpha\right] \tag{17–52a}$$

If a wave is symmetrical about a horizontal line, such as $B_0 = +5$ in Fig. 17–19 of Problem 17–30, there will be no even harmonics.

(3) If a wave is symmetrical about a point on the horizontal axis, so that with that point as origin $f(\alpha) = -f(-\alpha)$, then B_0 and all the cosine terms are zero (Fig. 17–4). That is, the wave may be represented by a series of sine terms only. This can be shown in the following manner: Equation (17–2) gives

$$f(\alpha) = B_0 + \sum_{n=1}^{n=\infty} A_n \sin n\alpha + \sum_{n=1}^{n=\infty} B_n \cos n\alpha$$

and

$$-f(-\alpha) = -\left[B_0 + \sum_{n=1}^{n=\infty} A_n \sin(-n\alpha) + \sum_{n=1}^{n=\infty} B_n \cos(-n\alpha)\right]$$

$$= -B_0 + \sum_{n=1}^{n=\infty} A_n \sin n\alpha - \sum_{n=1}^{n=\infty} B_n \cos n\alpha$$

Setting $f(\alpha)$ identically equal to $-f(-\alpha)$, we see that the sine terms cancel each other. The result is

$$2B_0 + 2\sum_{n=1}^{n=\infty} B_n \cos n\alpha \equiv 0$$

This is true when $B_0 = 0$ and all values of B_n are equal to zero.

(4) If a wave is symmetrical about a vertical line and the origin is so chosen that this vertical line passes through it, making $f(\alpha)$ equal to $f(-\alpha)$, then all the sine terms are zero. Choosing the origin at $\frac{\pi}{2}$ in Fig. 17–2 will give this symmetry. Of course, for such cases as that in Fig. 17–2 where the wave also has "half-wave" symmetry, then B_0 is also zero. This can be demonstrated quite easily in a manner similar to the one given for Rule (3). For $f(-\alpha)$, equation (17–2) becomes

$$f(-\alpha) = B_0 + \sum_{n=1}^{n=\infty} A_n \sin(-n\alpha) + \sum_{n=1}^{n=\infty} B_n \cos(-n\alpha)$$

$$= B_0 - \sum_{n=1}^{n=\infty} A_n \sin n\alpha + \sum_{n=1}^{n=\infty} B_n \cos n\alpha$$

Setting $f(\alpha)$ identically equal to $f(-\alpha)$, we see that the B_0 terms and the cosine terms cancel out, the result being

$$2\sum_{n=1}^{n=\infty} A_n \sin n\alpha \equiv 0$$

which is true when all values of A_n are equal to zero.

(5) If a wave has "half-wave" symmetry because $f(\alpha) = -f(\alpha+\pi)$ and also has "quarter-wave" symmetry because $f(\alpha) = f(\pi-\alpha)$, then the wave can be described by either a series of odd cosine terms only or a series of odd sine terms only, the choice depending on the position of the origin. The wave in Fig. 17–7 has both "half-wave" symmetry and "quarter-wave" symmetry. If the origin is chosen as shown in Fig. 17–7, the wave may be represented by a series of odd sine terms only. If the origin is chosen at the point $\frac{\pi}{2}$ in Fig. 17–7, the expression for the wave is a series of odd cosine terms only.

17–15. Graphical Method.*—The values of the constants B_0, A_n, and B_n may be found by means of a graphical method. This method is useful when portions of the wave cannot be expressed by mathematical equations.

* See the Fischer-Hinnen Method of analyzing a periodic wave in books on alternating-current circuits, such as "Principles of Alternating Currents" by R. R. Lawrence.

(a) *Asymmetrical Wave:* Let an asymmetrical wave represented by $y = f(\alpha)$ be plotted as shown in Fig. 17–12. Divide the range from 0 to 2π along the α-axis into m equal parts and erect an ordinate at the middle of each division, as y_k. If y_2, which is above the α-axis, is assumed to be positive, then y_{11}, which is below the α-axis, is negative.

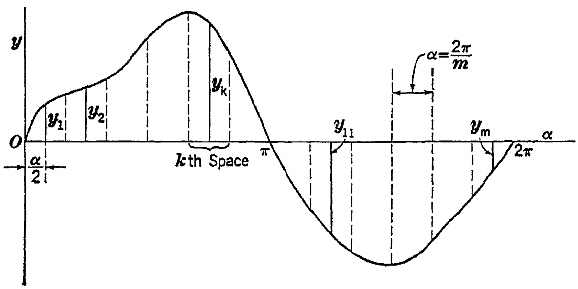


FIG. 17–12

From equation (17–48), the average ordinate is

$$B_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\, d\alpha \tag{17–48}$$

This may be expressed approximately as follows:

$$B_0 = \frac{1}{m}(y_1 + y_2 + \ldots + y_k + \ldots + y_{11} + \ldots + y_m)$$

or

$$B_0 = \frac{1}{m}\sum_{k=1}^{k=m} y_k \tag{17–53}$$

Thus, when solving for the value of B_0, the mechanical work of integration consists of tabulating the value of the ordinate at the middle of each of the m divisions with the proper plus or minus sign before each ordinate, adding these ordinates algebraically, and dividing the algebraic sum by the number of ordinates, or m.

Similarly, from equation (17–50),

$$A_1 = 2\left[\frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\sin\alpha\, d\alpha\right] \tag{17–54}$$

This value may be interpreted as twice the average height of the

$f(\alpha)$ sin α are chosen at the same points as the old ordinates in Fig. 17–12, so that the same subscripts can be used for the new ordinates, then:

$$y_1' = y_1 \sin\left(\frac{1}{2}\times\frac{2\pi}{m}\right)$$

$$y_2' = y_2 \sin\left(\frac{3}{2}\times\frac{2\pi}{m}\right)$$

$$y_m' = y_m \sin\left(\frac{2m-1}{2}\times\frac{2\pi}{m}\right)$$

The coefficient A_1 may now be written approximately as follows:

$$A_1 = 2\left[\frac{1}{m}\left\{y_1 \sin\left(\frac{1}{2}\times\frac{2\pi}{m}\right) + y_2 \sin\left(\frac{3}{2}\times\frac{2\pi}{m}\right) + y_3 \sin\left(\frac{5}{2}\times\frac{2\pi}{m}\right) + \dots + y_m \sin\left(\frac{2m-1}{2}\times\frac{2\pi}{m}\right)\right\}\right]$$

or

$$A_1 = 2\left[\frac{1}{m}\sum_{k=1}^{k=m} y_k \sin\left(k-\frac{1}{2}\right)\frac{2\pi}{m}\right] \qquad (17\text{–}55)$$

That is, A_1 is twice the average height of the ordinates which are obtained by multiplying y_1 by the sine of the angle measured from the origin to the ordinate y_1, multiplying y_2 by the sine of the angle measured from the origin to the ordinate y_2, etc.

For the general case, equation (17–50) gives

$$A_n = 2\left[\frac{1}{2\pi}\int_0^{2\pi} f(\alpha) \sin n\alpha \, d\alpha\right]$$

The coefficient A_n is given approximately by the expression

$$A_n = 2\left[\frac{1}{m}\sum_{k=1}^{k=m} y_k \sin\left(k-\frac{1}{2}\right)n\frac{2\pi}{m}\right] \qquad (17\text{–}56)$$

or

$$A_n = \frac{2}{m}\left\{y_1 \sin\left(\frac{1}{2}n\frac{2\pi}{m}\right) + y_2 \sin\left(\frac{3}{2}n\frac{2\pi}{m}\right) + y_3 \sin\left(\frac{5}{2}n\frac{2\pi}{m}\right) + \dots + y_m \sin\left(\frac{2m-1}{2}n\frac{2\pi}{m}\right)\right\} \qquad (17\text{–}56a)$$

By following a similar line of reasoning, we can obtain:

$$B_n = 2\left[\frac{1}{m}\sum_{k=1}^{k=m} y_k \cos\left(k-\frac{1}{2}\right)n\frac{2\pi}{m}\right] \qquad (17\text{–}57)$$

or $$B_n=\frac{2}{m}\left\{y_1 \cos\left(\frac{1}{2}n\frac{2\pi}{m}\right)+y_2 \cos\left(\frac{3}{2}n\frac{2\pi}{m}\right)\right.$$

$$\left.+y_3 \cos\left(\frac{5}{2}n\frac{2\pi}{m}\right)+\ \ldots\ +y_m \cos\left(\frac{2m-1}{2}n\frac{2\pi}{m}\right)\right\} \qquad (17\text{–}57a)$$

If the equation of the wave is desired with sine terms alone, as in equation (17–5), then compute C_n and θ_n by the relations

$$C_n=\sqrt{A_n^2+B_n^2}$$

$$\theta_n=\tan^{-1}\frac{B_n}{A_n}$$

(*b*) *Symmetrical Wave:* If the wave is symmetrical, there will be no B_0 term and no even harmonics, and all "mechanical" integrations may be made for 180° only. For ordinary cases, it is sufficiently accurate to take 10-degree intervals, or to take $m=36$ for the limits 0 and 2π; and this procedure will be assumed to simplify further the work which follows. Thus, for the limits 0 and π, there will be 18 mid-ordinates and 18 intervals, and each interval will be the same as for the complete cycle, as $\frac{\pi}{18}=\frac{2\pi}{36}$. Then the coefficients A_n and B_n may be determined from equations (17–56) and (17–57) by letting $m=18$. Thus,

$$A_1=\frac{2}{18}\sum_{k=1}^{k=18} y_k \sin\left(k-\frac{1}{2}\right)\frac{\pi}{18} \qquad (17\text{–}58)$$

or $$A_1=\frac{2}{18}\left(y_1 \sin 5°+y_2 \sin 15°+y_3 \sin 25°+\ \ldots\right) \qquad (17\text{–}58a)$$

$$B_1=\frac{2}{18}\sum_{k=1}^{k=18} y_k \cos\left(k-\frac{1}{2}\right)\frac{\pi}{18} \qquad (17\text{–}59)$$

or $$B_1=\frac{2}{18}\left(y_1 \cos 5°+y_2 \cos 15°+y_3 \cos 25°+\ \ldots\right) \qquad (17\text{–}59a)$$

$$A_3=\frac{2}{18}\sum_{k=1}^{k=18} y_k \sin\left(k-\frac{1}{2}\right)3\times\frac{\pi}{18} \qquad (17\text{–}60)$$

or $$A_3=\frac{2}{18}\left(y_1 \sin 15°+y_2 \sin 45°+y_3 \sin 75°+\ \ldots\right) \qquad (17\text{–}60a)$$

$$B_3 = \frac{2}{18} \sum_{k=1}^{k=18} y_k \cos\left(k - \frac{1}{2}\right) 3 \times \frac{\pi}{18} \qquad (17\text{–}61)$$

or $$B_3 = \frac{2}{18} \left(y_1 \cos 15° + y_2 \cos 45° + y_3 \cos 75° + \ldots\right) \qquad (17\text{–}61a)$$

$$A_n = \frac{2}{18} \sum_{k=1}^{k=18} y_k \sin\left(k - \frac{1}{2}\right) n \frac{\pi}{18} \qquad (17\text{–}62)$$

$$B_n = \frac{2}{18} \sum_{k=1}^{k=18} y_k \cos\left(k - \frac{1}{2}\right) n \frac{\pi}{18} \qquad (17\text{–}63)$$

It should be noted that the precision of the constants A_n and B_n and the angle θ_n is increased by making m large; but, in so doing, the time for calculation is very much increased. The determination of the coefficients, such as A_1 and B_1, does not depend on how many harmonics are included. The graphical solution will give, in the form of a Fourier series, a function that will agree fairly well with the original plotted curve when m and n are sufficiently large numbers.

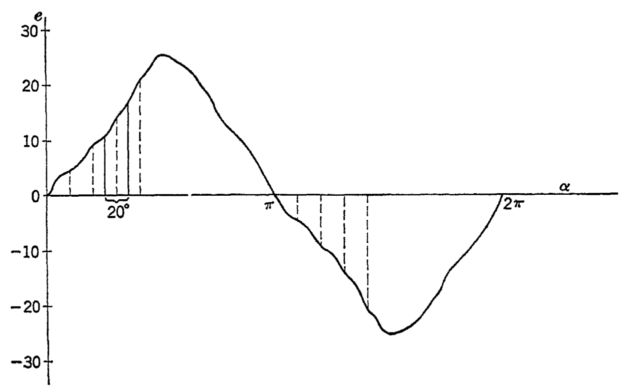


FIG. 17–13

Example 17–5.—A commercial wave, shown in Fig. 17–13, is assumed to be symmetrical. It is required to find its equation in terms of the first and third harmonics. Since the purpose of this example is to illustrate the theory just given, we shall use 20-degree intervals for simplicity instead of 10-degree intervals as suggested for better accuracy; that is, we will let $m = 9$ for the limits 0 and π. The first ordinate is taken at 10°, which is the middle of the first 20-degree interval.

FUNDAMENTAL

α degrees	e	$\sin\alpha$	$e \sin\alpha$ +	$e \sin\alpha$ −	$\cos\alpha$	$e \cos\alpha$ +	$e \cos\alpha$ −
10	2.0	0.1736	0.3472		0.9848	1.9696	
30	4.9	0.5000	2.4500		0.8660	4 2434	
50	10.0	0.7660	7.6600		0.6428	6 4280	
70	22.5	0.9397	21.1432		0.3420	7.6950	
90	30.0	1.0000	30.0000		0.0000	0.0000	
110	28.0	0.9397	26.3116		−0 3420		9.5760
130	25.0	0.7660	19.1500		−0.6428		16.0700
150	20.0	0.5000	10.0000		−0.8660		17.3200
170	7.0	0.1736	1.2152		−0 9848		6.8936
			118.2772			20.3360	49.8596

$$A_1 = \frac{2}{9}\sum_{k=1}^{k=9} e_k \sin\left(k - \frac{1}{2}\right)\frac{\pi}{9} = 2 \times \frac{118.2772}{9} = 26.28$$

$$B_1 = \frac{2}{9}\sum_{k=1}^{k=9} e_k \cos\left(k - \frac{1}{2}\right)\frac{\pi}{9} = \frac{2(20.3360 - 49.8596)}{9} = -6.54$$

$$C_1 = \sqrt{A_1^2 + B_1^2} = \sqrt{26.28^2 + (-6.54)^2} = 27.08$$

$$\theta_1 = \tan^{-1}\frac{B_1}{A_1} = \tan^{-1}\frac{-6.54}{26.28} = -14°$$

THIRD HARMONIC

α degrees	e	3α degrees	$\sin 3\alpha$	$e \sin 3\alpha$ +	$e \sin 3\alpha$ −	$\cos 3\alpha$	$e \cos 3\alpha$ +	$e \cos 3\alpha$ −
10	2.0	30	0.5000	1.0		0.8660	1.73	
30	4.9	90	1.0000	4.9		0.0000	0.00	
50	10.0	150	0.5000	5.0		−0.8660		8.66
70	22.5	210	−0.5000		11.25	−0.8660		19.50
90	30.0	270	−1.0000		30.00	0.0000	0.00	
110	28.0	330	−0.5000		14.00	0.8660	24.23	
130	25.0	390	0.5000	12.5		0.8660	21.65	
150	20.0	450	1.0000	20.0		0.0000	0.00	
170	7.0	510	0.5000	3.5		−0.8660		6.07
				46.9	55.25		47.61	34.23

$$A_3 = \frac{2}{9}\sum_{k=1}^{k=9} e_k \sin\left(k - \frac{1}{2}\right)3 \times \frac{\pi}{9} = \frac{2(46.9 - 55.25)}{9} = -1.86$$

$$B_3 = \frac{2}{9}\sum_{k=1}^{k=9} e_k \cos\left(k - \frac{1}{2}\right)3 \times \frac{\pi}{9} = \frac{2(47.61 - 34.23)}{9} = 2.98$$

$$C_3 = \sqrt{A_3^2 + B_3^2} = \sqrt{(-1.86)^2 + 2.98^2} = 3.51$$

$$\theta_3 = \tan^{-1}\frac{B_3}{A_3} = \tan^{-1}\frac{2.98}{-1.86} = 122^\circ$$

The equation of the wave is:

$$\begin{aligned} e &= C_1 \sin(\alpha + \theta_1) + C_3 \sin(3\alpha + \theta_3) + \ldots \\ &= 27.08 \sin(\alpha - 14^\circ) + 3.51 \sin(3\alpha + 122^\circ) + \ldots \\ &= 27.08 \sin(\alpha - 14^\circ) - 3.51 \sin(3\alpha - 58^\circ) + \ldots \end{aligned}$$

The student should plot this result and the original wave for comparison.

PROBLEMS

17–1. The expression for an emf wave is $e = K + 10 \sin 377t$. Draw to scale all component waves and the resultant wave with: (*a*) $K = +5$ and (*b*) $K = -5$. Do the waves have "half-wave" symmetry?

17–2. Draw to scale all component waves and the resultant wave for the emf wave $e = 10 + 30 \sin(157t + \alpha)$ when: (*a*) $\alpha = 30^\circ$ and (*b*) $\alpha = -60^\circ$.

17–3. Draw to scale all component waves and the resultant wave for the emf wave $e = 200 \sin 377t + 50 \sin 754t$. Is the wave symmetrical?

17–4. Draw to scale all component waves and the resultant wave for the current $i = 50 \sin 157t - 15 \sin 314t$. Is the wave symmetrical?

17–5. Draw to scale all component waves and the resultant wave for the current $i = 50 \sin 377t + 20 \sin 754t$.

17–6. Draw to scale all component waves and the resultant wave for the current $i = 100 \sin(157t + \alpha) - 30 \sin(314t + \beta)$ when: (*a*) $\alpha = 30^\circ$ and $\beta = 60^\circ$; (*b*) $\alpha = 30^\circ$ and $\beta = -120^\circ$; (*c*) $\alpha = -60^\circ$ and $\beta = 90^\circ$; and (*d*) $\alpha = -60^\circ$ and $\beta = -40^\circ$.

17–7. Draw to scale all component waves and the resultant wave for the current $i = 75 \sin(377t - 30^\circ) + 25 \sin(1131t + 60^\circ)$.

17–8. Plot the emf wave given by the equation $e = 100 \sin(377t + \alpha) + 50 \sin(1131t - \beta)$ when: (*a*) $\alpha = 20^\circ$ and $\beta = 75^\circ$; (*b*) $\alpha = -45^\circ$ and $\beta = 120^\circ$; (*c*) $\alpha = 10^\circ$ and $\beta = -90^\circ$.

17–9. Plot the current wave given by the equation $i = 30 + 50 \sin(157t - \alpha) - 20 \sin(314t + \beta)$ when: (*a*) $\alpha = 30^\circ$ and $\beta = 60^\circ$; (*b*) $\alpha = 75^\circ$ and $\beta = -50^\circ$; (*c*) $\alpha = -15^\circ$ and $\beta = 240^\circ$.

17–10. A voltage wave is given by the expression $e = 10 + 20 \sin \omega t$. If the origin or reference point is changed so that the voltage wave starts at zero and is increasing positively, find the new expression for e.

17–11. Find the average value of the following voltage wave over a complete cycle:

$$e = 10 + 100\sqrt{2} \sin(377t + 30^\circ) - 50\sqrt{2} \sin 1131t$$

17–12. Two sine terms of like frequency, namely, $10 \sin(157t + 60^\circ)$ and $5 \sin(157t + 40^\circ)$, form a product $G(\alpha) = 10 \sin(157t + 60^\circ) \times 5 \sin(157t + 40^\circ)$.

Show, by integration, that $\frac{1}{2\pi}\int_0^{2\pi} G(\alpha)\, d\alpha = \frac{10 \times 5}{2} \cos(60^\circ - 40^\circ)$.

17–13. Two sine terms of unlike frequencies, namely, 20 sin $(157t+20°)$ and 10 sin $(314t+60°)$, comprise the product $H(\alpha)=20 \sin (157t+20°)\times 10 \sin (314t+60°)$. Show, by integration, that $\frac{1}{2\pi}\int_0^{2\pi} H(\alpha)\, d\alpha=0$.

17–14. The current in a series circuit is $i=10+20 \sin 377t-5 \sin (754t+30°)$. (*a*) What will an a-c ammeter placed in the circuit read? (*b*) What will a d-c ammeter read if such a meter reads the average value?

17–15. A carbon-grain telephone transmitter is connected in series with a 50-volt battery and a load resistance of 200 ohms. The transmitter has an average resistance of 200 ohms which varies sinusoidally between the limits 180 and 220 ohms when a tuning fork is sounded in front of the transmitter. Find the effective value of the current flowing.

17–16. The no-load current wave of a transformer, as shown by an oscillograph, is symmetrical. It contains a fundamental and third and fifth harmonics that are, respectively, 30 and 8 per cent of the fundamental. An ammeter in the primary reads 5 amp. What are the effective values of the fundamental and each of the harmonic components?

17–17. The emf and current waves for a given circuit are: $e=200 \sin (377t+10°)+50 \sin (1131t+60°)$ and $i=20 \sin (377t+5°)+6 \sin (1131t+10°)$. Calculate the average power.

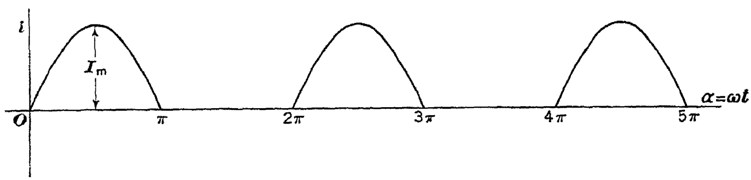


FIG. 17–14

17–18. Show that in Problem 17–17 the fundamental component of the emf and the third harmonic component of the current contribute no average power. Also, show that the third harmonic component of the emf and the fundamental of the current contribute no average power.

17–19. In a circuit the waves of emf and current are: $e=100 \sin (314t-20°)-50 \sin (942t+90°)$ and $i=25 \sin (314t-40°)+20 \sin (628t+10°)-10 \sin (942t+15°)$ Find: (*a*) the effective value of the voltage; (*b*) the effective value of the current; (*c*) the power taken by the circuit; (*d*) the power factor of the circuit.

17–20. The current flowing in a non-linear circuit is $i=100 \sin (377t+50°)+20 \sin (1131t-30°)$. Calculate: (*a*) the effective value; (*b*) the maximum value of the equivalent sine wave; and (*c*) the crest factor.

17–21. The waves of applied emf and current for a series circuit are: $e=100\sqrt{2} \sin \alpha+50\sqrt{2} \sin (3\alpha+30°)$ and $i=50\sqrt{2} \sin \alpha+25\sqrt{2} \sin (3\alpha+30°)$. Calculate the power factor of the circuit.

17–22. A current flowing in a series circuit, which consists essentially of a fundamental and a third harmonic, is $i=I_{m1} \sin 377t-I_{m3} \sin 1131t$. Its maximum or peak value is 75 amp and an ammeter connected in the circuit reads 50 amp. What are the values of I_{m1} and I_{m3}?

17–23. What is the equation of the current from a half-wave rectifier which is shown in Fig. 17–14? Plot the combined curve of the first four terms; and, on the same graph sheet, compare this curve with the original curve.

17–24. Find the equation of the current wave in Fig. 17–14 when the origin is chosen at the maximum value of the wave so that $f(\alpha) = f(-\alpha)$. Compare this expression with that of Problem 17–23.

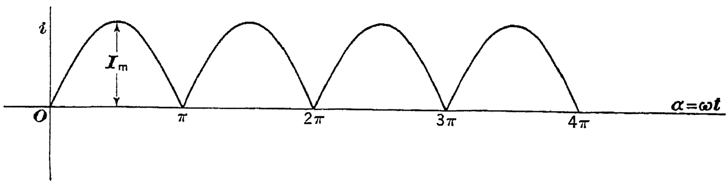


FIG. 17–15

17–25. What is the equation of the current from a full-wave rectifier which is shown in Fig. 17–15?

17–26. Repeat Problem 17–25 with the origin chosen at the maximum value of the wave so that $f(\alpha) = f(-\alpha)$.

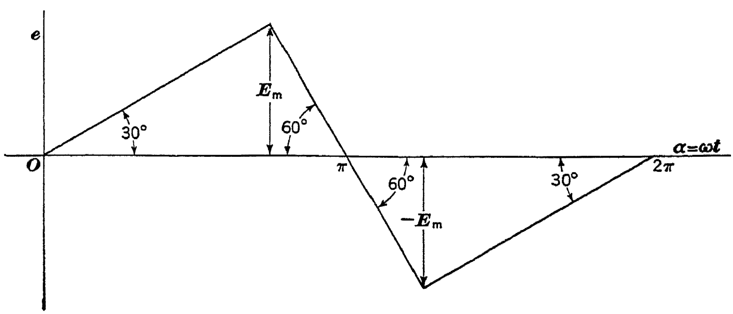


FIG. 17–16

17–27. What is the equation of the triangular wave of Fig. 17–16?

17–28. The no-load current wave of a transformer, as recorded by a Rosa Curve Tracer, is symmetrical. The values at the mid-points of the 18 divisions of one alternation are as shown in the accompanying tabulation.

Angle	*i*	*Angle*	*i*	*Angle*	*i*
5°	0.08	65°	0.79	125°	5.62
15°	0.18	75°	1.45	135°	4.14
25°	0.20	85°	2.72	145°	2.65
35°	0.21	95°	4.30	155°	1.28
45°	0.28	105°	5.65	165°	0.50
55°	0.40	115°	6.20	175°	0.13

What is the equation of the current wave? What is its effective value?

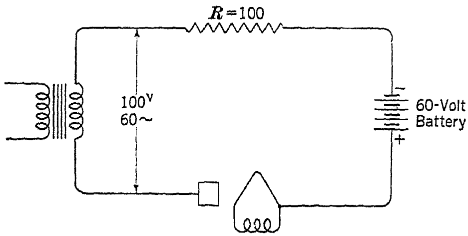


FIG. 17–17

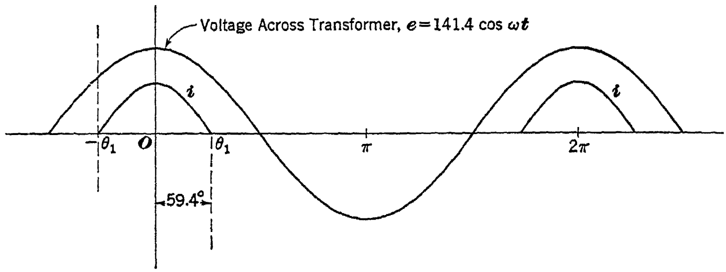


FIG. 17–18

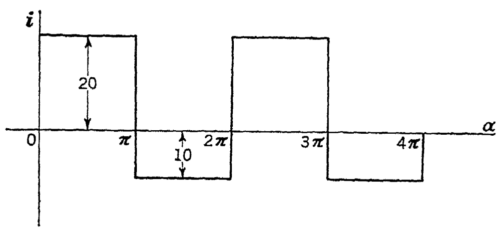


FIG. 17–19

17–29. A circuit containing a rectifier tube is shown in Fig. 17–17. The current flowing in the circuit, as shown in Fig. 17–18, is as follows:

$$\text{For } -\theta_1 < \omega t < \theta_1 \ldots i = \frac{141.4 \cos \omega t - 72}{100}$$

$$\text{For } \theta_1 < \omega t < (2\pi - \theta_1) \ldots i = 0$$

$$\theta_1 = \cos^{-1} \frac{72}{141.4} = 59.4°$$

Calculate: (*a*) the d-c component of the current wave and (*b*) the effective value.

17–30. Determine the equation for the current shown in Fig. 17–19. Choose your origin for a simple expression.

17–31. Find the expression for the current wave given in Fig. 17–20. Select your origin for a simple equation.

17–32. Find the equation for the current (pulse) wave indicated in Fig. 17–21. Choose your origin for a simple expression.

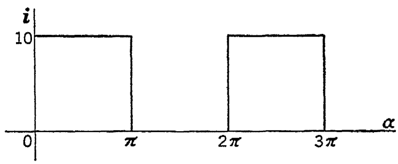


FIG. 17–20

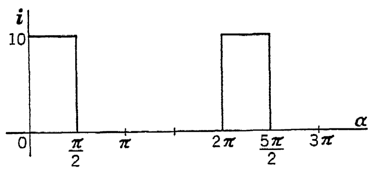


FIG. 17–21

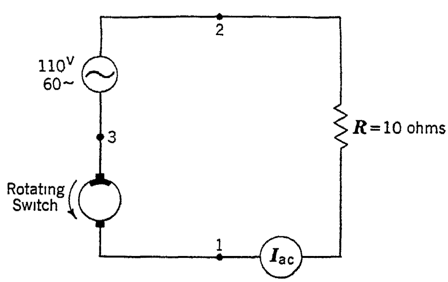


FIG. 17–22

17–33. A current flowing in a series circuit is

$$i = I_{m1} \sin 377t - I_{m3} \sin 1131t$$

The maximum ordinate of the current wave is 75 amp and an ammeter connected in the circuit reads 50 amp. What are the values of I_{m1} and I_{m3}?

17–34. Fig. 17–22 shows a circuit containing a rotating mechanical contact switch. It is revolving at 900 rpm and makes contact for one-fourth of each revolution (is open for three-fourths of a revolution). Assume that the rotating switch closes the circuit when the generator emf is going through zero and is increasing positively. (*a*) What is the fundamental frequency of the resulting non-sinusoidal wave of voltage between points 1 and 2? (*b*) Sketch the wave form for the voltage between points 1 and 3. (*c*) What will be the reading of the a-c ammeter?

CHAPTER 18

CIRCUITS WITH NON-SINUSOIDAL VOLTAGES AND CURRENTS

Although it is simpler and more convenient to work with pure sine waves, non-sinusoidal waves are generally obtained from commercial alternators and other electrical equipment which is capable of generating alternating emfs. It is assumed in this chapter that resistance, self-inductance, and capacitance are constant (that is, they do not vary with temperature, frequency, or current strength) unless it is otherwise stated. The Theorem of Superposition applies here. Thus, a problem may be resolved into several problems, the number depending on the number of harmonics and constant terms appearing in the impressed emf. The final result for the current is, of course, the sum of the values from all these separate solutions.

It must be remembered that not all circuits are linear; that is, all circuits do not have constant elements. In circuits containing iron, it is possible to have harmonics in the current wave that are not present in the impressed voltage wave. Such is the case with the no-load current wave of a transformer when a sinusoidal emf is impressed across the primary winding of the transformer. This is an indication that the circuit is non-linear.

18–1. Emf Relations in a Series Circuit Containing R, L, and C With Non-Sinusoidal Current Flowing.—(*a*) *Differential Equation Method:* The conditions in a circuit containing resistance, self-inductance, and capacitance in series are completely determined by the equation

$$e = e_R + e_L + e_C = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \qquad \textbf{(18–1)}$$

where e = impressed emf;

Ri = ohmic drop;

$L\dfrac{di}{dt}$ = drop due to self-induction;

$\dfrac{1}{C}\displaystyle\int i\,dt$ = drop due to the condenser.

Equation (18–1) holds irrespective of wave form.

It is convenient to find e_R, e_L, and e_C and then e by first assuming an expression for the current. Let the current flowing in the series circuit be

$$i = I_{m1} \sin (\omega t + \theta_1) + I_{m3} \sin (3\omega t + \theta_3) + I_{m5} \sin (5\omega t + \theta_5) + \ldots + I_{mn} \sin (n\omega t + \theta_n) \qquad (18\text{–}2)$$

The resistance drop will be proportional to the resistance, as

$$e_R = Ri = RI_{m1} \sin (\omega t + \theta_1) + RI_{m3} \sin (3\omega t + \theta_3) + \ldots + RI_{mn} \sin (n\omega t + \theta_n) \qquad (18\text{–}3)$$

The effective value of the resistance drop is

$$E_R = \sqrt{\frac{(RI_{m1})^2 + (RI_{m3})^2 + (RI_{m5})^2 + \ldots + (RI_{mn})^2}{2}} = RI \qquad (18\text{–}4)$$

where

$$I = \sqrt{\frac{I_{m1}^2 + I_{m3}^2 + I_{m5}^2 + \ldots + I_{mn}^2}{2}}$$

or the effective value of the current is

$$I = \sqrt{I_1^2 + I_3^2 + I_5^2 + \ldots + I_n^2} \qquad (18\text{–}5)$$

The drop due to self-induction is

$$e_L = L\frac{di}{dt} = L\frac{d}{dt}[I_{m1} \sin (\omega t + \theta_1) + I_{m3} \sin (3\omega t + \theta_3) + \ldots + I_{mn} \sin (n\omega t + \theta_n)]$$

This may be written as

$$e_L = \omega L I_{m1} \cos (\omega t + \theta_1) + 3\omega L I_{m3} \cos (3\omega t + \theta_3) + \ldots + n\omega L I_{mn} \cos (n\omega t + \theta_n) \qquad (18\text{–}6)$$

or

$$e_L = X_L I_{m1} \cos (\omega t + \theta_1) + 3X_L I_{m3} \cos (3\omega t + \theta_3) + \ldots + nX_L I_{mn} \cos (n\omega t + \theta_n) \qquad (18\text{–}7)$$

where $X_L = \omega L$.

Equation (18–7) has an effective value of

$$E_L = \sqrt{\frac{(X_L I_{m1})^2 + (3X_L I_{m3})^2 + \ldots + (nX_L I_{mn})^2}{2}}$$

or

$$E_L = \sqrt{(X_L I_1)^2 + (3X_L I_3)^2 + \ldots + (nX_L I_n)^2} \qquad (18\text{–}8)$$

The drop across the condenser* is

$$e_C = \frac{1}{C}\int i\,dt = \frac{1}{C}\int [I_{m1}\sin(\omega t+\theta_1) + I_{m3}\sin(3\omega t+\theta_3) + \dots + I_{mn}\sin(n\omega t+\theta_n)]\,dt$$

This may be written as

$$e_C = \frac{-I_{m1}}{\omega C}\cos(\omega t+\theta_1) - \frac{I_{m3}}{3\omega C}\cos(3\omega t+\theta_3) - \dots - \frac{I_{mn}}{n\omega C}\cos(n\omega t+\theta_n) \quad \textbf{(18–9)}$$

or

$$e_C = -X_C I_{m1}\cos(\omega t+\theta_1) - \frac{X_C I_{m3}}{3}\cos(3\omega t+\theta_3) - \dots - \frac{X_C I_{mn}}{n}\cos(n\omega t+\theta_n) \quad \textbf{(18–10)}$$

where $X_C = \frac{1}{\omega C}$.

The effective value is

$$E_C = \sqrt{\frac{(-X_C I_{m1})^2 + \left(\frac{-X_C I_{m3}}{3}\right)^2 + \dots + \left(\frac{-X_C I_{mn}}{n}\right)^2}{2}} \quad \textbf{(18–11)}$$

By substituting the values for e_R, e_L, and e_C in equation (18–1) and rearranging terms, we obtain

$$\begin{aligned} e = {} & RI_{m1}\sin(\omega t+\theta_1) + X_L I_{m1}\cos(\omega t+\theta_1) - X_C I_{m1}\cos(\omega t+\theta_1) \\ & + RI_{m3}\sin(3\omega t+\theta_3) + 3X_L I_{m3}\cos(3\omega t+\theta_3) - \frac{X_C I_{m3}}{3}\cos(3\omega t+\theta_3) \\ & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & + RI_{mn}\sin(n\omega t+\theta_n) + nX_L I_{mn}\cos(n\omega t+\theta_n) - \frac{X_C I_{mn}}{n}\cos(n\omega t+\theta_n) \end{aligned}$$

This can be written as

$$\begin{aligned} e = {} & \sqrt{R^2 + (X_L - X_C)^2}\, I_{m1}\sin(\omega t+\theta_1+\beta_1) \\ & + \sqrt{R^2 + \left(3X_L - \frac{X_C}{3}\right)^2}\, I_{m3}\sin(3\omega t+\theta_3+\beta_3) \\ & + \dots + \sqrt{R^2 + \left(nX_L - \frac{X_C}{n}\right)^2}\, I_{mn}\sin(n\omega t+\theta_n+\beta_n) \quad \textbf{(18–12)} \end{aligned}$$

* It is assumed that the condition is such that the constant of integration need not be considered in the steady state. If this is not so, then a constant must be added to the equations.

or
$$e = Z_1 I_{m1} \sin(\omega t + \theta_1 + \beta_1) + Z_3 I_{m3} \sin(3\omega t + \theta_3 + \beta_3) + \dots + Z_n I_{mn} \sin(n\omega t + \theta_n + \beta_n) \quad (18\text{–}13)$$

where

$$\left.\begin{aligned} Z_n &= \sqrt{R^2 + \left(nX_L - \frac{X_C}{n}\right)^2} \\ X_L &= 2\pi f L \text{ when } f = \text{fundamental frequency} \\ X_C &= \frac{1}{2\pi f C} \text{ when } f = \text{fundamental frequency} \\ \beta_n &= \tan^{-1} \frac{nX_L - \dfrac{X_C}{n}}{R} = \text{phase angle} \end{aligned}\right\} \quad (18\text{–}14)$$

The effective value of the impressed emf is

$$E = \sqrt{(Z_1 I_1)^2 + (Z_3 I_3)^2 + \dots + (Z_n I_n)^2} \quad (18\text{–}15)$$

The power factor of the circuit is

$$\text{P.F.} = \frac{I^2 R}{EI} = \frac{E_R}{E} \quad (18\text{–}16)$$

(*b*) *Superposition Theorem Method:* Since the circuit is linear, the foregoing problem can also be solved by the use of the Theorem of Superposition. That is, the final solution is the sum of n separate solutions, it being assumed that each harmonic voltage drop is due to the corresponding harmonic current flowing in the same R, L, and C series circuit. Whenever it is permissible to use the Superposition Theorem, this method should be used in preference to the Differential Equation Method. In equation (18–13) let

$$e_1 = Z_1 I_{m1} \sin(\omega t + \theta_1 + \beta_1) \quad (18\text{–}17a)$$

$$e_3 = Z_3 I_{m3} \sin(3\omega t + \theta_3 + \beta_3) \quad (18\text{–}17b)$$

$$e_5 = Z_5 I_{m5} \sin(5\omega t + \theta_5 + \beta_5) \quad (18\text{–}17c)$$

$$e_n = Z_n I_{mn} \sin(n\omega t + \theta_n + \beta_n) \quad (18\text{–}17d)$$

If it is considered that the fundamental voltage drop e_1 is due to the fundamental current $I_{m1} \sin(\omega t + \theta_1)$ flowing through Z_1, the third harmonic voltage drop e_3 is due to the third harmonic current $I_{m3} \sin(3\omega t + \theta_3)$ flowing through Z_3, and so on, then

$$e = e_1 + e_3 + e_5 + \dots + e_n$$

or
$$e = Z_1 I_{m1} \sin(\omega t + \theta_1 + \beta_1) + Z_3 I_{m3} \sin(3\omega t + \theta_3 + \beta_3) + \dots + Z_n I_{mn} \sin(n\omega t + \theta_n + \beta_n) \quad (18\text{–}13)$$

Example 18–1.—The current flowing in an inductive coil containing a resistance $R = 3$ ohms and self-inductance giving $X_L = 4$ ohms at fundamental frequency, Fig. 18–1(*a*), is

$$i = 100 \sin (377t + 30°) + 50 \sin (1131t + 90°)$$

Find the equation of the emf impressed across the circuit and the power factor of this circuit.

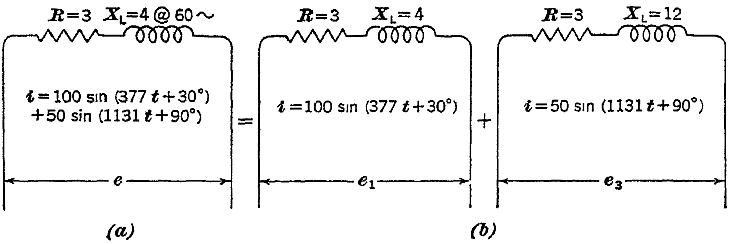


FIG. 18–1

Solution.—Since the circuit elements are constant, the Superposition Theorem applies and the solution to Fig. 18–1(*a*) may be considered as the sum of the solutions to the circuits in Fig. 18–1(*b*).

From equation (18–14),

$$Z_1 = \sqrt{3^2 + 4^2} = 5 \qquad \beta_1 = \tan^{-1}\frac{4}{3} = 53.2°$$

$$Z_3 = \sqrt{3^2 + (3 \times 4)^2} = 12.4 \qquad \beta_3 = \tan^{-1}\frac{12}{3} = 76°$$

Applying Ohm's Law to the circuits in Fig. 18–1(*b*), we obtain:

$$e_1 = Z_1 I_{m1} \sin (\omega t + \theta_1 + \beta_1) = 5 \times 100 \sin (377t + 30° + 53.2°)$$
$$= 500 \sin (377t + 83.2°)$$

$$e_3 = Z_3 I_{m3} \sin (3\omega t + \theta_3 + \beta_3) = 12.4 \times 50 \sin (1131t + 90° + 76°)$$
$$= 620 \sin (1131t + 166°)$$

By the Theorem of Superposition,

$$e = e_1 + e_3 = 500 \sin (377t + 83.2°) + 620 \sin (1131t + 166°)$$

The effective values of the current and the voltage are, respectively,

$$I = \sqrt{\frac{I_{m1}^2 + I_{m3}^2}{2}} = \sqrt{\frac{100^2 + 50^2}{2}} = 79.0 \text{ amp}$$

$$E = \sqrt{\frac{E_{m1}^2 + E_{m3}^2}{2}} = \sqrt{\frac{500^2 + 620^2}{2}} = 563.2 \text{ volts}$$

The power factor of the circuit is

$$\text{P.F.} = \frac{I^2 R}{EI} = \frac{79^2 \times 3}{563.2 \times 79} = 0.42$$

Example 18-2.—The inductive coil in Example 18–1 is replaced by a series combination of $R=3$ ohms and a condenser with $X_C=\frac{1}{2\pi fC}=4$ ohms at fundamental frequency, Fig. 18–2(*a*). Find the equation of the emf impressed across this series combination and the power factor of the circuit with the same current flowing.

Solution.—From equation (18–14),

$$Z_1=\sqrt{3^2+(-4)^2}=5 \qquad \beta_1=\tan^{-1}\frac{-4}{3}=-53.2°$$

$$Z_3=\sqrt{3^2+\left(\frac{-4}{3}\right)^2}=3.3 \qquad \beta_3=\tan^{-1}\frac{-4/3}{3}=-24°$$

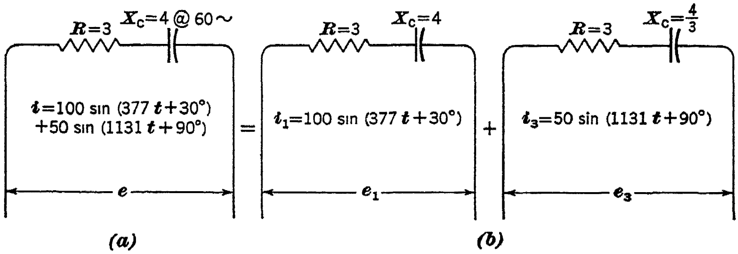


FIG. 18–2

In Fig. 18–2(*b*),

$$e_1=5\times 100 \sin(377t+30°-53.2°)$$
$$e_3=3.3\times 50 \sin(1131t+90°-24°)$$

By the Theorem of Superposition,

$$e=e_1+e_3=500\sin(377t-23.2°)+165\sin(1131t+66°)$$

The effective value of the voltage is

$$E=\sqrt{\frac{500^2+165^2}{2}}=373 \text{ volts}$$

The power factor is

$$\text{P.F.}=\frac{I^2R}{EI}=\frac{79^2\times 3}{373\times 79}=0.64$$

18–2. Current Wave Produced by a Non-Sinusoidal Wave of Emf Impressed on a Circuit Having *R*, *L*, and *C* in Series.

—Let the emf wave impressed across a circuit having R, L, and C in series be

$$e=E_{m1}\sin(\omega t+\theta_1)+E_{m3}\sin(3\omega t+\theta_3)+\ldots+E_{mn}\sin(n\omega t+\theta_n)$$

The components of current due to the different components of emf are then as follows:

(*a*) For the term $E_{m1} \sin(\omega t+\theta_1)$,

$$i_1=\frac{E_{m1}}{Z_1}\sin(\omega t+\theta_1-\beta_1) \qquad \textbf{(18–18a)}$$

(*b*) For the term $E_{m3} \sin(3\omega t+\theta_3)$,

$$i_3=\frac{E_{m3}}{Z_3}\sin(3\omega t+\theta_3-\beta_3) \qquad \textbf{(18–18b)}$$

(*c*) For the term $E_{mn} \sin(n\omega t+\theta_n)$,

$$i_n=\frac{E_{mn}}{Z_n}\sin(n\omega t+\theta_n-\beta_n) \qquad \textbf{(18–18c)}$$

where

$$Z_n=\sqrt{R^2+\left(nX_L-\frac{X_C}{n}\right)^2}$$

$$\beta_n=\tan^{-1}\frac{nX_L-\dfrac{X_C}{n}}{R}=\text{phase angle}$$

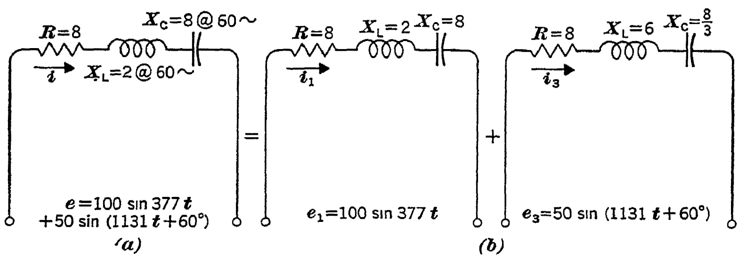


Fig. 18–3

By the Theorem of Superposition, the current flowing in the series circuit is

$$i=\frac{E_{m1}}{Z_1}\sin(\omega t+\theta_1-\beta_1)+\frac{E_{m3}}{Z_3}\sin(3\omega t+\theta_3-\beta_3)+\ \ldots$$

$$+\frac{E_{mn}}{Z_n}\sin(n\omega t+\theta_n-\beta_n) \qquad \textbf{(18–19)}$$

Example 18–3.—The emf wave impressed across a series circuit, Fig. 18–3(*a*), containing $R=8$, $X_L=2$ at 60 cycles, and $X_C=8$ at 60 cycles is

$$e=100\sin 377t+50\sin(1131t+60°)$$

Find the current flowing in the circuit.

Solution.—From equation (18–14),

$$Z_1 = \sqrt{8^2 + (2-8)^2} = 10 \qquad \beta_1 = \tan^{-1}\frac{-6}{8} = -36.8°$$

$$Z_3 = \sqrt{8^2 + \left[(3\times 2) - \frac{8}{3}\right]^2} = 8\ 65 \qquad \beta_3 = \tan^{-1}\frac{3.33}{8} = 22.6°$$

In Fig. 18–3(*b*),

$$i_1 = \frac{100}{10}\sin\,(377t + 36\ 8°)$$

$$i_3 = \frac{50}{8.65}\sin\,(1131t + 60° - 22.6°)$$

By the Theorem of Superposition,

$$i = i_1 + i_3 = 10\sin\,(377t + 36.8°) + 6\ 78\sin\,(1131t + 37.4°)$$

18–3. Resonance in a Series Circuit Consisting of *R*, *L*, and *C* With a Non-Sinusoidal Voltage Impressed.—Let the voltage wave impressed across a series circuit having resistance, inductance, and capacitance be

$$e = E_{m1}\sin\,(\omega t + \theta_1) + E_{m3}\sin\,(3\omega t + \theta_3) + \ldots + E_{mn}\sin\,(n\omega t + \theta_n)$$

Then the current flowing at any instant is

$$i = \frac{E_{m1}}{Z_1}\sin\,(\omega t + \theta_1 - \beta_1) + \frac{E_{m3}}{Z_3}\sin\,(3\omega t + \theta_3 - \beta_3) + \ldots + \frac{E_{mn}}{Z_n}\sin\,(n\omega t + \theta_n - \beta_n)$$

where

$$Z_n = \sqrt{R^2 + \left(nX_L - \frac{X_C}{n}\right)^2}$$

$$\tan\beta_n = \frac{nX_L - \dfrac{X_C}{n}}{R}$$

The condition of resonance for the nth harmonic is given by the relation

$$nX_L = \frac{X_C}{n} \tag{18–20}$$

Also, the effective value of the nth harmonic I_n, since $Z_n = R$, will be

$$I_n = \frac{E_n}{R} \tag{18–21}$$

where E_n is the effective value of the nth harmonic of the voltage wave.

If R is small relative to X_C and X_L, the nth harmonic of current will be very prominent in the total current wave and a large nth harmonic will be produced in the voltage drop across the condenser as well as in the voltage drop across the inductance.

18–4. Non-Sinusoidal Voltage Impressed on a Parallel Circuit Each Branch of Which Has R, L, and C in Series.—The line current can be obtained by finding each branch current separately and adding the results. In Fig. 18–4 is shown a parallel circuit each branch of which contains R, L, and C in series. Let the emf wave impressed on this circuit be

$$e = E_{m1} \sin(\omega t + \theta_1) + E_{m3} \sin(3\omega t + \theta_3) + \ldots + E_{mn} \sin(n\omega t + \theta_n)$$

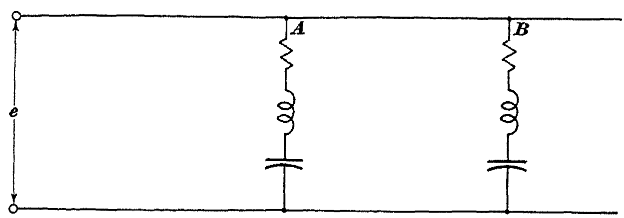


FIG. 18–4

The current flowing in each branch is indicated by the following relations:

(*a*) The current flowing in branch A is

$$i_A = \frac{E_{m1}}{Z_{A1}} \sin(\omega t + \Phi_{A1}) + \frac{E_{m3}}{Z_{A3}} \sin(3\omega t + \Phi_{A3}) + \ldots + \frac{E_{mn}}{Z_{An}} \sin(n\omega t + \Phi_{An}) \quad (18\text{–}22)$$

where

$$Z_{An} = \sqrt{R_A^2 + \left(nX_{LA} - \frac{X_{CA}}{n}\right)^2}$$

$$\Phi_{An} = \theta_n - \beta_{An}$$

$$\beta_{An} = \tan^{-1} \frac{nX_{LA} - \dfrac{X_{CA}}{n}}{R_A}$$

When the sine terms are expanded, equation (18–22) becomes

$$\begin{aligned} i_A = {} & \frac{E_{m1}}{Z_{A1}} \cos \Phi_{A1} \sin \omega t + \frac{E_{m1}}{Z_{A1}} \sin \Phi_{A1} \cos \omega t \\ & + \frac{E_{m3}}{Z_{A3}} \cos \Phi_{A3} \sin 3\omega t + \frac{E_{m3}}{Z_{A3}} \sin \Phi_{A3} \cos 3\omega t \\ & + \cdots \\ & + \frac{E_{mn}}{Z_{An}} \cos \Phi_{An} \sin n\omega t + \frac{E_{mn}}{Z_{An}} \sin \Phi_{An} \cos n\omega t \end{aligned} \tag{18–23}$$

(*b*) In branch B, a similar expression is obtained for the current i_B. Thus,

$$\begin{aligned} i_B = {} & \frac{E_{m1}}{Z_{B1}} \sin (\omega t + \Phi_{B1}) + \frac{E_{m3}}{Z_{B3}} \sin (3\omega t + \Phi_{B3}) + \cdots \\ & + \frac{E_{mn}}{Z_{Bn}} \sin (n\omega t + \Phi_{Bn}) \end{aligned} \tag{18–24}$$

or

$$\begin{aligned} i_B = {} & \frac{E_{m1}}{Z_{B1}} \cos \Phi_{B1} \sin \omega t + \frac{E_{m1}}{Z_{B1}} \sin \Phi_{B1} \cos \omega t \\ & + \frac{E_{m3}}{Z_{B3}} \cos \Phi_{B3} \sin 3\omega t + \frac{E_{m3}}{Z_{B3}} \sin \Phi_{B3} \cos 3\omega t \\ & + \cdots \\ & + \frac{E_{mn}}{Z_{Bn}} \cos \Phi_{Bn} \sin n\omega t + \frac{E_{mn}}{Z_{Bn}} \sin \Phi_{Bn} \cos n\omega t \end{aligned} \tag{18–25}$$

The line current is the sum of i_A and i_B. This sum is obtained by adding the coefficients of like terms of the same frequency, as $\sin \omega t$, $\cos \omega t$, $\sin 3\omega t$, etc. Hence,

$$\begin{aligned} i = i_A + i_B = {} & \left(\frac{E_{m1}}{Z_{A1}} \cos \Phi_{A1} + \frac{E_{m1}}{Z_{B1}} \cos \Phi_{B1} \right) \sin \omega t \\ & + \left(\frac{E_{m1}}{Z_{A1}} \sin \Phi_{A1} + \frac{E_{m1}}{Z_{B1}} \sin \Phi_{B1} \right) \cos \omega t \\ & + \left(\frac{E_{m3}}{Z_{A3}} \cos \Phi_{A3} + \frac{E_{m3}}{Z_{B3}} \cos \Phi_{B3} \right) \sin 3\omega t \\ & + \left(\frac{E_{m3}}{Z_{A3}} \sin \Phi_{A3} + \frac{E_{m3}}{Z_{B3}} \sin \Phi_{B3} \right) \cos 3\omega t \\ & + \cdots \\ & + \left(\frac{E_{mn}}{Z_{An}} \cos \Phi_{An} + \frac{E_{mn}}{Z_{Bn}} \cos \Phi_{Bn} \right) \sin n\omega t \\ & + \left(\frac{E_{mn}}{Z_{An}} \sin \Phi_{An} + \frac{E_{mn}}{Z_{Bn}} \sin \Phi_{Bn} \right) \cos n\omega t \end{aligned} \tag{18–26}$$

Example 18–4.—The voltage wave impressed on two impedances in parallel is

$$e = 200 \sin (377t + 10°) + 100 \sin (1131t + 30°)$$

As shown in Fig. 18–5, the first impedance has a resistance of 10 ohms and the second has a resistance of 3 ohms and an inductive reactance of 4 ohms at 60 cycles. What is the reading of an ammeter indicating the total line current?

Solution.—As branch *A* contains only resistance, the harmonic components of the current will be in phase with the harmonic components of the voltage. Hence,

$$i_A = \frac{200}{10} \sin (377t + 10°) + \frac{100}{10} \sin (1131t + 30°)$$

$$= 20 \sin (377t + 10°) + 10 \sin (1131t + 30°)$$

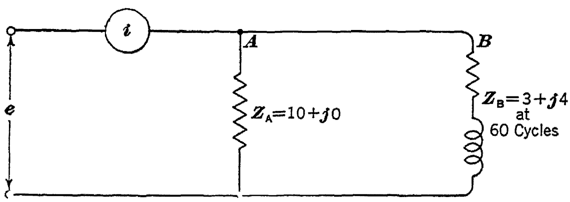


FIG. 18–5

As branch *B* contains inductive reactance, the harmonic components of the current will lag behind the harmonic components of the voltage. The angles are

$$\beta_1 = \tan^{-1} \frac{4}{3} = 53.2°$$

$$\beta_3 = \tan^{-1} 3 \times \frac{4}{3} = 76°$$

The current flowing in branch *B* is

$$i_B = \frac{200}{\sqrt{3^2 + 4^2}} \sin (377t + 10° - 53.2°) + \frac{100}{\sqrt{3^2 + (3 \times 4)^2}} \sin (1131t + 30° - 76°)$$

$$= 40 \sin (377t - 43.2°) + 8.08 \sin (1131t - 46°)$$

Expanding the expressions for i_A and i_B gives

$$i_A = 20(\cos 10° \sin 377t + \sin 10° \cos 377t)$$
$$+ 10(\cos 30° \sin 1131t + \sin 30° \cos 1131t)$$
$$= 19.70 \sin 377t + 3.48 \cos 377t + 8.66 \sin 1131t + 5 \cos 1131t$$

$$i_B = 40 [\cos (-43.2°) \sin 377t + \sin (-43.2°) \cos 377t]$$
$$+ 8.08 [\cos (-46°) \sin 1131t + \sin (-46°) \cos 1131t]$$
$$= 29.24 \sin 377t - 27.28 \cos 377t + 5\ 62 \sin 1131t - 5.81 \cos 1131t$$

The line current i, which is found by adding i_A and i_B, is

$$i = 48.94 \sin 377t - 23.8 \cos 377t + 14.28 \sin 1131t - 0.81 \cos 1131t$$
$$= 54.4 \sin (377t - 26°) + 14.3 \sin (1131t - 3.25°)$$

Its effective value is

$$I = \sqrt{\frac{54.4^2 + 14.3^2}{2}} = 39.7 \text{ amp}$$

18–5. Harmonics in Balanced Three-Phase Sources.—Let e_{aA}, e_{bB}, and e_{cC} be the instantaneous voltages generated in the windings of a balanced three-phase alternator, Fig. 18–6, where

$$e_{aA} = E_{m1} \sin \omega t + E_{m3} \sin 3\omega t + E_{m5} \sin 5\omega t + \dots \quad (18\text{–}27)$$

$$e_{bB} = E_{m1} \sin (\omega t - 120°) + E_{m3} \sin 3(\omega t - 120°) + E_{m5} \sin 5(\omega t - 120°) + \dots \quad (18\text{–}28)$$

$$e_{cC} = E_{m1} \sin (\omega t + 120°) + E_{m3} \sin 3(\omega t + 120°) + E_{m5} \sin 5(\omega t + 120°) + \dots \quad (18\text{–}29)$$

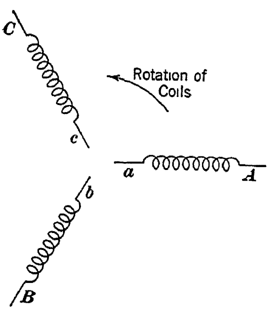


Fig. 18–6

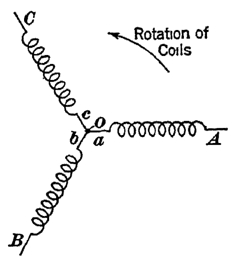


Fig. 18–7

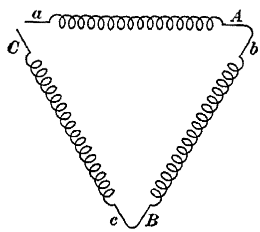


Fig. 18–8

For convenience the first, fifth, seventh, eleventh, etc. may be called the non-triplen harmonics; and the third, ninth, fifteenth, etc. may be called the triplen harmonics.

(*a*) *Coils Connected in Wye:* When the windings are connected in wye, as shown in Fig. 18–7, the triplen harmonics cancel each other in the line voltages but not in the line-to-neutral voltages. The line voltages are:

$$e_{AB} = e_{bB} - e_{aA}$$
$$= E_{m1} \sin (\omega t - 120°) - E_{m1} \sin \omega t + E_{m5} \sin 5(\omega t - 120°)$$
$$- E_{m5} \sin 5\omega t + E_{m7} \sin 7(\omega t - 120°) - E_{m7} \sin 7\omega t + \dots$$

or

$$e_{AB}=\sqrt{3}E_{m1}\sin(\omega t-150°)+\sqrt{3}E_{m5}\sin(5\omega t+150°)+\sqrt{3}E_{m7}\sin(7\omega t-150°)+\ldots \quad (18\text{–}30)$$

$$\begin{aligned}e_{BC}&=e_{cC}-e_{bB}\\&=E_{m1}\sin(\omega t+120°)-E_{m1}\sin(\omega t-120°)+E_{m5}\sin 5(\omega t+120°)\\&\quad-E_{m5}\sin 5(\omega t-120°)+E_{m7}\sin(7\omega t+120°)\\&\quad-E_{m7}\sin 7(\omega t-120°)+\ldots\end{aligned}$$

or

$$e_{BC}=\sqrt{3}E_{m1}\sin(\omega t+90°)+\sqrt{3}E_{m5}\sin(5\omega t-90°)+\sqrt{3}E_{m7}\sin(7\omega t+90°)+\ldots \quad (18\text{–}31)$$

$$\begin{aligned}e_{CA}&=e_{aA}-e_{cC}\\&=E_{m1}\sin\omega t-E_{m1}\sin(\omega t+120°)+E_{m5}\sin 5\omega t\\&\quad-E_{m5}\sin 5(\omega t+120°)+E_{m7}\sin 7\omega t\\&\quad-E_{m7}\sin 7(\omega t+120°)+\ldots\end{aligned}$$

or

$$e_{CA}=\sqrt{3}E_{m1}\sin(\omega t-30°)+\sqrt{3}E_{m5}\sin(5\omega t+30°)+\sqrt{3}E_{m7}\sin(7\omega t-30°)+\ldots \quad (18\text{–}32)$$

It should be noted from inspecting equations (18–27), (18–28), and (18–29) that all the triplen harmonics are in phase, since adding 360° or its multiples does not change the phase. As differences in equations (18–30), (18–31), and (18–32) are considered, the triplen harmonics cancel out. Hence, the line voltages of a three-phase alternator whose coils are connected in wye do not contain triplen harmonics. The wave shape of the line voltage will be different from that of the line-to-neutral voltage E_{OA}. Owing to the elimination of the triplen harmonics, it must be true for non-sinusoidal waves that

$$E_{AB}<\sqrt{3}E_{OA} \quad (18\text{–}33)$$

where E_{AB} and E_{OA} are effective values.

(*b*) *Coils Connected in Delta:* If the coils are connected in delta, as shown in Fig. 18–8, the voltage across aC is equal to the sum of the three coil voltages. It can be seen from equations (18–27) to (18–29) that the voltage e_{aC} contains no fundamental component since the components of the fundamental are 120° apart and they add to zero. The components of each non-triplen harmonic are also 120° apart and they add to zero. The components of each triplen harmonic are in phase and they add instead of cancelling. Thus,

$$e_{aC} = e_{aA} + e_{bB} + e_{cC}$$

or $e_{aC} = 3E_{m3} \sin 3\omega t + 3E_{m9} \sin 9\omega t + 3E_{m15} \sin 15\omega t + \ldots$ **(18–34)**

since

$$E_{m1} [\sin \omega t + \sin (\omega t - 120°) + \sin (\omega t + 120°)] = 0$$
$$E_{m5} [\sin 5\omega t + \sin 5(\omega t - 120°) + \sin 5(\omega t + 120°)] = 0$$

and so on.

If a is connected to C, the voltage e_{aC} will cause a circulating current to flow in the closed delta which in turn produces a drop in the three coils equal to the voltage given by equation (18–34). The drop across each coil is

$$e_Z = \tfrac{1}{3} e_{aC} = E_{m3} \sin 3\omega t + E_{m9} \sin 9\omega t + E_{m15} \sin 15\omega t + \ldots \quad \textbf{(18–35)}$$

When this drop e_Z is subtracted from the generated voltage in each coil, as given by equations (18–27) to (18–29), the triplen harmonics cancel out and the line voltages are:

$$e_{AB} = e_{bB} - e_Z$$

or $$e_{AB} = E_{m1} \sin (\omega t - 120°) + E_{m5} \sin 5(\omega t - 120°) + E_{m7} \sin 7(\omega t - 120°) + \ldots \quad \textbf{(18–36)}$$

$$e_{BC} = e_{cC} - e_Z$$

or $$e_{BC} = E_{m1} \sin (\omega t + 120°) + E_{m5} \sin 5(\omega t + 120°) + E_{m7} \sin 7(\omega t + 120°) + \ldots \quad \textbf{(18–37)}$$

$$e_{CA} = e_{aA} - e_Z$$

or $$e_{CA} = E_{m1} \sin \omega t + E_{m5} \sin 5\omega t + E_{m7} \sin 7\omega t + \ldots \quad \textbf{(18–38)}$$

As in the case of the Y-connected generator, the line voltages of a Δ-connected alternator do not contain triplen harmonics.

PROBLEMS

18–1. A current $i = 25 \sin (377t - 10°) + 10 \sin (1131t + 45°)$ is flowing through a coil containing $R = 10$ ohms and $L = 0.10$ henry. Find the equation for the voltage drop across the coil and the power taken by the coil. Plot the waves.

18–2. A coil having a resistance $R = 3$ ohms and an inductive reactance $X_L = 4$ ohms measured at 25 cps is connected to a source of emf. If the current flowing is $i = 20 \sin (377t + 20°) - 5 \sin (1131t - 60°)$, what is the expression for the voltage drop across the coil?

18–3. A current $i = 100 \sin 377t + 25 \sin (754t - 20°) - 10 \sin (1131t + 90°)$ is flowing through a series circuit containing $R = 10$ ohms, $X_L = 5$ ohms measured at 25 cycles, and $X_C = 7$ ohms measured at 25 cycles. Find the expression for the voltage drop across the entire circuit. What is the reading of a wattmeter connected in the circuit to read the total power?

18–4. A current $i = 50 \sin (377t + \alpha) - 20 \sin (1131t - \beta)$ is flowing through a coil having $R = 5$ ohms and $L = 0.2$ henry. Determine the expression for the voltage drop across the coil and the power taken by the coil: (*a*) when $\alpha = 10°$ and $\beta = 75°$; (*b*) when $\alpha = -30°$ and $\beta = 90°$.

18–5. An emf $e = 100 \sin (157t + 25°) + 30 \sin (471t - 87°)$ is impressed across a coil having $R = 10$ ohms and $X_L = 7$ ohms measured at 60 cps. What is the equation for the current? Plot the waves.

18–6. An emf $e = 200 \sin (377t - 30°) + 50 \sin (754t + 40°)$ is impressed across a series circuit containing $R = 5$ ohms, $L = 0.25$ henry, and $C = 100$ microfarads. Find the expression for the current and plot the emf and current waves. What is the reading of an ammeter placed in the circuit?

18–7. An emf $e = 300 \sin (377t + 60°) - 100 \sin (1131t - 45°)$ is impressed across a circuit consisting of $R = 20$ ohms and $C = 200$ microfarads in series. Find: (*a*) the reading of an ammeter placed in the circuit; (*b*) the reading of a wattmeter connected to measure the total power. Also, plot the emf and current waves.

18–8. A series circuit containing constant elements is connected to an emf of the form $e = 200 \sin 377t + 50 \sin (1131t + 60°)$. If the expression for the current is $i = 40 \sin 377t + 5 \sin (1131t + \theta_3)$, what are the constants of the circuit? What is the value for θ_3?

18–9. A voltmeter of the electrodynamometer type with a 150-volt scale has a resistance of 3500 ohms and an inductance of 0.1 henry. The instrument is calibrated on a 60-cycle, sinusoidal emf at 120 volts. What is the effective value of the fundamental when the meter reads 120 volts on a circuit which has a 35 per cent third harmonic in its potential?

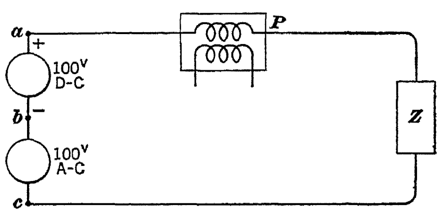


FIG. 18–9

18–10. A direct-current generator and a 60-cycle alternating-current generator are connected in series, as shown in Fig. 18–9. The terminal voltages of the generators are constant and each is 100 volts. Find the effective value of the current when : (*a*) $\dot{Z} = 10 + j0$; (*b*) $\dot{Z} = 6 + j8$; (*c*) $\dot{Z} = 8 - j6$.

18–11. In each of the three cases in Problem 18–10, what will be the readings of the wattmeter P when its potential coil terminals are connected across a and b and then across b and c?

18–12. An emf $e = 200 \sin 377t + 75 \sin 1131t$ is impressed on a series circuit containing a resistance of 10 ohms, an inductance of 0.1 henry, and a variable condenser. Plot a curve showing the effective values of the current against capacitance.

18–13. An impedance coil having a resistance of 4 ohms and an inductance of 0.6 henry is connected in series with a variable condenser. The impressed emf to this series combination is $e = 100 \sin 377t + 10 \sin 1131t$ For what value of the capacitance will the current have its greatest value? What is the greatest value of the current?

18–14. An emf $e = 200 \sin 157t - 100 \sin (314t + 40°)$ is impressed across a circuit consisting of a resistance $R = 10$ ohms and a capacitance $C = 100$ microfarads in parallel. What are the currents in the branches and the current in the line? Plot the three current waves.

18–15. A resistor and an air-core coil are connected in parallel across an emf of the form $e = 100 \sin 377t - 30 \sin (1131t - 45°)$. The resistance of the resistor is 10 ohms and the constants of the coil are $R = 3$ ohms and $X_L = 4$ ohms measured at 60 cps. Find the current in each branch and also the current in the line. What is the power taken by each branch?

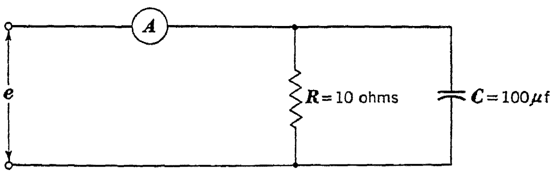


FIG 18–10

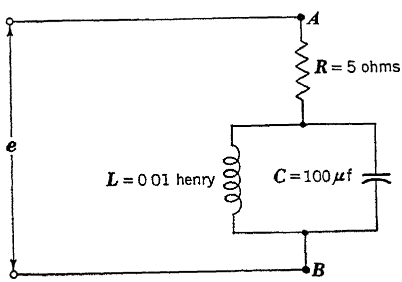


FIG. 18–11

18–16. A coil having $R = 3$ ohms and $L = 0.2$ henry is connected in parallel with a variable condenser. If the emf impressed on the combination is $e = 200 \sin 377t + 30 \sin 1131t$, what must be the value of the capacitance when the line current is a minimum?

18–17. In Fig. 18–10,

$$e = 10 + 100\sqrt{2} \sin 1000t$$

What is the effective value of the line current as read by an a-c ammeter?

18–18. Find the average power taken by the load AB in Fig. 18–11 if

$$e = 10 + 100\sqrt{2} \sin 1000t$$

18–19. What is the average power delivered to the *RLC* circuit shown in Fig. 18–12?

18–20. In the circuit shown in Fig. 18–13, determine: (*a*) the expression for the generator current *i*; (*b*) the effective value of the generator current; (*c*) the average power delivered by the generator to the load.

18–21. In Fig. 18–14,

$$i = 3 + 10\sqrt{2}\ \sin 1000t$$

Determine the expression for *e*.

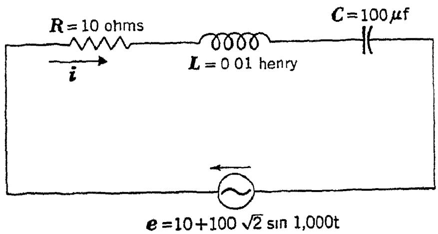


FIG. 18–12

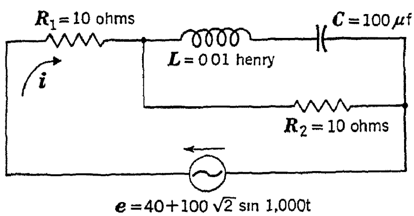


FIG. 18–13

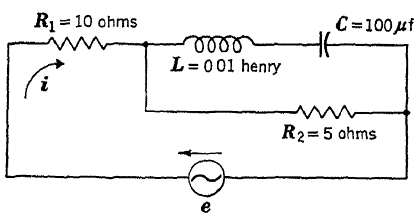


FIG. 18–14

18–22. The generated voltage of one coil of a balanced three-phase alternator, Fig. 18–6, is

$$e_{aA} = 100\sqrt{2}\ \sin \omega t + 50\sqrt{2}\ \sin 3\omega t + 5\sqrt{2}\ \sin 5\omega t$$

Write the expressions for the other two coil voltages: (*a*) for sequence *ABC*; (*b*) for sequence *ACB*.

18–23. Consider the three-phase alternator in Problem 18–22 with generated voltages in the *ABC* sequence, and assume that the coils are connected in wye, as shown in Fig. 18–7. (*a*) What is the effective value of the line-to-line voltage? (*b*) What is the ratio of the line-to-line voltage to the line-to-neutral voltage?

18–24. Consider the three-phase alternator in Problem 18–22 with generated voltages in the ABC sequence. If the coils are connected in delta, as shown in Fig. 18–8, what is the reading of a voltmeter with its terminals across a and C?

18–25. Fig. 18–15 shows a network which consists of three elements, namely, one resistance, one inductance, and one capacitance. The emf applied to the network is

$$e = 90 + 100\sqrt{2} \sin 1000t + 60 \sin 3000t$$

and the current that flows is

$$i = 10 + 20 \sin (1000t - 45°)$$

Determine the values of the elements R, L, and C, and their arrangement in the network.

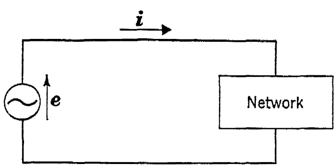


FIG. 18–15

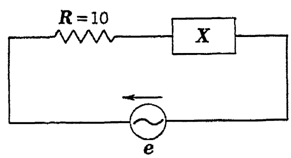


FIG. 18–16

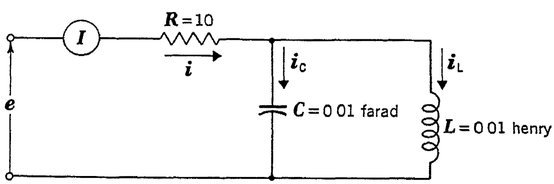


FIG. 18–17

18–26. The emf impressed across the terminals of a circuit containing some combination of R, L, and C in series is

$$e = 100 + 100\sqrt{2} \sin 377t$$

The resistance of the circuit is 20 ohms and the effective value of the current is 6.41 amp. Find the value or values of the reactive circuit elements.

18–27. Fig. 18–16 shows a resistance in series with a single reactive element X which is not equal to zero and not equal to infinity; that is, $X \neq 0$ and $X \neq \infty$. The emf is $e = 100 + 200 \sin 100t$, the effective value of the current is 10 amp, and the average power taken by the entire circuit is 1000 watts. Find the value and type of X.

18–28. The current flowing through the inductance L in Fig. 18–17 is

$$i_L = 10 + 10\sqrt{2} \sin 100t$$

Find the reading of the ammeter in the line.

CHAPTER 19

DIFFERENTIAL EQUATIONS

Transient electric phenomena, in both machines and networks, occupy a position of considerable importance in the successful operation of power and communication systems. The commercial importance generally increases with the size and complexity of the system. Electric transients, as the term implies, are usually of short duration and relate to what occurs in an electric circuit between periods of stable conditions. Any change, such as the starting or stopping of a refrigerator motor or the turning on of a lamp, necessitates a readjustment of the energy content in the entire system and produces electric transients just as truly as a stroke of lightning or a short circuit.

In order to gain a clear conception of the fundamental principles of electric transients and their application to the solution of quantitative problems, it is necessary to have some knowledge of differential equations. Thus, it will be helpful to the student, before taking up the topic of transient phenomena formally, to study a few of the differential equations that occur perhaps more often in electrical engineering than in any other kind of work, together with the methods of solution. More emphasis will be placed on how these solutions are obtained than on rigorous proofs of the methods. Since the circuits being treated in Chapters 20 and 21 are those with *constant elements*, only linear differential equations of the nth order with *constant coefficients* will be considered here.

For the sake of brevity, the derivations of some of the formulas given in the following articles are omitted; the student is referred to books* and articles on differential equations for these derivations, if desired. Electric circuit problems are used as illustrative examples to stimulate the student's interest when going over this chapter.

* *Differential Equations*, by A. Murray; *Differential Equations for Electrical Engineers*, by Philip Franklin; *The Elementary Theory of Operational Mathematics*, by Eugene Stephens; *Mathematics of Modern Engineering*, by Doherty and Keller; *Advanced Mathematics for Engineers*, by Reddick and Miller.

The aims of this chapter are: (*a*) to show that the complete solution of a linear differential equation of order n with constant coefficients may be given as the sum of two parts, namely, a complementary function (transient component of current in circuit problems) and a particular integral (steady state component of current in circuit problems); (*b*) to determine the forms of the complementary function when the auxiliary equation has distinct roots, equal roots, and imaginary roots; and (*c*) to indicate how to use the steady state solution as a particular integral in the final complete solution.

19–1. Differential Equation Defined.—A differential equation between any two variables, such as i and t, is an equation involving in any manner one or more of the derivatives of the dependent variable i with respect to the independent variable t, together with one or both of the variables i and t.

The order of a differential equation is the order of the highest derivative appearing in it. Thus, $\frac{d^3i}{dt^3}+\frac{di}{dt}=0$ is a differential equation of the third order.

The degree of a differential equation is the power to which the highest derivative is raised when the differential coefficients are free from radicals and fractions. The following examples are second-degree equations:

$$\left(\frac{di}{dt}\right)^2+i=0$$

$$i=K\sqrt{1+\left(\frac{di}{dt}\right)^2} \quad \text{where } K \text{ is a constant}$$

$$i=a\,\frac{di}{dt}+\frac{b}{di/dt} \quad \text{where } a \text{ and } b \text{ are constants}$$

19–2. Linear Equations of First Order.—A differential equation of the first order is said to be *linear* when the dependent variable and its first derivative appear only in the first degree (that is, in the first power). Such an equation may be written in the general form

$$\frac{di}{dt}+Pi=Q \tag{19–1}$$

where P and Q are functions of t or constants. An example is the differential equation for an RL circuit connected to a battery of voltage E. This equation is (see Example 19–1)

$$\frac{di}{dt}+\frac{R}{L}i=\frac{E}{L} \tag{19–2}$$

In this case, $P=\frac{R}{L}$ and $Q=\frac{E}{L}$.

It will now be shown that the complete solution of equation (19–1) may be given in the form

$$i=u+Kv \tag{19–3}$$

where u and v are functions of t determined from P and Q, and K is an arbitrary constant. That is, the solution consists of the sum of two parts, u and Kv. This method of analysis will be used in the study of linear equations of the nth order.

In some cases, the solution of a first-order differential equation can be simplified by noting that some combinations of terms may be made integrable by the introduction of a factor on both sides of the equation. Experience gained in solving differential equations will do much to aid the student in finding such "short cuts." The following method can be found in books on differential equations. If equation (19–1) is multiplied by the factor $e^{\int P\,dt}$, the result is

$$e^{\int P\,dt}\left[\frac{di}{dt}+Pi\right]=e^{\int P\,dt}Q \tag{19–4}$$

It can be seen that the left-hand side is the exact derivative of a product. Thus,

$$\frac{d}{dt}\left[e^{\int P\,dt}\cdot i\right]=e^{\int P\,dt}\left[\frac{di}{dt}+Pi\right]=e^{\int P\,dt}Q \tag{19–5}$$

The integration of an exact derivative of a product is a simple matter, and here lies the advantage of introducing the factor $e^{\int P\,dt}$. Then, integration of equation (19–5) gives

$$ie^{\int P\,dt}=\int e^{\int P\,dt}Q\,dt+K \tag{19–6}$$

from which

$$i=e^{-\int P\,dt}\int e^{\int P\,dt}Q\,dt+Ke^{-\int P\,dt}=u+Kv \tag{19–7}$$

where

$$K = \text{constant of integration}$$

$$u = e^{-\int P\,dt}\int e^{\int P\,dt}\,Q\,dt \tag{19-8}$$

$$v = e^{-\int P\,dt} \tag{19-9}$$

Since P and Q are either constants or functions of t, the right-hand sides of equations (19–8) and (19–9) can be determined as indicated in Example 19–1. Equation (19–7) states that the complete solution of equation (19–1) may be considered as consisting of the sum of part u and part Kv.

The following observations should be made concerning Kv and u in equation (19–7):

(*a*) Since $Kv = Ke^{-\int P\,dt}$, then substituting this value for i in equation (19–1) gives

$$\frac{d(Kv)}{dt} + P(Kv) = -KPe^{-\int P\,dt} + KPe^{-\int P\,dt} = 0$$

This indicates that the substitution of Kv for i in equation (19–1) yields zero on the right-hand side. It is reasonable to believe, then, that $Kv = Ke^{-\int P\,dt}$ may be obtained from the original equation (19–1) by setting the right-hand side equal to zero (see Art. 19–4); that is,

$$\frac{di}{dt} + Pi = 0$$

(*b*) The part u may be considered as a particular solution of equation (19–1); that is, the substitution of u for i will satisfy the original equation (19–1). If u is given by equation (19–8), then

$$\frac{du}{dt} = e^{-\int P\,dt}\left[e^{\int P\,dt}Q\right] + \int e^{\int P\,dt}Q\,dt\left[-Pe^{-\int P\,dt}\right]$$

or

$$\frac{du}{dt} = Q - Pe^{-\int P\,dt}\int e^{\int P\,dt}Q\,dt = Q - Pu$$

and

$$\frac{du}{dt} + Pu = (Q - Pu) + Pu = Q$$

This result indicates that the substitution of u for i in equation (19–1) gives Q on the right-hand side (see Art. 19–9).

(*c*) If Kv gives zero on the right-hand side and u gives Q on the right-hand side, it is reasonable to believe that the sum of these two parts is a solution of equation (19–1). It must be kept in mind that, although u satisfies the original equation (19–1), it is not a complete solution of equation (19–1) unless $Kv=0$ (see Arts. 19–9 and 19–10).

(*d*) It can be seen from equations (19–7) to (19–9) that u is free of any arbitrary constant and Kv contains the constant of integration.

Example 19–1.—An unvarying emf E is suddenly applied across an RL circuit. Find the expression for the current.

Solution.—The differential equation for the circuit may be written as follows:

$$Ri+L\frac{di}{dt}=E \text{ or } \frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}$$

In this case, $P=\frac{R}{L}$ and $Q=\frac{E}{L}$. From equation (19–8), the particular solution (or steady-state value) is

$$u=e^{-\int\frac{R}{L}dt}\int e^{\int\frac{R}{L}dt}\frac{E}{L}dt=e^{-\frac{Rt}{L}}\left[\frac{E}{L}\cdot\frac{L}{R}e^{\frac{Rt}{L}}\right]=\frac{E}{R}$$

From equation (19–9), the exponential term (or transient term) is

$$Kv=Ke^{-\int\frac{R}{L}dt}=Ke^{-\frac{Rt}{L}}$$

The substitution of u for i gives $Q=\frac{E}{L}$ on the right-hand side, or

$$\frac{d}{dt}\left(\frac{E}{R}\right)+\frac{R}{L}\left(\frac{E}{R}\right)=\frac{E}{L}$$

and the substitution of Kv for i gives 0 on the right-hand side, or

$$\frac{d}{dt}\left(Ke^{-\frac{Rt}{L}}\right)+\frac{R}{L}Ke^{-\frac{Rt}{L}}=0$$

The complete solution which will satisfy the original equation is

$$i=u+Kv=\frac{E}{R}+Ke^{-\frac{Rt}{L}}$$

where the determination of the arbitrary constant K will be given in the chapters on transient phenomena. In a circuit problem, the value of the arbitrary constant K depends on the boundary conditions, such as the initial conditions. It should be noted that, although $u=\frac{E}{R}$ satisfies the original differential equation, it is not the complete solution unless $Ke^{-\frac{Rt}{L}}=0$.

19-3. Linear Equations of *n*th Order With Constant Coefficients.—A differential equation of the nth order is said to be *linear* when the dependent variable and its derivatives appear only in the first power. Thus, a linear differential equation is formed by equating a linear expression in i and its derivatives to a given quantity X, as in the following case:

$$A_0 \frac{d^3i}{dt^3} + A_1 \frac{d^2i}{dt^2} + A_2 \frac{di}{dt} + A_3 i = X \qquad \textbf{(19-10)}$$

where the coefficients A_0, A_1, A_2, A_3 are constants and X may be a function of t or a constant.

In general, the coefficients of actual circuits containing iron and those of rotating alternating-current machines are variables. We shall confine ourselves, however, to simple cases where the circuit elements are constants.

If the highest derivative which appears in the linear expression is $\frac{d^ni}{dt^n}$, the equation is said to be of the nth order. Such an equation is

$$A_0 \frac{d^ni}{dt^n} + A_1 \frac{d^{n-1}i}{dt^{n-1}} + A_2 \frac{d^{n-2}i}{dt^{n-2}} + \ldots + A_n i = X \qquad \textbf{(19-11)}$$

where $A_0 \neq 0$.

An example of a second-order linear differential equation is that of an RLC circuit connected to a battery of voltage E, in which case

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = E$$

Differentiating and rearranging gives

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0$$

In Art. 19-2, it has been demonstrated that the solution of a linear differential equation of the first order may be considered as consisting of the sum of two parts. This same property is possessed by linear equations of order n with constant coefficients. It is the purpose of the remainder of this chapter to indicate methods of obtaining the two parts of the complete solution of equation (19-11). The part satisfying this equation is called a particular integral and the part satisfying equation (19-11) with X replaced by zero is called the complementary function. The complementary function will be considered first.

19–4. The Complementary Function.—The complementary function is the part Kv of the complete solution $i=u+Kv$; and it contains the arbitrary constant. The complementary function is the solution of the equation

$$A_0\frac{d^ni}{dt^n}+A_1\frac{d^{n-1}i}{dt^{n-1}}+A_2\frac{d^{n-2}i}{dt^{n-2}}+\ \dots\ +A_ni=0 \qquad \textbf{(19–12)}$$

This is the same as equation (19–11) except that X is replaced by zero. It contains no term which is independent of i. It will be seen that the general solution of equation (19–12) contains n arbitrary constants if $A_0\neq 0$. The determination of the complementary function will now be considered.

For simplicity, let us consider a linear differential equation of the second order with constant coefficients and the right-hand member zero. This equation is

$$A_0\frac{d^2i}{dt^2}+A_1\frac{di}{dt}+A_2i=0 \qquad \textbf{(19–13)}$$

One method of solving this differential equation is to assume a solution and then determine its correctness by substituting back in the original equation. Since equation (19–13) is linear, it will be satisfied if $\frac{d^2i}{dt^2}$ and $\frac{di}{dt}$ are proportional to i. This fact suggests that i may have the form $i=e^{mt}$. The derivatives of the exponential function $i=e^{mt}$, when m is a constant, are proportional to i and we are led to try e^{mt} as a solution.

Thus, to solve equation (19–13), it has been found convenient to let $i=e^{mt}$. Then

$$\frac{di}{dt}=me^{mt}$$

$$\frac{d^2i}{dt^2}=m^2e^{mt}$$

When these values are substituted in equation (19–13), the result is

$$(A_0m^2+A_1m+A_2)e^{mt}=0 \qquad \textbf{(19–14)}$$

Since, in an actual problem, $i=e^{mt}$ cannot be set equal to zero to have a useful solution, then

$$A_0m^2+A_1m+A_2=0 \qquad \textbf{(19–15)}$$

Equation (19–15) is called an *auxiliary equation.* The mechanics involved in setting up this equation from equation (19–13) is to substitute 1 for i, m for $\frac{di}{dt}$, and m^2 for $\frac{d^2i}{dt^2}$.

The advantage of going from equation (19–13) to equation (19–15) is that we have obtained an algebraic equation whose roots are such values of m that equation (19–13) is satisfied by $i=Ke^{mt}$, where K is a constant. Equation (19–15) has two roots, m_1 and m_2. When m has a value of m_1 which satisfies the auxiliary equation, then $K_1e^{m_1t}$ will satisfy equation (19–13). If m_2 is another distinct root of equation (19–15), then $K_2e^{m_2t}$ will satisfy equation (19–13). It can be shown by substituting values back in the original equation that $(K_1e^{m_1t}+K_2e^{m_2t})$ is also a solution of equation (19–13). Hence, the complete solution of equation (19–13) is

$$i=K_1e^{m_1t}+K_2e^{m_2t} \tag{19–16}$$

where m_1 and m_2 are the two distinct roots of equation (19–15) and K_1 and K_2 are the arbitrary constants. In a circuit problem, K_1 and K_2 are determined from the boundary conditions.

The cases where m_1 and m_2 are equal roots or imaginary roots will be considered in Arts. 19–7 and 19–8. Since equation (19–13) is a second-order equation, its general solution must contain two arbitrary constants as indicated in equation (19–16). For a third-order equation, the general solution must contain three arbitrary constants. For order n, with n distinct roots in the auxiliary equation, the general solution is

$$i=K_1e^{m_1t}+K_2e^{m_2t}+\ \ldots\ +K_ne^{m_nt} \tag{19–17}$$

The determination of the values of the arbitrary constants will be given in the chapters on transient phenomena, and it will be shown that these values depend on the boundary conditions.

Example 19–2.—An unvarying emf of 10 volts is suddenly impressed across a series circuit of $R=\frac{5}{2}$ ohms, $L=\frac{1}{2}$ henry, and $C=\frac{1}{3}$ farad. Determine the current as a function of time with $t=0$ when the circuit is closed.

Solution.—The differential equation of the circuit is

$$\frac{5}{2}i+\frac{1}{2}\frac{di}{dt}+3\int i\,dt=10$$

Whenever an equation contains a term with an integral sign, it is convenient to differentiate with respect to t and to simplify the result. In this case,

$$\frac{d^2i}{dt^2}+5\frac{di}{dt}+6i=0$$

The solution of the second equation will be a solution of the first equation. Then it is sufficient to consider only the second equation. The auxiliary equation is

$$m^2+5m+6=0$$

the roots of which are

$$m_1=-2 \quad \text{and} \quad m_2=-3$$

It will be shown by substitution that. (a) K_1e^{-2t} will satisfy the equation; (b) K_2e^{-3t} is another solution of the equation; and (c) $K_1e^{-2t}+K_2e^{-3t}$ will also satisfy the equation.

(a) If $i=K_1e^{-2t}$, then $\frac{di}{dt}=-2K_1e^{-2t}$ and $\frac{d^2i}{dt^2}=4K_1e^{-2t}$. Substituting these values in the equation gives

$$4K_1e^{-2t}-10K_1e^{-2t}+6K_1e^{-2t}=0$$

(b) If $i=K_2e^{-3t}$, then $\frac{di}{dt}=-3K_2e^{-3t}$ and $\frac{d^2i}{dt^2}=9K_2e^{-3t}$. Substitution gives

$$9K_2e^{-3t}-15K_2e^{-3t}+6K_2e^{-3t}=0$$

(c) If $i=K_1e^{-2t}+K_2e^{-3t}$, then $\frac{di}{dt}=-2K_1e^{-2t}-3K_2e^{-3t}$ and $\frac{d^2i}{dt^2}=4K_1e^{-2t}+9K_2e^{-3t}$. Substitution gives

$$(4K_1e^{-2t}+9K_2e^{-3t})-5(2K_1e^{-2t}+3K_2e^{-3t})+6(K_1e^{-2t}+K_2e^{-3t})=0$$

Since the right-hand member of the second equation is 0, the complete solution is

$$i=K_1e^{-2t}+K_2e^{-3t}$$

19–5. The Operator $p\equiv\frac{d}{dt}$.—To solve a linear equation with constant coefficients, the concept of a derivative operator has been introduced. Where the independent variable is x, the symbol $D=\frac{d}{dx}$ is used. When t is the independent variable, another symbol has been adopted. The definition of the operator p is

$$P\equiv\frac{d}{dt} \tag{19–18}$$

From the definition p is not a quantity, but is an operator. When it is placed to the left of any function of t, it indicates that the function is to be differentiated. Thus, it follows that

$$pi = \frac{d}{dt}(i) = \frac{di}{dt}$$

$$p(pi) = p^2 i = \frac{d}{dt}\left(\frac{di}{dt}\right) = \frac{d^2 i}{dt^2}$$

$$p^n i = \frac{d^n i}{dt^n} \tag{19–19}$$

That is, equation (19–19) shows that $p^n i$ calls for n successive differentiations of i with respect to t.

When it is desirable to introduce an integrating operator, it is usually represented by $\frac{1}{p}$. Thus,

$$\frac{1}{p} i = \int_0^t i \, dt \tag{19–20}$$

For two successive integrations,

$$\frac{1}{p^2} i = \frac{1}{p}\left(\frac{1}{p} i\right) = \int_0^t \left[\int_0^t i \, dt\right] dt \tag{19–21}$$

A property of the derivative operator and the integrating operator is

$$p\left(\frac{1}{p} i\right) = \frac{d}{dt}\int_0^t i \, dt = i \tag{19–22}$$

That is, the application of p removes the effect of applying $\frac{1}{p}$ to a current.

Although the operators p and $\frac{1}{p}$ are not algebraic quantities, they have many of the properties of such quantities. The reader is referred to books on operational mathematics for detailed discussions of the properties of these operators. For the cases treated in the chapters on direct-current and alternating-current transients, the operators and the sum of these operators with constant coefficients behave as if they were algebraic quantities. For example, the derivative operators may be factored as follows:

$$\frac{d^2 i}{dt^2} - (a+b)\frac{di}{dt} + abi = 0$$

$$[p^2 - (a+b)p + ab]i = 0$$

$$(p-a)[(p-b)i] = (p-b)[(p-a)i] = 0$$

where a and b are constants. When symbolic factors are used as operators, they must operate on something. With i located on the extreme right, the operating factors are taken in the order from right to left.

19–6. The Use of p in the Solution of the Complementary Function.—Equation (19–13) may be written as follows:

$$\left(A_0 \frac{d^2}{dt^2} + A_1 \frac{d}{dt} + A_2\right)i = 0$$

or

$$(A_0p^2 + A_1p + A_2)i = 0 \tag{19–23}$$

This expression may be written briefly in the form

$$f(p)i = 0 \tag{19–24}$$

where

$$f(p) = A_0p^2 + A_1p + A_2 \tag{19–25}$$

If m is substituted for p and $f(m)$ is then set equal to zero, the result is

$$f(m) = A_0m^2 + A_1m + A_2 = 0 \tag{19–26}$$

The expression thus obtained is similar to that of equation (19–15). The substitution of m for p gives a better interpretation for the roots of equation (19–26). Hence, the complementary function is, from equation (19–16),

$$i_{CF} = K_1e^{m_1t} + K_2e^{m_2t} \tag{19–27}$$

where m_1 and m_2 are the distinct roots of equation (19–26).

The advantage of this method of finding the *complementary function* of equation (19–12) lies in the fact that it is necessary only to remember the following process:

(*a*) Write the equation in the symbolic form

$$(A_0p^n + A_1p^{n-1} + A_2p^{n-2} + \ \ldots \ + A_n)i = 0 \tag{19–28}$$

(*b*) Substitute m for p and set $f(m)$ equal to zero, the result being

$$A_0m^n + A_1m^{n-1} + A_2m^{n-2} + \ \ldots \ + A_n = 0 \tag{19–29}$$

(*c*) Find the roots of equation (19–29).

(*d*) Write the solution as follows:

$$i = K_1e^{m_1t} + K_2e^{m_2t} + K_3e^{m_3t} + \ \ldots \ + K_ne^{m_nt} \tag{19–30}$$

where, m_1, m_2, ... m_n are the distinct roots of equation (19–29).

In circuit problems, the different solutions of the complementary function, or $K_1e^{m_1t}$, $K_2e^{m_2t}$, . . . $K_ne^{m_nt}$, are the *transient terms.* If, for example, m_1 is negative, then the transient term $K_1e^{m_1t}$ decreases to zero exponentially with increasing value of t.

Example 19–3.—A constant emf of 12 volts is suddenly applied across the terminals of a series circuit consisting of a resistance $R=4$ ohms and an inductance $L=2$ henrys. Find the transient term of the current flowing in the circuit, considering time to be zero at the instant when the circuit is completed.

Solution.—The differential equation for this circuit is

$$2\frac{di}{dt}+4i=12$$

or

$$\frac{di}{dt}+2i=6$$

The transient term is found by solving for the complementary function, which is obtained by setting the left-hand side equal to zero. Thus,

$$\frac{di}{dt}+2i=0$$

or

$$(p+2)i=f(p)i=0$$

where $f(p)=p+2$.

Substituting m for p in $f(p)=p+2$ and setting $f(m)=0$ gives the auxiliary equation

$$f(m)=m+2=0$$

The root is $m_1=-2$. Substituting this value in equation (19–30) gives the transient term i_t; that is,

$$i_t=i_{CF}=K_1e^{-2t}$$

Since m is negative, this term decreases exponentially to zero as t increases.

This value of i_t is not the complete solution of the original equation, since the right-hand side of the equation considered is not zero. (See Example 19–7 for the complete solution.)

19–7. When the Auxiliary Equation Has Equal Roots.—A differential equation of the nth order may have the form

$$(A_0p^n+A_1p^{n-1}+A_2p^{n-2}+\ .\ .\ .\ +A_n)i=0 \qquad \textbf{(19–28)}$$

Its general solution *must contain n arbitrary constants.* When two roots of equation (19–29) are equal, or $m_1=m_2=m_0$, a solution is

$$i=(K_1+K_2)e^{m_0t}+K_3e^{m_3t}+\ .\ .\ .\ +K_ne^{m_nt}$$

or

$$i=Ke^{m_0t}+K_3e^{m_3t}+\ .\ .\ .\ +K_ne^{m_nt} \qquad \textbf{(19–31)}$$

since (K_1+K_2) is equivalent to a single constant K. The solution of equation (19–31) has $(n-1)$ arbitrary constants and, hence, is not a general solution of equation (19–28).

If we try $i = te^{m_0 t}$, we will find by substitution that it satisfies equation (19–28). Thus, the general solution of equation (19–28) when equation (19–29) contains two equal roots is

$$i = K_1 e^{m_0 t} + K_2 t e^{m_0 t} + K_3 e^{m_3 t} + \dots + K_n e^{m_n t} \qquad (19\text{–}32)$$

When the auxiliary equation (19–29) contains three equal roots, the solution of equation (19–28) is

$$i = K_1 e^{m_0 t} + K_2 t e^{m_0 t} + K_3 t^2 e^{m_0 t} + K_4 e^{m_4 t} + \dots + K_n e^{m_n t} \qquad (19\text{–}33)$$

Example 19–4.—Find the solution of

$$\frac{d^2 i}{dt^2} + 4\frac{di}{dt} + 4i = 0$$

Solution.—The differential equation may be written in the form

$$(p^2 + 4p + 4)i = f(p)i = 0$$

where $f(p) = p^2 + 4p + 4$. Substituting m for p in $f(p) = p^2 + 4p + 4$ and setting $f(m)$ equal to 0 gives the auxiliary equation

$$m^2 + 4m + 4 = 0$$

The roots of this equation are

$$m_1 = -2 \quad \text{and} \quad m_2 = -2$$

Since the right-hand side of the original equation is 0, substitution of the values of m_1 and m_2 in equation (19–32) gives the general solution

$$i = K_1 e^{-2t} + K_2 t e^{-2t}$$

It is of interest to show that $K_2 t e^{-2t}$, on substitution, satisfies the original equation. Let

$$i = K_2 t e^{-2t}$$

Then

$$\frac{di}{dt} = K_2[t \cdot (-2e^{-2t}) + e^{-2t} \cdot 1] = -2K_2 t e^{-2t} + K_2 e^{-2t}$$

and

$$\frac{d^2 i}{dt^2} = 4K_2 t e^{-2t} - 4K_2 e^{-2t}$$

Substituting these values in the equation gives

$$(4K_2 t e^{-2t} - 4K_2 e^{-2t}) + 4(-2K_2 t e^{-2t} + K_2 e^{-2t}) + 4(K_2 t e^{-2t}) = 0$$

19–8. When the Auxiliary Equation Has Imaginary Roots.—If the auxiliary equation has a pair of imaginary roots,* $m_1 = a + jb$ and $m_2 = a - jb$, the solution of equation (19–28) is:

$$i = K_1 e^{at + jbt} + K_2 e^{at - jbt} + K_3 e^{m_3 t} + \dots$$
$$= K_1 e^{at} e^{jbt} + K_2 e^{at} e^{-jbt} + K_3 e^{m_3 t} + \dots$$

or

$$i = e^{at}(K_1 e^{jbt} + K_2 e^{-jbt}) + K_3 e^{m_3 t} + \dots \qquad (19\text{–}34)$$

* Imaginary roots come in conjugate pairs, as $a + jb$ and $a - jb$. That is, a cubic equation may have three real roots or one real root and two imaginary roots. See *First Course in the Theory of Equations* by L. E. Dickson.

Also,

$$K_1 e^{jbt} = K_1 \cos bt + jK_1 \sin bt$$
$$K_2 e^{-jbt} = K_2 \cos bt - jK_2 \sin bt$$

These values may be added to give

$$K_1 e^{jbt} + K_2 e^{-jbt} = (K_1 + K_2) \cos bt + (jK_1 - jK_2) \sin bt$$
$$= A \cos bt + B \sin bt$$

where $A = K_1 + K_2$ and $B = j(K_1 - K_2)$. Thus, equation (19–34) may be written as follows:

$$i = e^{at} (A \cos bt + B \sin bt) + K_3 e^{m_3 t} + \ldots \qquad (19\text{–}35)$$

Example 19–5.—Solve

$$\frac{d^2 i}{dt^2} + 2 \frac{di}{dt} + 10i = 0$$

Solution.—The auxiliary equation is

$$m^2 + 2m + 10 = 0$$

and the roots of this equation are

$$m_1 = -1 + j3 \quad \text{and} \quad m_2 = -1 - j3$$

Substituting values in equation (19–34) gives

$$i = e^{-t}(K_1 e^{j3t} + K_2 e^{-j3t})$$

From equation (19–35),

$$i = e^{-t}(A \cos 3t + B \sin 3t)$$

where $A = K_1 + K_2$ and $B = j(K_1 - K_2)$.

19–9. Particular Integral.—A *particular integral* is the part u of the complete solution $i = u + Kv$, and it contains no arbitrary constant. It is a particular solution i_{PI} of equation (19–11), which is

$$A_0 \frac{d^n i}{dt^n} + A_1 \frac{d^{n-1} i}{dt^{n-1}} + A_2 \frac{d^{n-2} i}{dt^{n-2}} + \ldots + A_n i = X \qquad (19\text{–}11)$$

That is, when i_{PI} is substituted in the equation, the result is X on the right-hand side.

When the transient terms in Kv become negligible in an actual circuit problem, the remaining part of the solution is u or the steady-state value. Thus, the steady-state solution will satisfy the differential equation of the circuit. Since a particular integral is a particular solution, the method of determining a particular integral by the steady-state solution should be followed whenever it is convenient to do so.

Example 19–6.—Find a particular integral for the circuit in Example 19–3

Solution.—The steady-state solution i_s, by ordinary circuit theory (that is, when $t \to \infty$), is

$$i_s = i_{PI} = \frac{E}{R} = \frac{12}{4} = 3$$

It can be seen that, if 3 is substituted for i, the original equation $2\frac{di}{dt} + 4i = 12$ will be satisfied.

19–10. Complete Solution of Linear Equations of the *n*th Order With Constant Coefficients.—The general form given as equation (19–11) is

$$A_0 \frac{d^n i}{dt^n} + A_1 \frac{d^{n-1} i}{dt^{n-1}} + A_2 \frac{d^{n-2} i}{dt^{n-2}} + \ . \ . \ . \ + A_n i = X$$

Its general solution is the sum of two parts, namely, the complementary function i_{CF} and a particular integral i_{PI}. Thus,

$$i = i_{CF} + i_{PI} = i_t + i_s \qquad (19\text{–}36)$$

The complementary function is obtained by solving the differential equation (19–11) with X replaced by zero. It is that part of the solution which involves the arbitrary constants. In a differential equation of the nth order, there must be n arbitrary constants. In the solution of an electrical-circuit problem, the different solutions of the complementary function are the transient terms. If all the roots of the auxiliary equation are distinct, then the complementary function has the form

$$i_{CF} = K_1 e^{m_1 t} + K_2 e^{m_2 t} + \ . \ . \ . \ + K_n e^{m_n t} \qquad (19\text{–}17)$$

A particular integral i_{PI} is a particular solution of equation (19–11). It contains no arbitrary constant. In a circuit problem, the steady-state solution or the steady-state term is a particular integral. Thus, the complete solution of equation (19–11) may be written as follows:

$$i = K_1 e^{m_1 t} + K_2 e^{m_2 t} + \ . \ . \ . \ + K_n e^{m_n t} + i_{PI} \qquad (19\text{–}37)$$

This is obvious because the substitution of $(K_1 e^{m_1 t} + \ . \ . \ . \ + K_n e^{m_n t})$ will give zero on the right-hand side of equation (19–11); and i_{PI}, being a particular solution, will give X.

Example 19–7.—Find the complete solution of the current in the circuit of Example 19–3.

Solution.—From Example 19–3, the complementary function is

$$i_{CF} = i_t = K_1 e^{-2t}$$

From Example 19–6, which considers the circuit in Example 19–3, a particular integral is

$$i_{PI} = i_s = \frac{E}{R} = \frac{12}{4} = 3$$

Hence, the complete solution is

$$i = i_{CF} + i_{PI} = i_t + i_s = K_1 e^{-2t} + 3$$

PROBLEMS

19–1. Give an example of a differential equation, with constant coefficients and the right-hand member zero, of : (*a*) the first degree and fourth order and (*b*) the second degree and second order.

19–2. Give an example of a differential equation, with constant coefficients and the right-hand member a constant, of: (*a*) the second degree and third order and (*b*) the third degree and fifth order.

19–3. Give an example of a linear differential equation of the third order with constant coefficients and the right-hand member a sine function.

19–4. An unvarying emf of 10 volts is suddenly impressed across a series circuit consisting of $R = 2$ ohms and $L = 2$ henrys. Considering $t = 0$ when the circuit is closed, find the complementary function of the current i. Solve the problem by letting $i = e^{mt}$.

19–5. Show that there are two arbitrary constants in the general solution of the following second-order linear differential equation:

$$\frac{d^2i}{dt^2} - \frac{di}{dt} = 0$$

19–6. Show that there are three arbitrary constants in the general solution of a linear differential equation of the third order with constant coefficients.

19–7. The three solutions of $\frac{d^2i}{dt^2} - 2\frac{di}{dt} - 8i = 0$ are K_1e^{4t}, K_2e^{-2t}, and $K_1e^{4t} + K_2e^{-2t}$. Show, by substitution, that each satisfies the original equation. Which is the general solution?

19–8. Write each of the following equations in the symbolic form, or p-form:

(*a*) $\frac{d^2i}{dt^2} - 4i = 0$

(*b*) $\frac{d^3i}{dt^3} - 3\frac{d^2i}{dt^2} + 2\frac{di}{dt} = E$

(*c*) $\frac{d^2i}{dt^2} - 5\frac{di}{dt} + 4i = 0$

(*d*) $\frac{d^3i}{dt^3} - 6\frac{d^2i}{dt^2} + 11\frac{di}{dt} - 6i = E \sin \omega t$

(*e*) $\frac{d^2i}{dt^2} - 5\frac{di}{dt} + 4i = Ee^{j\omega t}$

19–9. Write the following equations in the symbolic form, or p-form:

$$(a)\quad Ri+L\frac{di}{dt}+\frac{1}{C}\int i\,dt=E$$

$$(b)\quad 10i+\frac{1}{2}\int i\,dt=100\sin 377t$$

$$(c)\quad 5i+3\frac{di}{dt}+\frac{1}{2}\int i\,dt=200e^{j377t}$$

19–10. Simplify the expressions in Problem 19–9 by differentiation, and write the results in the p-form.

19–11. Find the complementary function of each of the following:

$(a)\quad (p-5)(p-1)i=0$

$(b)\quad (p-5)(p-1)i=E$

$(c)\quad (p-5)(p-1)i=E\sin \omega t$

$(d)\quad (p-5)(p-1)i=Ee^{j\omega t}$

What conclusion can be drawn regarding the form of the complementary function?

19–12. Find the complementary functions of the equations in Problem 19–8.

19–13. Solve for the transient terms of the equation $\frac{d^2i}{dt^2}-10\frac{di}{dt}+25i=0$. What is the complementary function?

19–14. Determine the transient terms of the equation

$$(p-2)(p-2)(p-2)i=0$$

What is the complementary function?

19–15. Calculate the transient terms in the expression $\frac{d^2i}{dt^2}+3i=0$. What is the complementary function?

19–16. A battery of 50 volts is impressed across a series circuit consisting of $R=2$ ohms, $L=0.01$ henry, and $C=100\mu$f. Determine the part u, or a particular integral, by the "steady-state solution" method.

19–17. A coil consisting of $R=3$ ohms and $L=0.1$ henry is connected to a battery of $E=10$ volts. Find the part u, or a particular integral. Write the complete expression for the current without determining the value for the arbitrary constant.

19–18. Repeat Problem 19–17 if the battery is replaced by a sinusoidal source of emf for which $e=100\sin 377t$.

CHAPTER 20

DIRECT-CURRENT TRANSIENTS

In ordinary electrical problems, currents and potential differences, whether continuous or alternating, are treated for the steady-state condition. That is, a sufficient time has elapsed for the currents and the potential differences to reach their steady-state values after the circuit is established. Just after the moment of closing the circuit, the currents and potential differences go through a transition state. The transients also appear when the circuit is opened or the circuit conditions are changed in any other way, as by a change of load, a change of impedance, etc. Whether of voltage or of current, a transient is usually a non-repeating, exponential function of time if the steady-state condition is constant; or it is an exponential superimposed upon a harmonic function of time if the steady-state condition is harmonic in character.

Electric transients do not occur in circuits containing resistance only. In such an R circuit, the current rises at once to the steady-state value $\frac{E}{R}$ at the instant when an unvarying emf E is applied to the circuit. Transients appear only in circuits containing inductance or capacitance, which circuit elements represent storage of energy in the electromagnetic or electrostatic form.

Electric transients may be classified into two groups, namely, single-energy transients and double-energy transients. Single-energy transients occur in circuits in which energy is stored in one form only (either electromagnetic or electrostatic). These occur in circuits containing R and L or R and C. Double-energy transients occur in circuits containing R, L, and C, in which energy must be stored in both forms.

In general, transients accompany any change in either the amount or the form of the energy stored in an electric circuit. The more the resistance predominates, the less are the severity and duration of the transient terms. This condition is the result of the dissipation of energy by the resistance.

The purpose of this chapter is primarily to study the behavior of each of the following circuits when it is suddenly connected to a battery of voltage E: (*a*) an RL circuit, (*b*) an RC circuit, (*c*) an RLC circuit, (*d*) a parallel circuit, and (*e*) a series-parallel circuit.

20–1. The Transient Period.—When an automobile comes slowly to a stop after its brakes are applied, kinetic energy is being changed to heat through the agency of friction. Energy changes also take place in electric circuits. In a circuit containing a storage battery and an inductive coil, energy in the battery is transformed into magnetic energy. Such changes in the form of the energy, it seems, cannot take place instantly. For example, time is required for a battery to bring the current in an inductive circuit to its steady-state value; it must add gradually to the energy stored in the magnetic field as the current increases. The interval between periods of stable conditions is generally spoken of as the *transient period.*

The length of the transient period is indefinite. Each transient term is usually represented by an exponential function of time with a negative exponent. These terms do not become zero until the variable time is infinite. Thus, it becomes necessary to agree that the transient change has ceased when it is no longer measurable, or is negligible. In general, the duration of the transition state is but a very few seconds.

20–2. The *RL* Circuit.—In an RL circuit, both the elements R and L affect the current that will flow at any instant. For dynamic equilibrium, the sum of the Ri drop and the counter emf $L\frac{di}{dt}$ must at all times equal the applied emf.

An example of transient current in an inductive circuit is the changing of the field current in an electric generator when the field circuit is changed. Another example is the change in armature current of an alternator when there is a line-to-line short circuit.

The following two cases will be considered: (*a*) when the circuit is closed; and (*b*) when, after the steady-state condition has been reached, the impressed emf is removed and the circuit is shorted simultaneously.

20–3. Closing of an *RL* Circuit.—When an unvarying emf E is impressed across the terminals of a circuit containing R and L in series, Fig. 20–1(*a*), the self-inductance (electrical inertia) of the circuit requires that the current be zero at the instant at which the switch is closed. The current i then increases gradually to its steady-state value $\frac{E}{R}$, as shown in Fig. 20–1(b). This gradual increase is due to the induced emf $e_i = -L\frac{di}{dt}$ opposing the battery voltage.

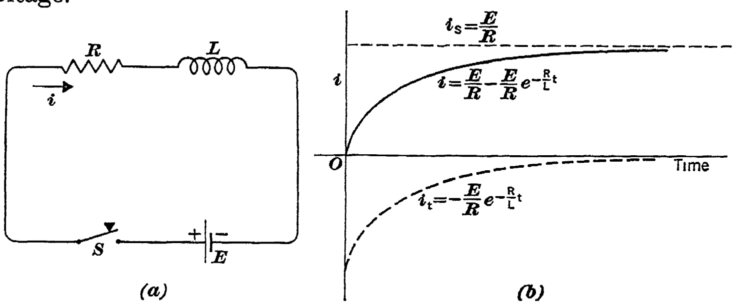


FIG. 20–1

When the current in the circuit is zero (at the instant at which the switch is closed), the Ri drop is zero and the entire battery voltage is across the inductance. With the current increasing gradually, part of the battery voltage is consumed in the resistance drop and the rest is consumed in the inductance. By Kirchhoff's Emf Law, the sum of the two counter voltages must at all times equal the applied emf, or

$$L\frac{di}{dt} + Ri = E \qquad (20\text{–}1)$$

(a) *Equation for Current:* Equation (20–1) may be written in the form

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \qquad (20\text{–}2)$$

or, symbolically, in the form

$$\left(p + \frac{R}{L}\right)i = f(p)i = \frac{E}{L} \qquad (20\text{–}3)$$

where $f(p) = p + \frac{R}{L}$. For the sake of simple analysis the actual current is separated into two components. Thus,

$$i = i_t + i_s \qquad (20\text{–}4)$$

where i_t = transient term, or the complementary function;
i_s = steady-state term, or the particular integral.

The transient term is found by substituting m for p in $f(p) = p+\frac{R}{L}$ and putting $f(m) = 0$. The result is

$$m+\frac{R}{L}=0$$

The root is

$$m_1 = -\frac{R}{L}$$

and, from equation (19–30),

$$i_t = Ke^{-\frac{Rt}{L}} \tag{20–5}$$

The steady-state term is the steady-state solution of the problem. The self-inductance of the circuit is apparent only when the current is increasing or decreasing. For the steady-state condition, the current is practically constant and the effect of the inductance L on current flow is negligible. Hence, the final value of the current is

$$i_s = \frac{E}{R} \tag{20–6}$$

and the complete equation for the current is

$$i = i_t + i_s = Ke^{-\frac{Rt}{L}} + \frac{E}{R} \tag{20–7}$$

The value of the constant* K may be determined from the physical set-up or the boundary conditions of the problem. At the instant when the switch S is closed or time $t=0$, the current is zero because of the inductance. With $i=0$, it is evident that the resistive drop Ri at that instant is zero. The applied emf must then, by Kirchhoff's Emf Law, be equal to the voltage drop $e_L = L\frac{di}{dt}$ across the inductance.

With the boundary conditions $t=0$, $i=0$, and $E = L\frac{di}{dt}$, equation (20–7) becomes

$$0 = K + \frac{E}{R}$$

* The constant K is determined with the complete equation of the current. It must not be determined with the complementary function when there is a steady-state solution.

from which

$$K = -\frac{E}{R}$$

Hence, the actual current flowing in the circuit after the closing of the switch is

$$i = \frac{E}{R} - \frac{E}{R} e^{-\frac{Rt}{L}} = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \tag{20–8}$$

The transient term, or the exponential term, approaches zero exponentially with time. Theoretically the actual current does not reach its final value $\frac{E}{R}$ until $t = \infty$; but actually, in most cases, it attains the so-called steady-state value within 1 second or less.

(*b*) *Time Constant of the Circuit:* It will be seen that the ratio $\frac{L}{R}$ governs the time required for the current determined by equation (20–8) to reach a certain percentage of its steady-state value. The larger the inductance of the circuit, the more time will the current take to reach its steady-state value and the longer will be the duration of the transient. The converse is true for the resistance.

The expression $\frac{L}{R}$ is called the time constant* of the circuit. When $t = \frac{L}{R}$, equation (20–8) reduces to

$$i = \frac{E}{R}(1 - e^{-1}) = \frac{E}{R} \cdot \frac{e - 1}{e} \tag{20–9}$$

or

$$i = \frac{2.718 - 1}{2.718} \cdot \frac{E}{R} = 0.632 \frac{E}{R} \tag{20–10}$$

That is, after an interval of time $t = \frac{L}{R}$, measured from $t = 0$, the current will reach a value which is 63.2 per cent of its final value $\frac{E}{R}$. Thus, the time constant may be defined as the length of time required for the current to reach 63.2 per cent of its final value. This time constant offers a convenient method of comparing *RL* circuits.

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* It can be shown by dimensional analysis that $\frac{L}{R}$ has the dimension of time.

The time constant of the circuit may also be defined as the time required for the current to reach the value $\frac{E}{R}$ if it increases at the same rate as the initial rate $\left(\frac{di}{dt}\right)_{t=0}$. The initial rate of current growth is the rate of growth at the instant the switch is closed. This is obtained by differentiating the current in equation (20–8) with respect to time and setting t equal to 0. The result is

$$\frac{di}{dt}=\frac{E}{R}\cdot\frac{R}{L}e^{-\frac{Rt}{L}}$$

from which

$$\left(\frac{di}{dt}\right)_{t=0}=\frac{E}{L} \tag{20–11}$$

Thus,

$$\text{Time constant}=\frac{\frac{E}{R}}{\left(\frac{di}{dt}\right)_{t=0}}=\frac{\frac{E}{R}}{\frac{E}{L}}=\frac{L}{R} \tag{20–12}$$

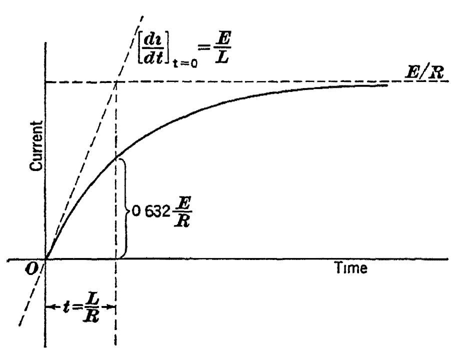

FIG. 20–2

It follows from examining Fig. 20–2 that the time constant is the length of time that would be required for the current to reach its maximum value if it continued to increase at the initial rate.

(*c*) *Voltage-Time Relations During Current Growth:* The voltage drop across the resistance is directly proportional to the current i and, of course, follows the same variation with respect to time as does i. Thus,

$$e_R=Ri=E\left(1-e^{-\frac{Rt}{L}}\right) \tag{20–13}$$

The voltage drop across the inductance at any time is

$$e_L = L\frac{di}{dt} = Ee^{-\frac{Rt}{L}} \tag{20–14}$$

The graphs showing the drops Ri and $L\frac{di}{dt}$ are given in Fig. 20–3. At any instant, the sum of these drops is equal to E.

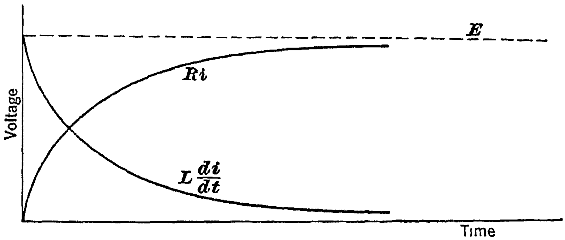

FIG. 20–3

(*d*) *Energy-Time Relation During the Current Growth:* When a battery is connected to an RL circuit, the flow of energy from the battery is proportional to the current. As the current begins to flow, part of the energy from the battery is stored in the inductance and the rest is dissipated in the resistance

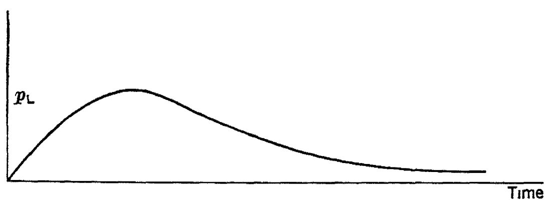

FIG. 20–4

The rate at which energy is being changed to heat is i^2R. The rate at which energy is stored in the magnetic field of the circuit is

$$p_L = e_L i = Li\frac{di}{dt} \tag{20–15}$$

Substituting for i its value from equation (20–8) gives

$$p_L = L\frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)\frac{E}{L}e^{-\frac{Rt}{L}}$$

or

$$p_L = \frac{E^2}{R}\left(e^{-\frac{Rt}{L}} - e^{-\frac{2Rt}{L}}\right) \tag{20–16}$$

The graph for equation (20–16) is shown in Fig. 20–4.

The instant at which the rate of energy storage in the magnetic field is a maximum may be found by differentiating equation (20–16) with respect to time, setting the result equal to zero, and then solving for t. Thus,

$$\frac{d}{dt}(p_L) = \frac{-E^2}{L} e^{-\frac{Rt}{L}} + \frac{2E^2}{L} e^{-\frac{2Rt}{L}} = 0$$

or

$$\frac{d}{dt}(p_L) = \frac{E^2}{L} e^{-\frac{Rt}{L}}\left(2e^{-\frac{Rt}{L}} - 1\right) = 0 \tag{20–17}$$

Hence,

$$\left(2e^{-\frac{Rt}{L}} - 1\right) = 0$$

from which

$$e^{-\frac{Rt}{L}} = 0.5$$

From tables of exponential functions,

$$\frac{Rt}{L} = 0.6931$$

and

$$t = 0.6931\,\frac{L}{R}$$

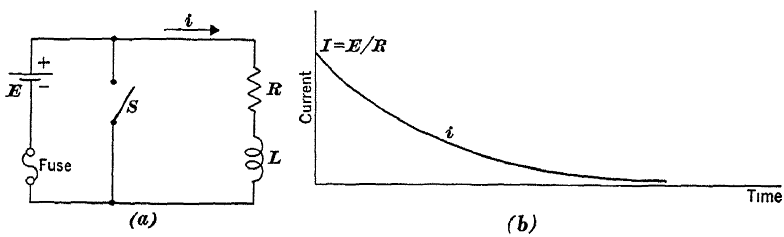

Fig. 20–5

Substituting 0.6931 for $\frac{Rt}{L}$ in equation (20–16) gives the maximum rate of energy storage. Thus,

$$(p_L)_{\max} = \frac{E^2}{R}(0.5 - 0.25) = 0.25I^2R \tag{20–18}$$

where $I = \frac{E}{R}$. Equation (20–18) states that the rate of energy storage in the magnetic field is never greater than 25 per cent of the maximum rate of energy dissipation which occurs when the current has reached its steady-state value.

The total energy stored in the magnetic field is the area under the curve shown in Fig. 20–4 and is equal to

$$W = \int_0^\infty p_L \, dt = \int_0^I Li \, di = \tfrac{1}{2}LI^2 \text{ joules} \qquad \textbf{(20–19)}$$

where L is assumed to be a constant.

20–4. Stopping of Current in an *RL* Circuit.—Let a steady current $I = \frac{E}{R}$ be flowing in the circuit, Fig. 20–5(*a*). At the instant at which the switch S is closed, or at time $t = 0$, the current will not at once fall to zero. The delay is due to the presence of inductance. As long as the steady-state current continues, energy ($W = \frac{1}{2}LI^2$) remains stored in the magnetic field. After the battery is removed from the circuit, the energy from the inductance tends to maintain the current.

It should be noted that, although the impressed emf may change instantly from 0 to E or from E to 0, the change in the current is gradual because of the inductance in the circuit. This is equivalent to inertia which prevents the current from changing its value from 0 to I or from I to 0 immediately.

With the applied emf E removed by the closing of switch S in Fig. 20–5(*a*), the equation of the circuit now reduces to

$$L\frac{di}{dt} + Ri = 0 \qquad \textbf{(20–20)}$$

or

$$(Lp + R)i = 0 \qquad \textbf{(20–21)}$$

The solution of equation (20–21) is

$$i = Ke^{-\frac{Rt}{L}} \qquad \textbf{(20–22)}$$

With the boundary conditions, or when $t = 0$, $i = \frac{E}{R}$, and $L\frac{di}{dt} = -Ri$, then the value for K is $\frac{E}{R}$. Hence, the solution of equation (20–20) is

$$i = \frac{E}{R}e^{-\frac{Rt}{L}} \qquad \textbf{(20–23)}$$

The graph of equation (20–23) is shown in Fig. 20–5(*b*).

20–5. The *RC* Circuit.—As in the case of the elements in an *RL* circuit, the elements R and C in an *RC* circuit affect the current flow at any instant. The two component voltages that will be operative in counterbalancing the applied emf are the Ri drop and the voltage drop $\frac{q}{C}$ of the condenser. The condenser is assumed *initially uncharged* in this chapter unless it is otherwise stated. Since charge is the accumulation of current, the voltage drop of the condenser may also be expressed as $\frac{1}{C}\int i\,dt$.

The following two cases will be considered: (*a*) when the condenser is charging, and (*b*) when the condenser is discharging.

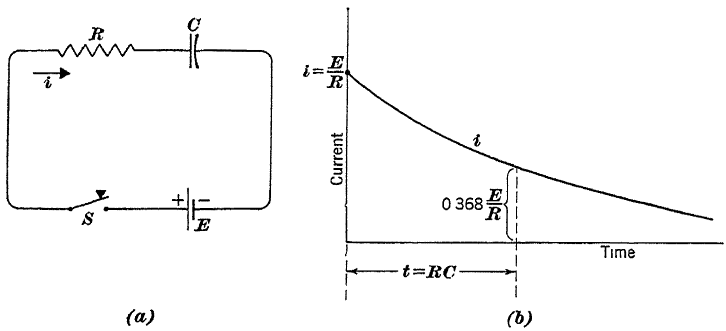

FIG. 20–6

20–6. Condenser Charging in an *RC* Circuit.—A continuous emf E is impressed upon a circuit containing R and C in series, as in Fig. 20–6(*a*). Since the circuit is assumed to possess zero self-inductance, the initial current cannot be predicted to be of zero value. At the instant when the switch S is closed, the condenser has no charge and acts as though its plates were short-circuited. Thus, for that instant, the circuit is one of pure resistance and the initial current is $i=\frac{E}{R}$. As time passes, charge accumulates on the plates of the condenser, until the condenser is fully charged and has a voltage between the plates equal and opposite to the impressed emf E. At the same time, the current is decreasing gradually to practically zero, as shown in Fig. 20–6(*b*), because of the increasing back voltage of the condenser.

The voltage drop across the resistance R is

$$e_R = Ri$$

and the voltage drop across the capacitance C is

$$e_C = \frac{1}{C}\int i\,dt$$

According to Kirchhoff's Emf Law, the fundamental differential equation for the circuit is

$$Ri + \frac{1}{C}\int i\,dt = E \qquad (20\text{–}24)$$

(*a*) *Current-Time Relation:* To find the expression for the current as a function of time, it is advantageous to differentiate equation (20–24) with respect to t. The result is

$$R\frac{di}{dt} + \frac{i}{C} = 0$$

or

$$\left(Rp + \frac{1}{C}\right)i = 0 \qquad (20\text{–}25)$$

The solution of this equation, which is also a solution of equation (20–24), is

$$i = Ke^{-\frac{t}{RC}} \qquad (20\text{–}26)$$

With the boundary conditions, or when $t=0$, then $e_C=0$ because the condenser is initially uncharged and $i = \frac{E}{R}$. The constant K is, therefore,

$$K = \frac{E}{R}$$

and

$$i = \frac{E}{R}e^{-\frac{t}{RC}} \qquad (20\text{–}27)$$

The graph of equation (20–27) is shown in Fig. 20–6(*b*).

(*b*) *Time Constant:* The time constant for this circuit is

$$\text{Time constant} = RC \qquad (20\text{–}28)$$

If this particular value of t is substituted in equation (20–27), the value of the current is

$$i = 0.368\frac{E}{R}$$

that is, RC seconds after the circuit is established, the current will be reduced to 36.8 per cent of its initial value.

(*c*) *Charge-Time Relation:* The condenser in the circuit, Fig. 20–6(*a*), acts as a storage for electrical energy. At the instant the circuit is closed, there will be no charge on the condenser, which is initially uncharged, and the voltage across it is zero. Since current is the flow of charge, the condenser plates will collect charge with the flow of current. As charge accumulates on the plates of the condenser, there will appear a voltage, $e_C = \frac{q}{C} = \frac{1}{C}\int i\,dt$, across the condenser. When the condenser voltage is equal and opposite to the applied emf, the current ceases to flow.

If we write $\frac{dq}{dt}$ for i, and q for $\int i\,dt$, equation (20–24) becomes

$$R\frac{dq}{dt} + \frac{q}{C} = E \tag{20–29}$$

This equation is similar to equation (20–1) with R corresponding to L and $\frac{1}{C}$ corresponding to R. Hence, the solution is of tne form (20–8), or

$$q = CE\left(1 - e^{-\frac{t}{RC}}\right) \tag{20–30}$$

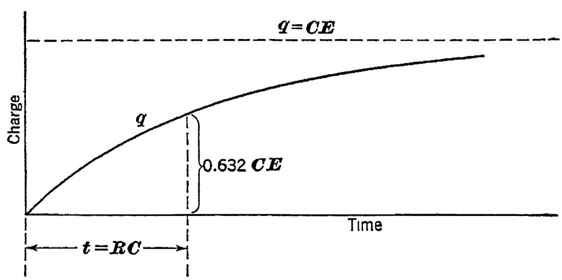

FIG. 20–7

Fig. 20–7 gives the curve for the charge. Since $i = \frac{dq}{dt}$, then differentiating equation (20–30) with respect to t gives equation (20–27).

If t is set equal to RC in equation (20–30),

$$q = (1 - 0.368)CE = 0.632CE \tag{20–31}$$

At this particular interval of time after the switch is closed, the charge is 63.2 per cent of its maximum value $q = CE$.

(*d*) *Voltage-Time Relation:* The voltage drop across the resistance R at any time is

$$e_R = Ri = Ee^{-\frac{t}{RC}} \tag{20–32}$$

Also, the voltage drop across the condenser at any time is proportional to the charge, or

$$e_C = \frac{q}{C} = E\left(1 - e^{-\frac{t}{RC}}\right) \tag{20–33}$$

(*e*) *Energy-Time Relations:* The rate of energy supply to the circuit is

$$p = Ei = \frac{E^2}{R} e^{-\frac{t}{RC}} \tag{20–34}$$

The power dissipated through the resistance is

$$p_R = Ri^2 = \frac{E^2}{R} e^{-\frac{2t}{RC}} \tag{20–35}$$

The rate of energy storage in the electric field of the condenser is

$$p_C = e_C i = \frac{E^2}{R}\left(e^{-\frac{t}{RC}} - e^{-\frac{2t}{RC}}\right) \tag{20–36}$$

The maximum rate of energy storage in the electric field occurs at an instant $t = 0.6931\ RC$ seconds after the circuit is closed. This interval is obtained in the following way:

$$\frac{d}{dt}(p_C) = -\frac{E^2}{R^2C} e^{-\frac{t}{RC}} + \frac{2E^2}{R^2C} e^{-\frac{2t}{RC}} = 0 \tag{20–37}$$

or

$$\frac{d}{dt}(p_C) = \frac{E^2}{R^2C} e^{-\frac{t}{RC}}\left(-1 + 2e^{-\frac{t}{RC}}\right) = 0$$

From this relation,

$$e^{-\frac{t}{RC}} = 0.5$$

or

$$t = 0.6931\ RC \text{ seconds} \tag{20–38}$$

The total energy stored in the electric field is the area under the curve for p_C, if plotted, and is equal to

$$W = \int e_C i\, dt = \int_0^Q \frac{q}{C}\, dq \tag{20–39}$$

or

$$W = \tfrac{1}{2}\frac{Q^2}{C} = \tfrac{1}{2}CE^2 \text{ joules} \tag{20–40}$$

20–7. Condenser Discharging in an RC Circuit.—Assume that the switch S, Fig. 20–8(a), has been closed in position M sufficiently long for the circuit to have reached the steady-state condition. That is, the voltage across the condenser is equal and opposite to the battery voltage E, the current is zero, and the energy stored in the condenser is $\frac{1}{2}CE^2$. When the switch is moved to position N, an initial current $i=\frac{E}{R}$ will flow through the short-circuited path. The direction of current flow will be opposite to that when the condenser is charging. If the positive sense of the charging current is retained, as indicated in Fig. 20–8(a), then the discharge current is considered negative. As the current flows, the charge on the condenser is reduced and the voltage is likewise decreased. With a reduction in the condenser voltage, the emf impressed across the resistance is less and, hence, the current is less. The continuation of the flow of current further discharges the condenser and so still further reduces the condenser voltage. The rate of discharge therefore decreases, owing to the decreasing potential, as shown in Fig. 20–8(b).

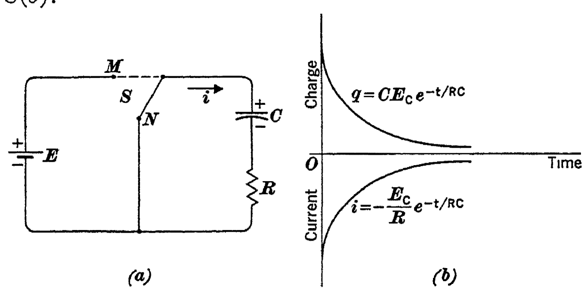

FIG. 20–8

(a) *Current-Time Relation:* When the switch in Fig. 20–8(a) is moved to position N, the expression for the current is

$$Ri+\frac{1}{C}\int i\,dt=0 \qquad (20\text{–}41)$$

Differentiating, we obtain

$$R\frac{di}{dt}+\frac{i}{C}=0$$

or

$$\left(Rp+\frac{1}{C}\right)i=0 \qquad (20\text{–}42)$$

The solution of this equation is

$$i = Ke^{-\frac{t}{RC}} \tag{20-43}$$

When $t=0$, then $i = -\frac{E_C}{R}$ (where E_C = voltage of condenser at the beginning of the discharge period and the minus sign indicates that the discharge current is flowing in the direction opposite to the assumed positive sense for the charging current). The value of K is

$$K = -\frac{E_C}{R}$$

Hence, the discharge current is

$$i = -\frac{E_C}{R} e^{-\frac{t}{RC}} \tag{20-44}$$

(*b*) *Charge-Time Relation:* The equation for the charge may be found by letting $i = \frac{dq}{dt}$ and $\int i\,dt = q$ in equation (20–41). Thus,

$$R\frac{dq}{dt} + \frac{q}{C} = 0 \tag{20-45}$$

or

$$\left(Rp + \frac{1}{C}\right)q = 0$$

The solution of this equation is

$$q = Ke^{-\frac{t}{RC}} \tag{20-46}$$

With the boundary conditions, or when $t=0$ and $q = CE_C$, then

$$K = CE_C$$

Hence, the equation for the charge on the condenser during discharge, as shown in Fig. 20–8(*b*), is

$$q = CE_Ce^{-\frac{t}{RC}} \tag{20-47}$$

20–8. The *RLC* Circuit.—Examination of the physical nature of an *RLC* circuit, Fig. 20–9, shows that the inductance requires the current to be zero at the instant at which the switch is closed. The series capacitance requires the final current also to be zero. The transient current which flows immediately after the closing of the switch will last for the period required for the condenser to acquire its steady-state charge. Under the steady-state condition, the voltage across the condenser is equal and opposite to the applied voltage. During the transient period there exists an interchange of energy between the emf source and L and C. It is possible to have free electric oscillations by selecting the proper values of L and C. The condenser is assumed to be initially uncharged.

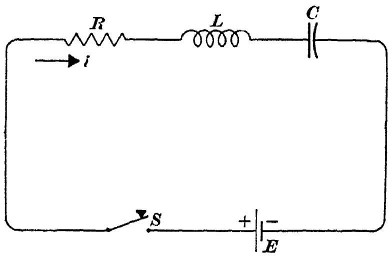

FIG. 20–9

The differential equation for a circuit of R, L, and C in series is

$$Ri+L\frac{di}{dt}+\frac{1}{C}\int i\,dt=E \tag{20–48}$$

Differentiating equation (20–48) and dividing through by L gives

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{i}{LC}=0 \tag{20–49}$$

or

$$\left(p^2+\frac{R}{L}p+\frac{1}{LC}\right)i=0 \tag{20–50}$$

The roots of the auxiliary equation $m^2+\frac{R}{L}m+\frac{1}{LC}=0$ are:

$$m_1=-\frac{R}{2L}+\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}=-a+b \tag{20–51}$$

$$m_2=-\frac{R}{2L}-\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}=-a-b \tag{20–52}$$

where

$$a=\frac{R}{2L} \tag{20–53}$$

$$b=\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}} \tag{20–54}$$

The transient current that flows may be a single surge, as shown in Fig. 20–10; or it may be a train of oscillations, as shown in Fig. 20–11. It is the amount of the resistance in the circuit, relative to the inductance and capacitance, that determines the shape of the transient current. There are three forms of solution, the proper one depending on whether $\left(\frac{R^2}{4L^2}\right)$ is greater than $\frac{1}{LC}$, equal to $\frac{1}{LC}$, or less than $\frac{1}{LC}$. These cases will be designated as follows:

(*a*) Non-oscillatory case: $\left(\frac{R^2}{4L^2}\right) > \frac{1}{LC}$

(*b*) Oscillatory case: $\left(\frac{R^2}{4L^2}\right) < \frac{1}{LC}$

(*c*) Critical case: $\left(\frac{R^2}{4L^2}\right) = \frac{1}{LC}$

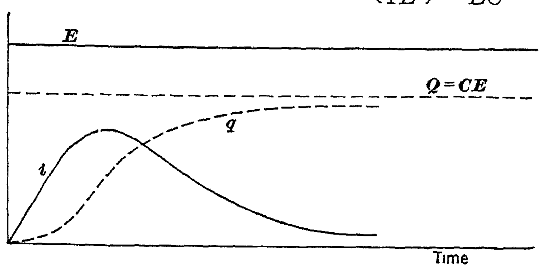

FIG. 20–10

When the roots of the auxiliary equation are real and unequal, as in case (*a*), the current is a single surge. If the roots are complex, as in case (*b*), the current is oscillatory. The change of form takes place in the critical case when the roots are real and equal.

20–9. Damping.—There is fundamentally no reason for selecting resistance as a factor which determines whether or not a circuit will allow the current to oscillate. But, for purposes of discussion, the element R has been chosen as the criterion.

In the analysis of the three cases mentioned in the preceding article, the factor $\epsilon^{-at} = \epsilon^{-\frac{Rt}{2L}}$ will appear in the expression for the currents. The exponent of ϵ in this factor is negative; this is physically necessary, for otherwise the current would increase without limit as t increases indefinitely. When R is large, the exponential factor $e^{-\frac{Rt}{2L}}$ is small.

If the exponential term $e^{-\frac{Rt}{2L}}$ did not appear in the current expression, there would be no damping. For this reason, $e^{-\frac{Rt}{2L}}$ is called the *damping factor* or *decrement factor*. When there is damping, as can be seen from Fig. 20–11, each crest is lower than the preceding one. Since the exponent contains the product of R and t, it is obvious that as R is increased the time of damping is decreased.

20–10. Non-Oscillatory Case.—The non-oscillatory case is sometimes called the overdamped case. The resistance in the circuit is greater than the critical value $R = 2\sqrt{\frac{L}{C}}$ which will make $\frac{R^2}{4L^2}$ equal to $\frac{1}{LC}$; and the form of the current is a single surge, as in Fig. 20–10.

At the instant at which the circuit is closed, the current is zero because of the inductance, the initially uncharged condenser acts like a short circuit, and the entire battery voltage is impressed across the inductance. The current will have a rapid rate of increase. As the current grows, there will appear resistance drop, voltage drop across the condenser due to the accumulation of charge, and inductance drop due to the change in current growth. All three of these voltage drops oppose the battery voltage during the interval when the current is increasing to its maximum value. After reaching this value, the current starts decreasing. The resistance drop and the condenser voltage are still opposing the battery voltage, and the energy stored in the inductance is actually helping the battery to force current through the circuit. As time increases indefinitely, the current decreases to zero and the condenser voltage is equal and opposite to the applied emf.

It is also interesting to consider the flow of energy in the circuit. The inductance receives energy from the battery while the current is increasing to its maximum value. This energy is gradually returned to the circuit as the current diminishes to zero. The condenser, on the other hand, receives energy continuously until it has stored the maximum value $\frac{1}{2}CE^2$. In the final analysis, the energy dissipated in the resistance comes from the battery.

The mathematical solution of this case involves two real and unequal roots of equation (20–50). This is so, since $\frac{R^2}{4L^2}$ is greater than $\frac{1}{LC}$. The solution is of the form

$$i = K_1 e^{(-a+b)t} + K_2 e^{(-a-b)t} \tag{20–55}$$

As two arbitrary constants are to be determined, it is necessary to set up two boundary conditions. When $t = 0$,

$$i = 0 \text{ and } E = L\frac{di}{dt}$$

The second condition is true because when $t = 0$, then $i = 0$, $e_R = Ri = 0$, and $e_C = 0$. Substituting the first boundary condition in equation (20–55) gives

$$0 = K_1 + K_2 \qquad \text{or} \qquad K_1 = -K_2 \tag{20–56}$$

To obtain $L\frac{di}{dt}$, differentiate equation (20–55) and multiply through by L. Thus,

$$L\frac{di}{dt} = L[K_1(-a+b)e^{(-a+b)t} + K_2(-a-b)e^{(-a-b)t}] \tag{20–57}$$

For $t = 0$, it reduces to

$$E = L\frac{di}{dt} = L[K_1(-a+b) + K_2(-a-b)]$$

$$= -aL(K_1+K_2) + bL(K_1 - K_2) = 2K_1 bL$$

or

$$K_1 = -K_2 = \frac{E}{2bL} \tag{20–58}$$

Substituting the values of K_1 and K_2 in equation (20–55) gives

$$i = \frac{E}{2bL}[e^{-at}\,e^{bt} - e^{-at}\,e^{-bt}] = \frac{Ee^{-at}}{bL}\left[\frac{e^{bt} - e^{-bt}}{2}\right]$$

or

$$i = \frac{Ee^{-at}}{bL}\sinh bt \tag{20–59}$$

where $\sinh bt$ = hyperbolic sine* of $bt = \frac{e^{bt} - e^{-bt}}{2}$.

The curve showing the variation of the current is given in Fig. 20–10. The charge on the condenser, or $q = \int i\,dt$, is the dotted curve. It can be seen from the current equation and Fig. 20–10 that, after the switch is closed, the current starts to increase at the

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* See Appendix G on Hyperbolic Functions.

rate of $\left(\frac{di}{dt}\right)_{t=0} = \frac{E}{L}$. It then increases with time to a maximum value and then decreases to zero. At the same time, q increases asymptotically to a maximum value $Q = CE$. Since $i = \frac{dq}{dt}$, the slope of the q-curve indicates the magnitude of the current. The potential across the condenser is equal and opposite to the applied emf when the steady-state condition is reached.

20–11. Oscillatory Case.—In the oscillatory case, the resistance in the circuit is less than the critical value, or $R < 2\sqrt{\frac{L}{C}}$ in order that $\frac{R^2}{4L^2} < \frac{1}{LC}$, and the form of the current is a train of oscillations, as shown in Fig. 20–11. At the instant when the circuit is closed, the current is zero. If the resistance is small, there is little opposition to the flow of current until a charge has accumulated on the condenser. The initial rate of current growth depends on the inductance in the circuit.

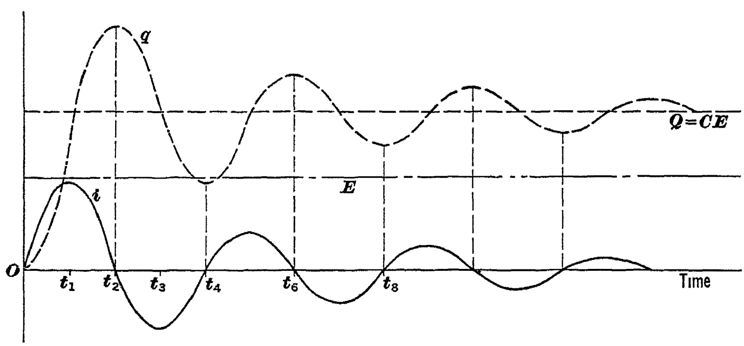


FIG. 20–11

While the current is growing during the interval from O to t_1, Fig. 20–11, energy is stored in the magnetic field and in the condenser. All three voltages, namely, the resistance drop, the inductance drop, and the condenser voltage drop, oppose the battery voltage. From t_1 to t_2, the current is decreasing and, since the slope $\frac{di}{dt}$ is now negative, the voltage across the inductance is opposite to that when the current is increasing. The energy $\frac{1}{2}Li^2$

stored in the inductance actually helps the battery to send current through the circuit and further charge the condenser. At time t_2, the condenser is charged to a voltage greater than the battery voltage; that is, $\frac{q}{C} > E$ or $q > CE$. The condenser must then discharge before dynamic equilibrium is reached. That is, since the battery voltage is not great enough to hold so much charge on the condenser, the charge will begin to flow out of the condenser and the result will be a negative current, as shown in the interval from t_2 to t_3. According to this line of reasoning, the oscillation of current continues indefinitely, with the amplitude of each crest being smaller than the preceding one. It seems that equilibrium is never quite reached.

The oscillatory nature of the current may be explained by considering the energy transfers that take place. When the current is decreasing, say from positive maximum to zero, energy is returned to the circuit from the magnetic field. This energy, minus a portion which is dissipated in the resistance, is stored in the electrostatic field. If at any time the energy stored in the electrostatic field exceeds the value $\frac{1}{2}CE^2$, the excess energy must be returned to the circuit before static equilibrium is obtained. Thus, the energy in the circuit, minus the portion in the form of i^2R loss, is stored alternately first in the inductance and then in the capacitance. The interchange of energy between the electromagnetic field and the electrostatic field thus causes the oscillatory current.

The mathematical solution may now be considered. When the resistance is less than the critical value, $\frac{R^2}{4L^2}$ is less than $\frac{1}{LC}$ and b has the form

$$b = \sqrt{-\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} = \sqrt{-1}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = jb'$$

where

$$b' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

The roots of the auxiliary equation, which are conjugate complex quantities, are

$$m_1 = -a + jb'$$
$$m_2 = -a - jb'$$

The solution, with the constants to be determined, is

$$i = K_1 e^{(-a+jb')t} + K_2 e^{(-a-jb')t} \tag{20–60}$$

Substituting the boundary conditions, or $t = 0$, $i = 0$, $e_R = Ri = 0$, $e_C = 0$, and $E = L\frac{di}{dt}$, as in the non-oscillatory case, we find that the constants have the following relation:

$$K_1 = -K_2$$

Differentiating equation (20–60) and multiplying through by L gives

$$L\frac{di}{dt} = L[K_1(-a+jb')e^{(-a+jb')t} + K_2(-a-jb')e^{(-a-jb')t}] \tag{20–61}$$

For $t = 0$, equation (20–61) reduces to

$$E = L\frac{di}{dt} = L[K_1(-a+jb') + K_2(-a-jb')]$$

$$= -aL(K_1+K_2) + jb'L(K_1-K_2) = 2K_1jb'L$$

from which

$$K_1 = -K_2 = \frac{E}{2jb'L}$$

Thus,

$$i = \frac{E}{2jb'L}[e^{-at}\,e^{jb't} - e^{-at}\,e^{-jb't}] = \frac{Ee^{-at}}{b'L}\left[\frac{e^{jb't} - e^{-jb't}}{2j}\right]$$

or

$$i = \frac{Ee^{-at}}{b'L}\sin b't \tag{20–62}$$

where

$$\sin b't = \frac{e^{jb't} - e^{-jb't}}{2j}$$

For this case, the current and charge curves are shown in Fig. 20–11. The current is oscillating and decreases to zero. The decreasing amplitudes of the successive current maxima are caused by the damping effect of the resistance in the circuit. The rate of decay is determined by the value of $\frac{R}{2L}$, which is sometimes called the attentuation constant of the circuit. The current has the form of a damped sinusoid, $\frac{E}{b'L}\sin b't$ multiplied by a damping factor e^{-at}. Its curve may be obtained by first drawing the sine wave $\frac{E}{b'L}\sin b't$ and then modifying the sine wave by e^{-at}. It can be seen in Fig. 20–11 that the time spacings from 0 to t_2, from t_2 to t_4, etc. along the time axis must be the same because the points of crossing are those of the sine wave. The frequency of oscillation may be considered as that given by the following expression (it being assumed that $\sin b't = \sin \omega t$):

$$\omega = 2\pi f = b' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

or

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \qquad (20\text{–}63)$$

It must be noted that the lobes of the curve in Fig. 20–11 are not lobes of a sine wave because of the decreasing value of e^{-at} with an increase in time, although they retain the general appearance of a sine wave.

The charge $q = \int_0^t i\,dt$ also oscillates, as shown in Fig. 20–11, approaching $Q = CE$ as a limiting value. Its curve may be obtained by plotting the area between the i-curve and the time axis against time as abscissa.

20–12. Critical Case.—The critical case is sometimes called the critically damped case. In the analysis of this case, the resistance R in the circuit is assumed to have the value $2\sqrt{\frac{L}{C}}$. The solution of this problem is not important since it is highly improbable that a circuit would ever have exactly the critical resistance.

The distinction between a circuit with critical resistance and another which is slightly overdamped is quite clear in the mathematical solution, but is not evident from the physical nature of the surges. If a series of oscillograms of the current surges were taken with the resistance decreased in steps to the critical value, there would be no distinguishing characteristic to identify the oscillogram for the critical case. The critical resistance is merely the lowest value that will permit a single surge in an RLC circuit.

The mathematical solution is quite simple. When $\frac{R^2}{4L^2} = \frac{1}{LC}$, the roots of the auxiliary equation are equal and the general form of solution is

$$i = K_1 e^{-at} + K_2 t e^{-at} \qquad (20\text{–}64)$$

With the boundary conditions, or when $t=0$, $i=0$, $e_R = Ri = 0$, $e_C = 0$, and $E = L\dfrac{di}{dt}$,

$$K_1 = 0$$

Equation (20–64) then reduces to

$$i = K_2 t e^{-at} \tag{20–65}$$

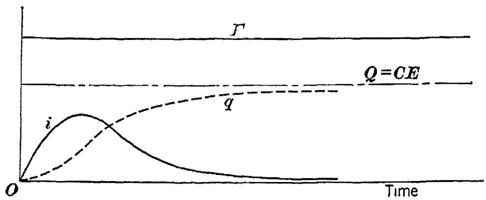


FIG. 20–12

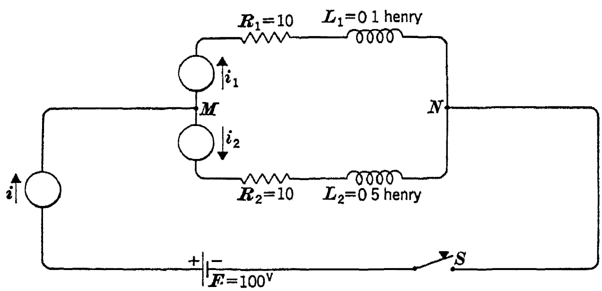


FIG. 20–13

Differentiating equation (20–65) and multiplying through by L gives

$$L\frac{di}{dt} = LK_2[-ate^{-at} + e^{-at}]$$

For $t=0$,

$$L\frac{di}{dt} = E = LK_2$$

or

$$K_2 = \frac{E}{L}$$

Hence, the complete solution for this case is

$$i = \frac{Et}{L}e^{-at} \tag{20–66}$$

The curves representing i and $q = \int i\,dt$ are shown in Fig. 20–12.

20–13. Parallel Circuits.—In the solution of parallel circuits, such as that shown in Fig. 20–13, it is only necessary to consider each branch separately and then apply Kirchhoff's Current Law at the point M to find the battery current.

Applying Kirchhoff's Emf Law, we have:

(a) Outer Circuit

$$R_1 i_1 + L_1 \frac{di_1}{dt} = E$$

$$\frac{di_1}{dt} + \frac{R_1}{L_1} i_1 = \frac{E}{L_1}$$

$$\frac{di_1}{dt} + 100\, i_1 = 1000$$

(b) Inner Circuit

$$R_2 i_2 + L_2 \frac{di_2}{dt} = E$$

$$\frac{di_2}{dt} + \frac{R_2}{L_2} i_2 = \frac{E}{L_2}$$

$$\frac{di_2}{dt} + 20\, i_2 = 200$$

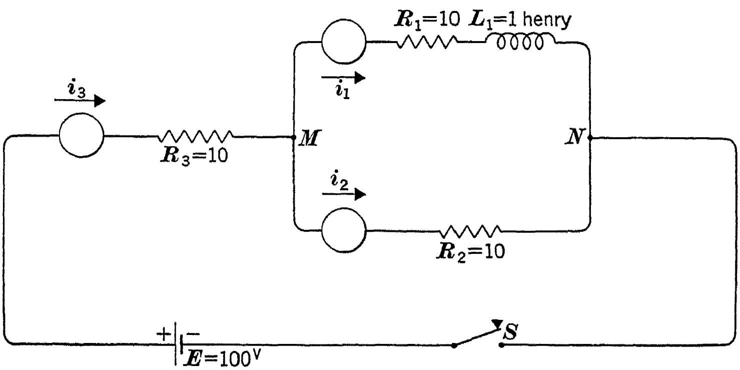


FIG. 20–14

These resulting equations are of the form of equation (20–2) and their solutions are given by equation (20–7). When values are substituted in equation (20–7), it is found that

$$i_1 = 10 + K_1 e^{-100t} \quad \text{and} \quad i_2 = 10 + K_2 e^{-20t}$$

Therefore,

$$L_1 \frac{di_1}{dt} = -10 K_1 e^{-100t} \quad \text{and} \quad L_2 \frac{di_2}{dt} = -10 K_2 e^{-20t}$$

When $t = 0$,

$$i_1 = 0 \quad \text{and} \quad L_1 \frac{di_1}{dt} = 100$$

Also,

$$i_2 = 0 \quad \text{and} \quad L_2 \frac{di_2}{dt} = 100$$

Then,

$$K_1 = -10 \quad \text{and} \quad K_2 = -10$$

Hence,

$$i_1 = 10 - 10e^{-100t}$$
$$i_2 = 10 - 10e^{-20t}$$

By applying Kirchhoff's Current Law at point M, we obtain

$$i = i_1 + i_2 = 20 - 10e^{-20t} - 10e^{-100t}$$

20–14. Series-Parallel Circuits.—A series-parallel circuit, Fig. 20–14, will now be considered. It is required to find the equations for the three currents. Applying Kirchhoff's Emf Law, we have:

(a) Outer Circuit

$$R_3i_3 + R_1i_1 + L_1\frac{di_1}{dt} = E$$

$$10i_3 + 10i_1 + \frac{di_1}{dt} = 100$$

(b) Inner Circuit

$$R_3i_3 + R_2i_2 = E$$

$$10i_3 + 10i_2 = 100$$

or $i_3 + i_2 = 10$

Applying Kirchhoff's Current Law at point M gives

$$i_3 = i_1 + i_2$$

If this value is substituted for i_3 and the expressions are written in the symbolic form, the circuit equations become

$$(20+p)i_1 + 10i_2 = 100$$
$$i_1 + 2i_2 = 10$$

These equations may be considered as simultaneous equations. The solution for i_1 is

$$i_1 = \frac{\begin{vmatrix} 100 & 10 \\ 10 & 2 \end{vmatrix}}{\begin{vmatrix} (20+p) & 10 \\ 1 & 2 \end{vmatrix}} = \frac{200-100}{40+2p-10} = \frac{100}{30+2p}$$

or

$$(30+2p)i_1 = 100$$

The complementary function, or the transient term, is

$$i_{1t} = K_1e^{-15t}$$

Also, the steady-state solution of i_1, by inspection of Fig. 20–14, is

$$i_{1s} = \frac{1}{2}\left(\frac{100}{10+5}\right) = \frac{10}{3}$$

Hence, the complete solution is

$$i_1 = \frac{10}{3} + K_1 e^{-15t}$$

With the boundary conditions, or when $t=0$, $i_1=0$, and $i_3=i_2=\frac{100}{10+10}=5$, the constant K_1 is $\frac{-10}{3}$. Therefore,

$$i_1 = \frac{10}{3} - \frac{10}{3} e^{-15t}$$

The current i_2 may be determined by solving the two simultaneous equations again. In this case, however, it can be found directly from the relation $i_1 + 2i_2 = 10$. Thus,

$$i_2 = \frac{10 - i_1}{2} = \frac{10}{3} + \frac{5}{3} e^{-15t}$$

Then,

$$i_3 = i_1 + i_2 = \frac{20}{3} - \frac{5}{3} e^{-15t}$$

PROBLEMS

20–1. An unvarying emf E is applied to an R circuit.

(*a*) Find the equation for the current, considering $t=0$ when the circuit is completed.

(*b*) Find the equation for the Ri drop across the resistance.

(*c*) Find the equation for the energy dissipated in the resistance.

(*d*) Sketch the graphs representing the equations in (*a*), (*b*), and (*c*). Use time as abscissa.

20–2. An unvarying emf E is applied to an L circuit.

(*a*) Find the equation for the current.

(*b*) Find the drop across the inductance at any instant.

(*c*) Find the equation for the energy stored in the magnetic field.

(*d*) Sketch graphs representing the current, the drop across the inductance, and the energy stored in the magnetic field.

(*e*) What is the initial rate of growth of the current?

20–3. An unvarying emf E is applied to a C circuit.

(*a*) What is the current-time relation?

(*b*) What is the charge-time relation?

(*c*) What is the relation for the voltage across the condenser at any instant?

(*d*) What is the relation giving the energy stored in the condenser at every instant?

(*e*) Sketch graphs for (*a*), (*b*), (*c*), and (*d*).

(*f*) What is the initial rate of current growth?

20–4. A coil having a resistance of 3 ohms and an inductance of 0.09 henry is connected across a battery of 20 volts. Plot the current and its two components. Assume that $t=0$ when the circuit is completed.

20–5. An air-core inductor for which $L = 0.05$ henry and the resistance is negligible is connected in series with the coil in Problem 20–4. Plot the currents of Problem 20–4 and Problem 20–5 on the same diagram Compare the rate of current growth and the duration of the transient.

20–6. A constant emf of 10 volts is applied to a coil for which $R = 5$ ohms and $L = 0.01$ henry. How long after the switch is closed will it take for the current to rise to one-half and three-fourths, respectively, of its final value? What is the time constant of this coil?

20–7. If, in Problem 20–6, $L = 2.0$ henrys, find the times as before. What is the time constant of the coil with this change? Draw the currents in Problems 20–6 and 20–7 on the same diagram. What conclusions can be made?

20–8. A coil has an inductance of 1 henry and a resistance of 5 ohms What resistance must be connected in series with the coil to have the current rise to one-half of its final value in 0.0694 second?

20–9. A circuit containing resistance and inductance is connected to a battery. The initial rate of growth of current is 20 amp per second, and the final value of current is 5 amp. What is the time constant?

20–10. A resistor of 5 ohms is connected in series with a coil having $R = 3$ ohms and $L = 0.01$ henry. If an unvarying emf of 50 volts is suddenly impressed across this combination, find the voltages across the resistor and the coil and plot these voltages as a function of time.

20–11. A coil for which $R = 3$ ohms and $L = 2$ henrys is connected across the terminals of a battery of 20 volts. The battery has negligible resistance. Find and plot: (*a*) the rate at which energy is dissipated in the resistance; (*b*) the rate at which energy is stored in the magnetic field; (*c*) the maximum amount of energy stored in the inductance. Explain in detail the shape of the power curve obtained in part (*b*).

20–12. A coil for which $R = 3$ ohms and $L = 0.2$ henry is connected to a 50-volt battery with negligible resistance. After the current has reached its steady-state value, the battery is removed and replaced simultaneously by a resistor of 2 ohms. Plot a graph showing the current growth and the decay of current. Also plot a curve showing the diminishing of the energy in the inductance.

20–13. A coil having a resistance of 5 ohms and an inductance of 1 henry is connected in series with a non-inductive resistance of 5 ohms. At $t = 0$, a constant emf of 100 volts is impressed on the circuit; 0.1 second later the applied emf is suddenly increased to 300 volts. How long will it take the current, from $t = 0$, to read 90 per cent of the final value with an applied voltage of 300 volts?

20–14. A resistor of 5 ohms and a condenser of 3 μf are connected in series to a 50-volt battery. The resistance of the battery is negligible. Plot the current as a function of time.

20–15. A resistor of 10 ohms and a condenser of 2 μf are connected in series across a 100-volt battery with negligible resistance. How long after the switch is closed will the current have one-half of its maximum value? What is the time constant of the circuit?

20–16. Repeat Problem 20–15 with a condenser of 100 μf. Plot on the same diagram the current-time and charge-time curves of Problems 20–15 and 20–16. What is the effect of increasing the capacitance of the condenser?

20–17. After the current in Problem 20–15 has attained its steady-state value, the battery is shorted out. Plot a graph showing the discharge current through the resistor.

20–18. A resistor of 5 ohms and a condenser of 50 μf are connected in series to an unvarying emf of 32 volts. Find and plot: (*a*) the rate at which energy is dissipated in the resistor; (*b*) the rate at which energy is stored in the condenser.

20–19. A 50-volt battery (negligible resistance) is connected across a circuit containing $R = 30$ ohms, $L = 0.05$ henry, and $C = 400$ μf in series. What case is this? Find the equations for the current i, the voltage $L\frac{di}{dt}$, and the charge q. Plot the three quantities as a function of time. If $L\frac{di}{dt}$ becomes negative, note the positions of the zero points with respect to the current. What is the action of a negative $L\frac{di}{dt}$ in the circuit?

20–20. The equation for the current in an overdamped circuit consisting of R, L, and C in series is

$$i = \frac{E}{2bL}(e^{-at}\,e^{bt} - e^{-at}\,e^{-bt})$$

Sketch a set of curves (the current and its two components) for the case when: (*a*) L is large and C is small; (*b*) L is small and C is large

20–21. In a slightly overdamped circuit where R is near $2\sqrt{\frac{L}{C}}$, sketch the current and its two components

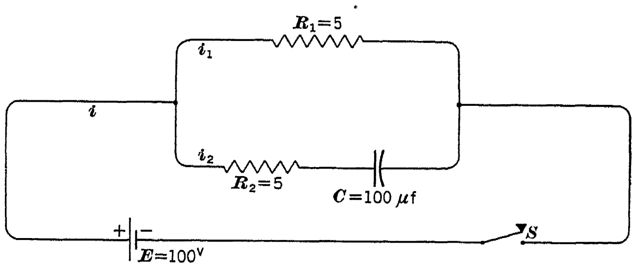


FIG. 20–15

20–22. A series circuit consisting of $R = 5$ ohms, $L = 0.1$ henry, and $C = 200$ μf is connected to a battery of 100 volts. What case is this? Find the expressions for the current and the charge. Plot the curves.

20–23. A circuit containing R, L, and C in series is connected to a battery of 32 volts. The inductance $L = 0.06$ henry, the capacitance $C = 300$ μf, and the resistance has the critical value. Find the equations for the current and the charge. Plot the curves.

20–24. Repeat Problem 20–23 with two values of resistance each slightly greater than the critical value. This problem is intended to show how the overdamped case approaches the critical case. Plot the curves for the three currents on one diagram and the curves for the three charges on another diagram.

20–25. A circuit has in series $R = 20$ ohms, $L = 0.5$ henry, $C = 20$ μf. The condenser is initially charged to a potential difference of 110 volts and then is allowed to discharge through this circuit. Write the equation for the transient current.

20–26. Determine the equations for the currents in the circuit shown in Fig. 20–15.

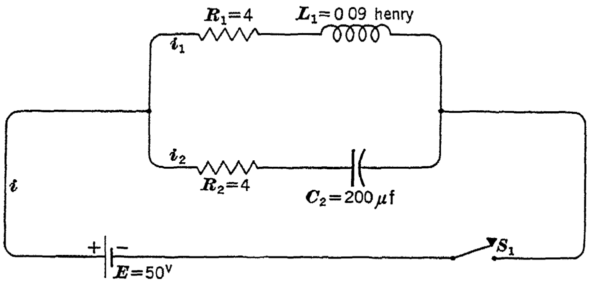


FIG. 20–16

20–27. Determine the equations for the currents in the circuit of Fig. 20–16.

20–28. Two non-inductive resistors of 20 ohms and 60 ohms, respectively, are connected in series across a 110-volt battery with negligible internal resistance. By means of a switch, a coil having $R = 50$ ohms and $L = 0.5$ henry is then suddenly shunted across the 60-ohm resistor. Consider time zero at the instant the switch is closed. (*a*) Find the equation for the potential drop across the coil as a function of time. (*b*) What is the initial rate of change of current in each of the resistors and the coil?

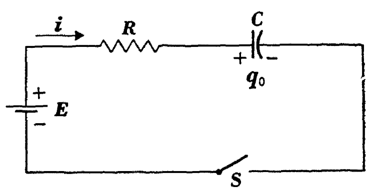


FIG. 20–17

20–29. A resistor of 5 ohms is inserted between the battery and the parallel circuit of Problem 20–27. Find the equations for the currents after the switch is closed.

20–30. Fig. 20–17 shows an RC circuit with an initial charge q_0 on the condenser. As indicated in the diagram, the initial charge on the condenser is such that the voltage $\frac{q_0}{C}$ opposes the battery voltage around the series loop. The instantaneous value of the charge is

$$q = \int_0^t i\,dt + q_0$$

Also, the circuit equation is

$$Ri + \frac{q}{C} = E$$

or

$$Ri + \frac{1}{C}\int_0^t i\,dt = E - \frac{q_0}{C}$$

Considering time $t = 0$ at the instant the switch is closed, show that the expression for the current is

$$i = \frac{E - \frac{q_0}{C}}{R} e^{-\frac{t}{RC}}$$

Discuss this expression when : (a) $\frac{q_0}{C} < E$; (b) $\frac{q_0}{C} > E$.

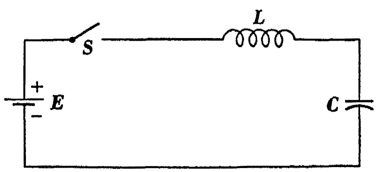


FIG. 20–18

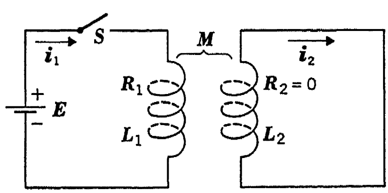


FIG. 20–19

20–31. For the conditions stated in Problem 20–30, find the expression for: (a) the voltage e_R across the resistance; (b) the voltage e_C across the condenser.

20–32. Repeat Problem 20–30 when the initial charge on the condenser is such that the voltage $\frac{q_0}{C}$ aids the battery voltage around the series loop.

20–33. In the non-oscillatory case of the RLC circuit, or when $\frac{R^2}{4L^2} > \frac{1}{LC}$, the battery current is

$$i = \frac{Ee^{-at}}{bL}\left[\frac{e^{bt} - e^{-bt}}{2}\right] = \frac{Ee^{-at}}{bL} \sinh bt$$

The graph for this current is shown in Fig 20-10 Show that the time corresponding to the maximum value of the current is

$$t = \frac{1}{b} \tanh^{-1} \frac{b}{a}$$

20-34. Fig 20-18 shows a circuit with no resistance and no initial charge on the condenser. Assume that time $t = 0$ when the switch S is closed. Find the expression for the energy stored in the inductance after the switch is closed

20-35. The coefficient of coupling of the two coils in Fig. 20-19 is $k = 0.5$. Considering time $t = 0$ when the switch S is closed, find the complete expression for the battery current i_1.

CHAPTER 21

ALTERNATING-CURRENT TRANSIENTS

When an alternating voltage is suddenly applied to a circuit, a certain interval of time is generally required for the current to adjust itself to the steady-state mode of variation. The form and amplitude of the steady-state current will depend on the applied emf, as is known from ordinary circuit theory. In general, as in the direct-current circuit, the transient terms will have the exponential form Ke^{mt}. These transient terms depend only on the physical nature of the circuit. The amplitude of the transient current in an alternating-current circuit is governed not only by the circuit elements and the amplitude of the applied voltage, but also by the instant of the alternating emf cycle at which the circuit is closed. An illustration of the effect of the time at which a circuit is closed is given in Art. 21–2.

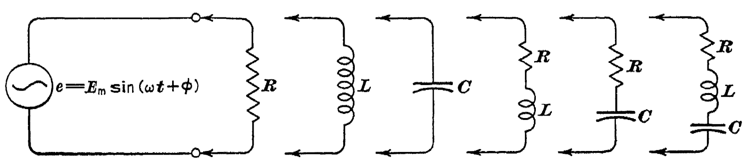


FIG. 21–1

In this chapter the circuit elements are assumed constant and only sinusoidal impressed emf waves will be considered. If the form of the applied emf differs materially from a sine wave, the original wave can be separated into sinusoidal components, as outlined in the chapter on non-sinusoidal waves. Components of current are then found for the different components of voltage and are added to obtain the resultant current. The Principle of Superposition applies when the circuit elements are constant. The circuits shown in Fig. 21–1 will be taken up in order from left to right.

21–1. The *R* Circuit.—A sinusoidal emf is impressed across a purely resistive circuit, Fig. 21–2. If $t = 0$ at the instant at which the switch is closed, and if the emf wave is going through zero and is increasing positively at that instant, as indicated in Fig. 21–3(*a*), the expression for the emf is $e = E_m \sin \omega t$. Kirchhoff's Emf Law gives

$$Ri = E_m \sin \omega t \tag{21–1}$$

from which

$$i = \frac{E_m}{R} \sin \omega t \tag{21–2}$$

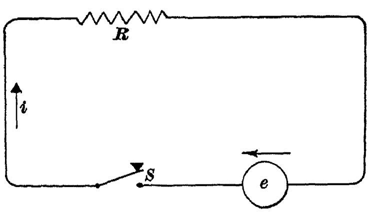


FIG. 21–2

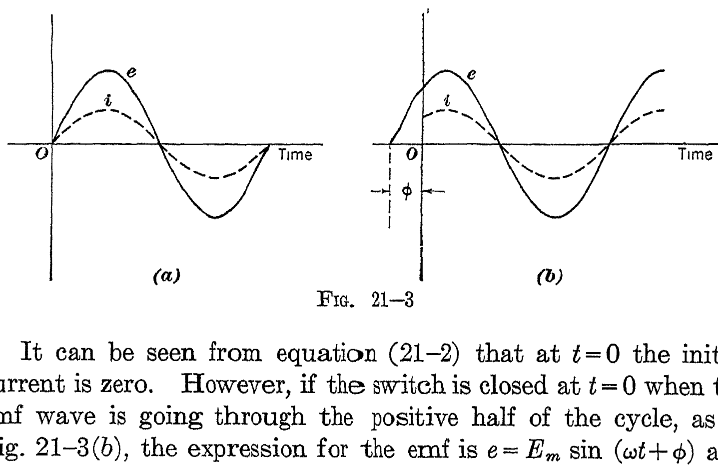


FIG. 21–3

It can be seen from equation (21–2) that at $t = 0$ the initial current is zero. However, if the switch is closed at $t = 0$ when the emf wave is going through the positive half of the cycle, as in Fig. 21–3(*b*), the expression for the emf is $e = E_m \sin (\omega t + \phi)$ and the initial current has a definite positive value. Equation (21–1) becomes

$$Ri = E_m \sin (\omega t + \phi) \tag{21–3}$$

and

$$i = \frac{E_m}{R} \sin (\omega t + \phi) \tag{21–4}$$

When the switch is closed during the negative half of the cycle, the initial current is negative. This means that ϕ has a value greater than 180 degrees.

Thus, it is seen that the time at which the switch is closed is a very important factor in determining the initial current. There will be no transient effect in an R circuit. Equation (21–4) gives the complete solution. After $t=0$, the current variation is of the form of the voltage variation.

21–2. The L Circuit.—If the resistance R is replaced by an inductance L in Fig. 21–2 and the switch S is closed at an instant when the initial phase of the emf wave is ϕ as indicated in Fig. 21–3(b), or $e=E_m \sin(\omega t+\phi)$, Kirchhoff's Emf Law gives

$$L\frac{di}{dt}=E_m \sin(\omega t+\phi) \tag{21–5}$$

Separating variables, we get

$$di=\frac{E_m}{L}\sin(\omega t+\phi)dt$$

Integrating then gives

$$i=\frac{-E_m}{\omega L}\cos(\omega t+\phi)+K \tag{21–6}$$

where K = constant of integration.

The presence of inductance requires that $i=0$ at $t=0$. Hence, substitution of values in equation (21–6) gives

$$K=\frac{E_m}{\omega L}\cos\phi \tag{21–7}$$

and

$$i=\frac{E_m}{\omega L}\cos\phi-\frac{E_m}{\omega L}\cos(\omega t+\phi) \tag{21–8}$$

The first term may be considered as a "switching" term which depends on whether the switch is closed during the positive or negative part of the emf wave and also on the phase of the emf wave at the instant of closure.

If the switch is closed at $t=0$ when $e=0\left(\frac{de}{dt}\text{ positive}\right)$, then $\phi=0$ and equation (21–8) becomes

$$i=\frac{E_m}{\omega L}(1-\cos\omega t) \tag{21–9}$$

The graph of this equation is given in Fig. 21–4(a). The switching component $\frac{E_m}{\omega L}$ remains the same in magnitude and persists indefinitely. The current i is never negative in this case.

If the switch is closed at $t=0$ when $e=E_m$, then $\phi=90°$ and equation (21–8) becomes

$$i=\frac{E_m}{\omega L}\sin\omega t \tag{21–10}$$

The variation of i is shown in Fig. 21–4(*b*). There is no switching component and the current reaches its steady-state value immediately after the circuit is closed.

The switching term $\frac{E_m}{\omega L}\cos\phi$ may be viewed as a d-c component which displaces the steady-state term from the time axis. Its amplitude depends on the value of ϕ or the point on the emf cycle at which the switch is closed (maximum when $e=0$ and zero when $e=E_m$). It persists indefinitely in a hypothetical L circuit, a phenomenon not encountered in actual circuits because of the presence of resistance.

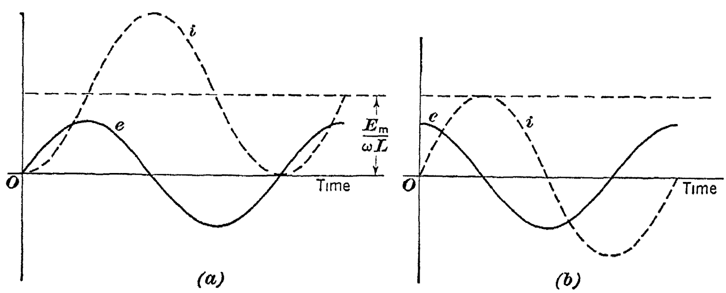


FIG. 21–4

21–3. The *C* Circuit.—If the resistance R in Fig. 21–2 is replaced by a condenser, the condition for dynamic equilibrium is

$$\frac{1}{C}\int i\,dt=E_m\sin(\omega t+\phi) \tag{21–11}$$

Differentiating and rearranging, we get

$$i=E_m\omega C\cos(\omega t+\phi) \tag{21–12}$$

Equation (21–12) is not the complete expression for the current in the circuit [see equation (21–26) as $R\to 0$]. For a condenser (initially uncharged) the current is infinite at $t=0$ when e is not zero. Equation (21–12) contains only the steady-state term.

When the switch is closed at an instant such that $e=0$ $\left(\frac{de}{dt}\text{ positive}\right)$, there is no switching term and equation (21–12) is the complete solution. The graph is shown in Fig. 21–5.

21–4. The *RL* Circuit.—A sinusoidal emf is impressed across the terminals of a series circuit consisting of resistance and inductance, as in Fig. 21–6. The current immediately after the circuit is established depends on the instantaneous value of the voltage

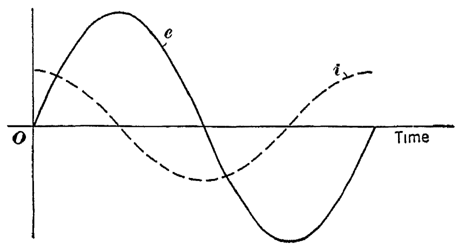


FIG. 21–5

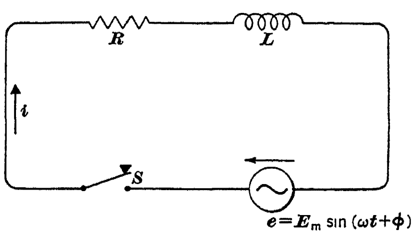


FIG. 21–6

when the switch S is closed. If the switch S is closed at an instant when the initial phase of the emf wave is ϕ, then the condition for dynamic equilibrium is

$$L\frac{di}{dt}+Ri=E_m \sin\ (\omega t+\phi) \tag{21–13}$$

or, symbolically,

$$\left(p+\frac{R}{L}\right)i=f(p)i=\frac{E_m}{L}\sin\ (\omega t+\phi) \tag{21–14}$$

where $f(p)=p+\frac{R}{L}$.

A general remark, which applies to the rest of this chapter, may be made at this point. The form of expression for the complementary function or transient terms will be the same for a given circuit whether the applied emf is d-c, a-c, or any function of time. This is so because the complementary function is obtained from the differential equation of the circuit with the right-hand side of the equation replaced by zero. Of course, the values of the arbitrary constants will depend on the boundary conditions.

Let the steady-state component of the current be known as i_s and the transient component be designated by i_t. Then the actual current in the circuit may be written as follows:

$$i = i_s + i_t \tag{21–15}$$

The transient component is obtained by finding the complementary function of equation (21–14). Substituting m for p in $f(p) = p + \frac{R}{L}$ and setting $f(m)$ equal to 0, we obtain

$$m + \frac{R}{L} = 0$$

The root is $m_1 = \frac{-R}{L}$, and the transient component is

$$i_t = Ke^{-\frac{Rt}{L}} \tag{21–16}$$

The steady-state component can be found from elementary a-c theory. Thus,

$$i_s = \frac{E_m}{Z} \sin\,(\omega t + \phi - \theta) \tag{21–17}$$

where the impedance $Z = \sqrt{R^2 + \omega^2 L^2}$ and the power factor angle $\theta = \tan^{-1} \frac{\omega L}{R}$.

The complete solution is

$$i = i_s + i_t = \frac{E_m}{Z} \sin\,(\omega t + \phi - \theta) + Ke^{-\frac{Rt}{L}} \tag{21–18}$$

To determine the constant K, apply the boundary conditions to the equation. When $t = 0$, $i = 0$ and $L\,\frac{di}{dt} = E_m \sin\,\phi$; and the equation becomes

$$0 = \frac{E_m}{Z} \sin\,(\phi - \theta) + K$$

from which

$$K = \frac{-E_m}{Z} \sin\,(\phi - \theta)$$

Hence, the actual current is

$$i = \frac{E_m}{Z}[\sin(\omega t + \phi - \theta) - e^{-\frac{Rt}{L}} \sin(\phi - \theta)] \qquad \textbf{(21–19)}$$

It will be observed that equation (21–19), if substituted in equation (21–13), satisfies the condition for dynamic equilibrium. The curves representing the actual current and its two components for the case when the power factor angle θ is greater than the initial phase angle ϕ, or $\theta > \phi$, are shown in Fig. 21–7. The steady-state component $i_s = \frac{E_m}{Z} \sin(\omega t + \phi - \theta)$ is a sinusoidal curve and it lags behind e by the phase angle θ, as in ordinary a-c circuit theory.

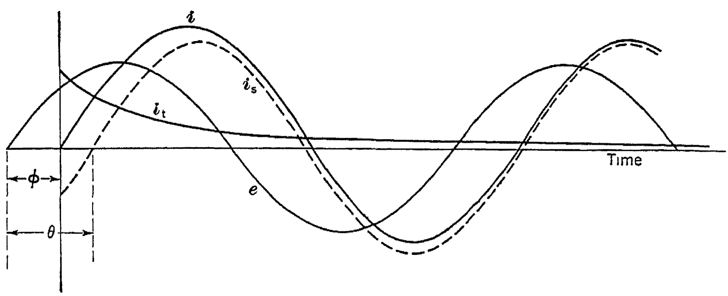


FIG. 21–7

The transient term $i_t = -e^{-\frac{Rt}{L}} \frac{E_m}{Z} \sin(\phi - \theta)$ is a decreasing exponential function. It should be noted that $\sin(\phi - \theta)$ is negative, since $\theta > \phi$. The actual current, which is the sum of the two, approaches the steady state as a limit.

Since the amplitude of the transient component depends on the time of the alternating voltage cycle at which the circuit is completed, it will be of interest to consider two instants: (*a*) when the transient component is zero and (*b*) when the transient component is a maximum.

On examination of equation (21–19), it can be seen that at $t = 0$ the transient term is equal in magnitude but opposite in sign to the steady-state term. That is, the transient component is zero when the circuit is closed at an instant when the steady-state current is zero; it is a maximum when the switch is closed at an instant when the steady-state current is a maximum. Mathematically, the transient term is zero when $(\phi-\theta)$ is 0, π, 2π, 3π, 4π, etc. It is maximum when $(\phi-\theta)$ is $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, etc.

The transient component may have a positive or a negative maximum value, the sign depending on whether $\sin(\phi-\theta)$ is equal to $+1$ or -1.

21–5. The *RC* Circuit.—Fig. 21–8 shows a series circuit containing resistance and capacitance. The condition for dynamic equilibrium is

$$Ri+\frac{1}{C}\int i\,dt=E_m\sin(\omega t+\phi) \tag{21–20}$$

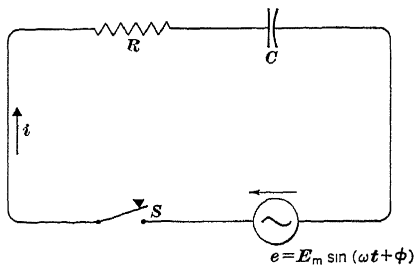


FIG. 21–8

The condenser is initially uncharged. When the switch is closed, the condenser acts like a short circuit. The circuit at $t=0$ is one with resistance only, and the current instantly acquires the value $\frac{e}{R}$. As a result of the current flow, the condenser acquires an electric charge. The condenser then exhibits its capacitive counter-voltage $\frac{q}{C}=\frac{1}{C}\int i\,dt$ and becomes an active member of the circuit.

Writing equation (21–20) in the symbolic form, we have

$$\left(R+\frac{1}{pC}\right)i=E_m\sin(\omega t+\phi) \tag{21–21}$$

By substituting m for p, the auxiliary equation $R+\frac{1}{mC}=0$ can easily be obtained. The root of the auxiliary equation is

$$m_1=\frac{-1}{RC}$$

and the transient term is

$$i_t = Ke^{-\frac{t}{RC}} \tag{21–22}$$

From the ordinary alternating-current circuit theory, the steady-state term is given by the relation

$$i_s = \frac{E}{Z} \sin(\omega t + \phi - \theta) \tag{21–23}$$

in which $Z = \sqrt{R^2 + \left(\frac{-1}{\omega C}\right)^2}$ and the power factor angle $\theta = \tan^{-1} \frac{-1/\omega C}{R}$.

The complete solution is

$$i = \frac{E_m}{Z} \sin(\omega t + \phi - \theta) + Ke^{-\frac{t}{RC}} \tag{21–24}$$

The constant K is determined by applying the boundary conditions. When $t = 0$, $e_C = 0$, $e = E_m \sin \phi$, and $i = \frac{E_m \sin \phi}{R}$, then

$$\frac{E_m \sin \phi}{R} = \frac{E_m}{Z} \sin(\phi - \theta) + K$$

or

$$K = \frac{E_m \sin \phi}{R} - \frac{E_m}{Z} \sin(\phi - \theta) = \frac{E_m \sin \phi}{R} - \frac{E_m}{R} \cdot \frac{R}{Z} \sin(\phi - \theta)$$

$$= \frac{E_m}{R} [\sin \phi - \cos \theta \sin(\phi - \theta)]$$

Since $\sin \phi = \sin[\theta + (\phi - \theta)] = \sin \theta \cos(\phi - \theta) + \cos \theta \sin(\phi - \theta)$,

$$K = \frac{E_m}{R} \sin \theta \cos(\phi - \theta) \tag{21–25}$$

When this value for K is substituted in equation (21–24), the equation for the actual current is found to be

$$i = \frac{E_m}{Z} \sin(\omega t + \phi - \theta) + \frac{E_m}{R} \sin \theta \cos(\phi - \theta) e^{-\frac{t}{RC}} \tag{21–26}$$

At $t = 0$, the current becomes

$$i = \frac{E_m}{R} \sin \phi \tag{21–27}$$

which is the boundary value $\frac{e}{R}$ when the circuit acts like an R circuit.

There are two conditions under which the transient component is zero. These conditions, by examination of the second term on the right-hand side of equation (21–26), are when $\theta = 0$ and $(\phi - \theta) = \frac{\pi}{2}, \frac{3\pi}{2}$, etc. For the case when $\theta = 0$, the circuit is a resistive circuit and there can be no transient current. For the second condition, the switch is closed at an instant when the steady-state component is a maximum. This is so because, when $(\phi - \theta) = \frac{\pi}{2}$, we have $i_s = \frac{E_m}{Z} \sin(\phi - \theta) = \frac{E_m}{Z}$ at $t = 0$.

In the general case, the transient component may have any value. Low resistance in the circuit, compared to the reactance, will permit a large transient component to flow. Closing the

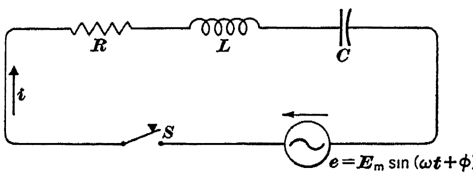


FIG. 21–9

switch at the maximum value of the applied emf will mean a high initial rush of current. If the resistance in the circuit is further reduced, the transient surge will be larger but of shorter duration.

21–6. The *RLC* Circuit.—Assume that a sinusoidal emf is impressed across the terminals of a series circuit consisting of resistance, inductance, and capacitance, as in Fig. 21–9, and that the condenser is initially uncharged. If the switch S is closed at an instant when the initial phase of the emf wave is ϕ, or $e = E_m \sin(\omega t + \phi)$, then the condition for dynamic equilibrium is

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = E_m \sin(\omega t + \phi) \qquad (21\text{–}28)$$

or, symbolically,

$$\left(R + Lp + \frac{1}{pC}\right)i = E_m \sin(\omega t + \phi) \qquad (21\text{–}29)$$

Some idea of the general nature of the circuit's behavior with sinusoidal applied emf may be obtained by reviewing or considering the same RLC circuit with a battery voltage E. The form of the expression for the complementary function or transient terms of the a-c case will be the same as that of the d-c case. Of course, in general, the values of the arbitrary constants will be different and the final expressions for the currents will be different. The steady-state term given by the ordinary alternating-current theory is

$$i_s=\frac{E_m}{Z}\sin(\omega t+\phi-\theta) \tag{21-30}$$

where

$$Z=\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}$$

$$\theta=\tan^{-1}\frac{\omega L-\dfrac{1}{\omega C}}{R}$$

The transient component is the complementary function obtained from equation (21–29). Substituting m for p and setting $f(m)$ equal to 0, we get

$$\left(R+Lm+\frac{1}{mC}\right)=0 \tag{21-31}$$

The roots are:

$$m_1=-\frac{R}{2L}+\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}=-a+b$$

$$m_2=-\frac{R}{2L}-\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}=-a-b$$

where

$$a=\frac{R}{2L}$$

$$b=\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}$$

As in the case of the direct-current problem, there are three cases to be considered:

(*a*) Non-oscillatory case, or $\dfrac{R^2}{4L^2}>\dfrac{1}{LC}$

(*b*) Critical case, or $\dfrac{R^2}{4L^2}=\dfrac{1}{LC}$

(*c*) Oscillatory case, or $\dfrac{R^2}{4L^2}<\dfrac{1}{LC}$

21–7. Non-Oscillatory Case.—In this case, $\frac{R^2}{4L^2} > \frac{1}{LC}$ and b is a real number. Therefore, both roots of equation (21–31) are real and unequal. The expression for the transient component i_t is of the form

$$i_t = K_1 e^{m_1 t} + K_2 e^{m_2 t} \qquad (21\text{–}32)$$

The complete expression for the current is

$$i = \frac{E_m}{Z} \sin\,(\omega t + \phi - \theta) + K_1 e^{m_1 t} + K_2 e^{m_2 t} \qquad (21\text{–}33)$$

or

$$i = \frac{E_m}{Z} \sin\,(\omega t + \phi - \theta) + K_1 e^{(-a+b)t} + K_2 e^{(-a-b)t} \qquad (21\text{–}34)$$

To determine the constants K_1 and K_2, it is necessary to have two simultaneous equations involving the constants. Apply the boundary conditions, for which $t = 0$, $i = 0$, $e_R = Ri = 0$, $e_C = 0$, and $L\frac{di}{dt} = E_m \sin \phi$. Equation (21–34) then becomes

$$0 = \frac{E_m}{Z} \sin\,(\phi - \theta) + K_1 + K_2$$

from which

$$K_1 + K_2 = \frac{-E_m}{Z} \sin\,(\phi - \theta) \qquad (21\text{–}35)$$

To find the second equation, differentiate equation (21–34) and multiply through by L. The result is

$$L\frac{di}{dt} = L\left[\frac{\omega E_m}{Z} \cos\,(\omega t + \phi - \theta) + (-a + b)K_1 e^{(-a+b)t} + (-a - b)K_2 e^{(-a-b)t}\right] \qquad (21\text{–}36)$$

For $t = 0$, equation (21–36) becomes

$$E_m \sin \phi = \frac{\omega L E_m}{Z} \cos\,(\phi - \theta) - aL(K_1 + K_2) + bL(K_1 - K_2)$$

$$= \frac{\omega L E_m}{Z} \cos\,(\phi - \theta) + \frac{aLE_m}{Z} \sin\,(\phi - \theta) + bL(K_1 - K_2)$$

from which

$$K_1 - K_2 = \frac{E_m \sin \phi - \frac{\omega L E_m}{Z} \cos\,(\phi - \theta) - \frac{aLE_m}{Z} \sin\,(\phi - \theta)}{bL} \qquad (21\text{–}37)$$

Solving equations (21–35) and (21–37) simultaneously, we obtain the following values for K_1 and K_2:

$$K_1 = \frac{E_m \sin\phi - \frac{\omega L E_m}{Z}\cos(\phi-\theta) - (a+b)\frac{LE_m}{Z}\sin(\phi-\theta)}{2bL}$$

or

$$K_1 = \frac{K_0}{2bL} - \frac{E_m}{2Z}\sin(\phi-\theta) \tag{21–38}$$

$$K_2 = \frac{\frac{\omega L E_m}{Z}\cos(\phi-\theta) + (a-b)\frac{LE_m}{Z}\sin(\phi-\theta) - E_m\sin\phi}{2bL}$$

or

$$K_2 = \frac{-K_0}{2bL} - \frac{E_m}{2Z}\sin(\phi-\theta) \tag{21–39}$$

where, for the sake of simplicity in writing,

$$K_0 = E_m\sin\phi - \frac{\omega L E_m}{Z}\cos(\phi-\theta) - \frac{aLE_m}{Z}\sin(\phi-\theta) \tag{21–40}$$

Hence, the final solution is

$$i = \frac{E_m}{Z}\sin(\omega t+\phi-\theta) + e^{-at}\left\{\left[\frac{K_0}{2bL} - \frac{E_m}{2Z}\sin(\phi-\theta)\right]e^{bt} + \left[\frac{-K_0}{2bL} - \frac{E_m}{2Z}\sin(\phi-\theta)\right]e^{-bt}\right\}$$

$$= \frac{E_m}{Z}\sin(\omega t+\phi-\theta) + e^{-at}\left\{\frac{K_0}{bL}\cdot\frac{e^{bt}-e^{-bt}}{2} - \frac{E_m}{Z}\sin(\phi-\theta)\cdot\frac{e^{bt}+e^{-bt}}{2}\right\}$$

or

$$i = \frac{E_m}{Z}\sin(\omega t+\phi-\theta) + e^{-at}\left\{\frac{K_0}{bL}\sinh bt - \frac{E_m}{Z}\sin(\phi-\theta)\cosh bt\right\} \tag{21–41}$$

The current has a steady-state term, which is $\frac{E_m}{Z}\sin(\omega t+\phi-\theta)$, and two transient terms. Because of the effect of the factor e^{-at}, one is an exponentially damped hyperbolic sine and the other is an exponentially damped hyperbolic cosine. At $t=0$, when $\omega t=0$, $e^{-at}=1$, $\sinh bt=0$, and $\cosh bt=1$, the transient component is equal in magnitude and opposite in sign to that of the steady-state component. Theoretically, the steady-state condition is not reached until $t=\infty$.

21–8. Critical Case.—When $\frac{R^2}{4L^2}=\frac{1}{LC}$, the quantity b is equal to zero and the two roots of equation (21–31) are equal. The complete solution is

$$i=\frac{E_m}{Z}\sin(\omega t+\phi-\theta)+K_1e^{-at}+K_2te^{-at} \tag{21–42}$$

To find the constants, apply the boundary conditions, for which $t=0$, $i=0$, $e_R=Ri=0, e_C=0$, and $L\frac{di}{dt}=E_m\sin\phi$. Equation (21–42) becomes

$$0=\frac{E_m}{Z}\sin(\phi-\theta)+K_1$$

from which

$$K_1=\frac{-E_m}{Z}\sin(\phi-\theta) \tag{21–43}$$

To determine K_2, differentiate equation (21–42) and multiply through by L The result is

$$L\frac{di}{dt}=L\left[\frac{\omega E_m}{Z}\cos(\omega t+\phi-\theta)-aK_1e^{-at}\right.$$
$$\left.+K_2(-ate^{-at}+e^{-at})\right] \tag{21–44}$$

For $t=0$, equation (21–44) reduces to

$$E_m\sin\phi=L\left[\frac{\omega E_m}{Z}\cos(\phi-\theta)-aK_1+K_2\right]$$

Substituting the value from equation (21–43) for K_1, we get

$$K_2=\frac{E_m}{L}\sin\phi-\frac{\omega E_m}{Z}\cos(\phi-\theta)-\frac{RE_m}{2LZ}\sin(\phi-\theta) \tag{21–45}$$

Hence, the complete solution of the current is

$$i=\frac{E_m}{Z}\sin(\omega t+\phi-\theta)-\frac{E_m}{Z}\sin(\phi-\theta)e^{-at}$$
$$+\left[\frac{E_m}{L}\sin\phi-\frac{\omega E_m}{Z}\cos(\phi-\theta)-\frac{RE_m}{2LZ}\sin(\phi-\theta)\right]te^{-at}$$

or

$$i=\frac{E_m}{Z}\sin(\omega t+\phi-\theta)+e^{-at}\left[\frac{K_0}{L}t-\frac{E_m}{Z}\sin(\phi-\theta)\right] \tag{21–46}$$

where K_0 is given by equation (21–40).

This case is often referred to as the critical case. It is the boundary-line case between the condition when $\frac{R^2}{4L^2} > \frac{1}{LC}$ and the condition when $\frac{R^2}{4L^2} < \frac{1}{LC}$.

21–9. Oscillatory Case.—If $\frac{R^2}{4L^2} < \frac{1}{LC}$, then the roots of equation (21–31) are conjugate imaginaries. Since b is an imaginary quantity, a change in nomenclature is desirable. Let

$$b = j\beta \tag{21–47}$$

where

$$\beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{21–48}$$

By substituting these values of b and β in equations (21–38), (21–39), and (21–34), and making use of the relations $e^{j\beta t} = \cos \beta t + j \sin \beta t$ and $e^{-j\beta t} = \cos \beta t - j \sin \beta t$, we obtain the following expression for the actual current:

$$i = \frac{E_m}{Z} \sin (\omega t + \phi - \theta) + e^{-at}\left[\frac{K_0}{\beta L} \sin \beta t - \frac{E_m}{Z} \sin (\phi - \theta) \cos \beta t\right] \tag{21–49}$$

where K_0 is given by equation (21–40).

The current consists of a steady-state term and two transient terms. Because of the effect of the factor e^{-at}, one is an exponentially damped sine wave and the other is an exponentially damped cosine wave. At $t = 0$, or when $\omega t = 0$, $e^{-at} = 1$, $\sin \beta t = 0$, and $\cos \beta t = 1$, the transient component is equal in magnitude and opposite in sign to the steady-state component.

For the sake of analysis, the two transient terms in equation (21–49) may be combined. The current expression then becomes

$$i = \frac{E_m}{Z} \sin (\omega t + \phi - \theta) + I_t e^{-at} \sin (\beta t - \Psi) \tag{21–50}$$

where

$$I_t = \sqrt{\left[\frac{K_0}{\beta L}\right]^2 + \left[\frac{E_m}{Z} \sin (\phi - \theta)\right]^2}$$

$$\Psi = \tan^{-1} \frac{E_m \beta L \sin (\phi - \theta)}{K_0 Z}$$

The current in the circuit, as expressed by equation (21–50), consists of a damped sinusoidal term added to an undamped sinusoidal term. The two components will, in general, be of different frequencies. The frequency $\frac{\omega}{2\pi}$ of the steady-state component is determined solely by the frequency of the applied emf; the frequency $\frac{\beta}{2\pi}$ of the transient component depends entirely on the elements R, L, and C of the circuit. The appearance of the resulting wave of the actual current may vary widely, its shape depending on the natural frequency of the circuit and the applied frequency. The closing of the switch at different points on the voltage wave will produce different effects.

In an RLC circuit, initially without stored energy, it is not possible to apply a sinusoidal emf without producing a transient component i_t in the current. However, if the condenser is initially charged, it will be possible to find a sinusoidal emf that will produce a zero transient component in the current expression. In this case, the initial charge must be correctly related to the initial phase angle and amplitude of the applied emf. This relation will be considered in the remainder of this article.

It can be seen from equation (21–50) that the transient component will be zero at every instant if $I_t = 0$. This means that $K_0 = 0$ and $\sin(\phi - \theta) = 0$. Substituting $\sin(\phi - \theta) = 0$ in equation (21–40) and then setting the resulting expression for K_0 equal to zero gives

$$K_0 = E_m \sin \phi - \frac{\omega L E_m}{Z} = 0 \tag{21–51}$$

from which

$$\sin \phi = \frac{\omega L}{Z} \tag{21–52}$$

Hence, for $I_t = 0$, the switch must be closed at an instant when the initial phase angle of the applied emf is $\phi = \sin^{-1} \frac{\omega L}{Z}$.

In the steady-state component i_s, as indicated in equation (21–30), the condition $\sin(\phi - \theta) = 0$ or $\phi = \theta$ at $t = 0$ requires that

$$\sin \phi = \sin \theta = \frac{\omega L - \frac{1}{\omega C}}{Z} \tag{21–53}$$

Because of the conflicting conditions for $\sin \phi$, as indicated in equations (21–52) and (21–53), it is not possible to have $I_t = 0$ unless there is initial energy stored in the circuit.

When there is an initial charge Q_0 on the condenser of an RLC circuit such that $L\frac{di}{dt} = E_m \sin \phi - \frac{Q_0}{C}$ at $t = 0$, it can be shown that the expression for K_0 of equation (21–40) takes the form

$$K_0 = E_m \sin \phi - \frac{\omega L E_m}{Z} \cos (\phi - \theta) - \frac{a L E_m}{Z} \sin (\phi - \theta) - \frac{Q_0}{C} \qquad \text{(21–54)}$$

Substituting $\sin (\phi - \theta) = 0$ in the last equation and then setting K_0 equal to 0 gives

$$E_m \sin \phi - \frac{\omega L E_m}{Z} - \frac{Q_0}{C} = 0$$

from which

$$\sin \phi = \frac{\frac{\omega L E_m}{Z} + \frac{Q_0}{C}}{E_m} = \frac{\omega L}{Z} + \frac{Q_0}{E_m C} \qquad \text{(21–55)}$$

Also, from equation (21–53),

$$\sin \phi = \frac{\omega L - \frac{1}{\omega C}}{Z} = \frac{\omega L}{Z} - \frac{1}{\omega C Z} \qquad \text{(21–56)}$$

Equation (21–56) may be made equal to equation (21–55) if the initial charge on the condenser is

$$Q_0 = \frac{-E_m}{\omega Z} \qquad \text{(21–57)}$$

PROBLEMS

21–1. A sinusoidal emf is impressed across a 5-ohm resistance at $t = 0$. The effective value of the voltage is 110 volts and the frequency is 60 cps. The switch is closed 0.003 second after the emf wave passes through zero $\left(\frac{de}{dt} \text{ positive}\right)$.

(*a*) What is the expression for the current?
(*b*) What is the initial rate of current growth?
(*c*) What is the maximum value of the current?
(*d*) Write an expression for the energy dissipated.
(*e*) Draw the graphs for (*a*) and (*d*) and the applied emf.

21–2. Repeat Problem 21–1 with the resistance replaced by an inductance of 0.02 henry. In part (*d*) the expression should be for the energy stored in the magnetic field.

21–3. An emf $e = 100 \sin (377t + \phi)$ is applied to a circuit with $L = 0.03$ henry. Find the expression for the current when the switch is closed at $\phi = 0°$, 60°, 90°, 180°. Plot these four current waves on the same diagram.

21–4. A condenser with a capacitance of 500 μf is connected across an emf of the form $e = 100 \sin (377t + \phi)$. The condenser is assumed initially uncharged, and the switch is closed at an instant when $e = 0 \left(\frac{de}{dt} \text{ positive}\right)$.

(a) What is the expression for the current?
(b) What is the initial rate of current growth?
(c) What is the maximum value of the current?
(d) What is the energy stored in the electrostatic field?

21–5. A 60-cycle sinusoidal emf of 220 volts (effective) is applied to a series circuit of $R = 10$ ohms and $L = 0.005$ henry. The switch is closed at the instant when the emf wave is going through zero $\left(\frac{de}{dt} \text{ positive}\right)$.

(a) Write the expression for the current.
(b) What is the maximum value of the current, and at what time will it occur?
(c) What is the initial rate of change of current?
(d) Plot curves showing the current, its components, and the applied emf.

21–6. A coil having $R = 5$ ohms and $L = 0.01$ henry is connected to a source of emf for which $e = 100 \sin (377t + \phi)$.

(a) Write the expression for the current.
(b) At what instant of the alternating emf cycle, when the circuit is closed, will there be no transient current?
(c) At what instant of the alternating emf cycle, when the circuit is closed, will the actual current have the maximum value?
(d) Draw the current curves for parts (b) and (c).

21–7. An RC circuit having $R = 5$ ohms and $C = 200$ μf is connected to a source of emf for which $e = 100 \sin (377t + 60°)$.

(a) What is the equation for the current?
(b) Plot the current and its components.

21–8. A series circuit consisting of $R = 4$ ohms and $C = 100$ μf is connected to an alternating emf $e = 100 \sin (157t + 90°)$.

(a) What is the equation for the charge?
(b) What is the equation for the current?
(c) Plot the curves of q and i.
(d) At what instant of the alternating emf cycle, when the circuit is closed, will the transient component be zero?

21–9. Repeat Problem 21–7 with the resistance reduced to 1 ohm. Compare the initial rush of current and the duration of the transient surge.

21–10. A circuit consisting of $R = 20$ ohms, $L = 0.01$ henry, and $C = 200$ μf is connected to a sinusoidal emf $e = 100 \sin 377t$. The condenser is initially uncharged, and the circuit is closed at an instant when the voltage wave is going through zero $\left(\frac{de}{dt} \text{ positive}\right)$. Find the expression for the current.

21–11. The resistance in the circuit of Problem 21–10 is changed to a value such that $\frac{R^2}{4L^2}=\frac{1}{LC}$. Find the expression for the current.

21–22. An RLC circuit where $R=2$ ohms, $L=0.05$ henry, and $C=20$ μf is connected to a sinusoidal emf at an instant such that $e=100 \sin(377t+90°)$. Find the expression for the current.

21–13. What is the expression for the current in an LC circuit when the applied emf is sinusoidal?

21–14. Derive a general expression for the current in an RLC circuit when the condenser has an initial charge Q_0.

21–15. A condenser of 100 μf is connected in parallel to a coil having $R=3$ ohms and $L=0.01$ henry. An alternating emf $e=100 \sin 377t$ is connected to this parallel combination at an instant when the emf wave is going through zero $\left(\frac{de}{dt}\text{ positive}\right)$. Find the equation for each of the three currents as a function of time.

CHAPTER 22

TRANSMISSION LINE CALCULATIONS

Heretofore circuits with lumped elements only have been considered. The resistance, inductance, and capacitance of a circuit were regarded as being concentrated at just one part of the circuit. The lead wires for completing the circuit were assumed to have negligible resistance and negligible inductance. In the solution of problems relating to ordinary circuits and short transmission lines,* the distributed elements may be replaced by concentrated elements. However, in the case of long transmission lines, it becomes essential that the elements of the transmission line be considered as distributed. The method presented in this chapter takes into account the distributed elements, and it is a *safe* one to use. If it is used for some transmission line problem in which one of the approximate methods would have yielded an acceptable result, nothing is lost except a little extra labor.

The current and voltage of a transmission line are functions of time and of position along the line. They may be considered in the steady state or in the transient state, the choice depending on the conditions of the problem. Only steady-state conditions will be treated in this chapter. For treating the theory of long, smooth lines (with uniformly distributed elements) in the *steady state*, a large number of methods are available. All these methods begin with the same premises. The differences between them result largely from differences in the methods adopted for handling the solution of the resulting differential equations. The results apply equally to alternating-current power and communication work and, by proper interpretation, to long lines carrying direct current in the steady state.

Although there is a tendency to adopt the rationalized MKS system of units, many existing tables and books give values of the line elements on the per-mile basis. Also, it must be understood that the adoption of the MKS system in a book does not prevent

* See books on transmission lines concerning approximations of short lines, such as the so-called nominal T line, the nominal π line, and Steinmetz's Representation.

the use of other units in problems. For the reason that a student should be able to work problems on either the per-meter basis or the per-mile basis, formulas given in this chapter will be expressed in these two units of length.

The purposes of this chapter are: (*a*) to present the concept of distributed elements and to indicate briefly how they can be be determined from the physical line; (*b*) to set up the basic differential equations of the exact solution of long lines and then to derive from them the expressions for voltage and current at any point along the line; (*c*) to give a very brief physical interpretation of the voltage and current equations; (*d*) to discuss the propagation constant, attenuation constant, wavelength constant, wavelength, and velocity of propagation; and (*e*) to give an expression for the characteristic or surge impedance.

22–1. The Electric Circuit.—A transmission line consists of two or more conductors for the purpose of transferring electrical energy from one place to another. For single-phase transmission and telephone work, the line usually consists of two metallic conductors. For three-phase transmission, electric power is transmitted over three-phase, three-wire lines or over three-phase, four-wire lines. If the polyphase system is balanced, computations may be made for one phase (line to neutral) by assuming that the system is a Y-Y circuit, with a Y-connected generator and a Y-connected load. For the Y-Y circuit, the line-to-neutral voltage is considered the phase voltage and the line current is considered the phase current.

Although the electrical problems which arise from power and communication lines are fundamentally alike, some of the differences should be noted:

(*a*) In purpose, a power line differs from a telephone line in that the former is a circuit for the transmission of a large amount of power while the latter is a communication circuit for the transmission of information.

(*b*) In construction, the power lines usually have larger conductors, larger spacings of conductors, and larger insulators than telephone lines.

(*c*) In operation, power lines ordinarily are operated at one frequency, such as 25, 50, or 60 cps; whereas telephone lines must transmit a large number of frequencies of varying amplitudes at the same time, for example, voice frequencies from 200 to 5000 cps. Also, since power lines transfer a large amount of electrical energy, it is necessary that such circuits be of high efficiency; while the primary requirement of a telephone line is to have a faithful reproduction of the original wave at the receiving end. Although the efficiency of a communication circuit is relatively low, the network is usually designed for maximum power transfer to the receiver.

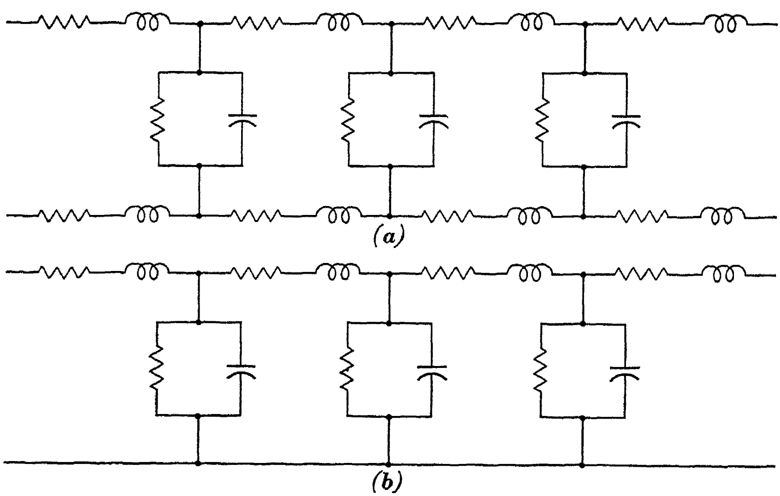


FIG. 22–1

If the elements of a two-parallel-wire line are uniformly distributed, the line is called a *smooth* line. Such a line may be considered as consisting of an infinite number of infinitesimal sections with series and parallel branches, as shown in Fig. 22–1(*a*). This represents the conditions in a line with constant series resistance, series inductance, shunt capacitance, and shunt leakage conductance per unit length of line. For purposes of calculation, it may be convenient to represent the line by a series of infinitesimal sections, as indicated in Fig. 22–1(*b*). The advantage of this representation is that it can readily be used for one phase (line to neutral) of a balanced three-phase, Y-Y system.

In general, the potential from conductor to conductor at any point along the line differs from that at another point both in magnitude and phase. Likewise, the current in the conductor varies from point to point.

22–2. Calculation of *R*, *L*, and *C* of a Two-Wire Line.*—The elements of a conventional two-parallel-wire line are the series resistance, series inductance, shunt capacitance, and shunt leakage conductance, all being taken per unit length of line. The calculation of the elements R, L, and C will be considered in this article. The shunt conductance of the two-wire line and that of the three-phase line will be discussed in Art. 22–4.

(*a*) *Resistance:* The so-called ohmic or d-c resistance R_{dc} of a single conductor per mile may be found from wire tables. This can be converted into ohms per meter, since a mile contains 1609.4 meters. When a wire carries alternating current, a non-uniform current distribution over the cross-section of the wire results in a larger power loss for a given rms alternating current than for the same value of direct current flowing in the same wire. This non-uniformity of the current density over the cross-section of the wire is due to "skin effect" and "proximity effect." The phenomenon known as *skin effect* is the increase in resistance of the conductor because of the tendency for the current to flow with greater density near the outside or surface of the conductors. The *proximity effect* is the increase in resistance caused by the crowding of the current on one side of a conductor because of the current flowing in adjacent conductors.

The larger power loss resulting from non-uniform current distribution is usually taken care of by the use of the effective alternating-current resistance, which is defined as the average power loss in the conductor divided by the square of the effective value of the alternating current flowing. At ordinary commercial power frequencies, it is permissible to neglect the skin effect in small wires of non-magnetic materials, such as copper or aluminum conductors with diameters less than 0.0127 meter or areas less than 250,000

* For derivations of formulas, see "Principles of Electric Power Transmission," by L. F. Woodruff; "Electrical Power Transmission," by E. A. Loew; "Introductory Study of Electrical Characteristics of Power and Telephone Transmission Lines," by Norris and Bingham; "Electrical Transmission of Power and Signals," by Edward W. Kimbark.

circular mils. For ordinary power frequencies and the usual spacings of conductors in transmission lines, the proximity effect may also be neglected. The skin-effect resistance ratio $\frac{R_e}{R_{dc}}$ (where R_e = resistance of conductor including skin effect) increases with the frequency, the permeability of the conductor material, and the cross-sectional area of the conductor.* The resistance of a transmission line will also depend on the temperature and on whether the conductors are solid wires or stranded wires.

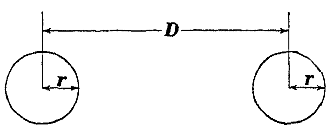


FIG. 22–2

It is obvious that the total resistance R of a single conductor is equal to the resistance per unit length multiplied by the entire length of the single conductor. The total resistance for the line of two wires is $2R$.

(b) *Inductance:* When the spacing D between conductors is large compared with the radius r of the conductor, as in Fig. 22–2, the inductance per unit length of a line of two wires is

$$L = 4 \times 10^{-7}\left[\ln\frac{D}{r} + \frac{1}{4}\frac{\mu}{\mu_0}\right] \text{ henrys per meter (two wires)} \qquad \textbf{(22–1)}$$

or

$$L = 1.482 \log_{10}\frac{D}{r} + 0.1609\frac{\mu}{\mu_0} \text{ millihenrys per mile (two wires)} \qquad \textbf{(22–1a)}$$

where $\ln = \log_e$ (natural logarithm, base $e = 2.718$);

D = distance between centers of conductors;
r = radius of conductors;
μ_0 = permeability of medium between conductors
$= 1.257 \times 10^{-6}$ henry per meter for free space;
$\frac{\mu}{\mu_0}$ = relative permeability, or ratio between the actual permeability of the conductor material and that of free space.

Since D and r form a ratio, it is only necessary that they be expressed in the same unit. Although equation (22–1) appears to be dimensionless, the numerical factor 4×10^{-7} includes the dimension of L; that is, $4 \times 10^{-7} = \text{constant} \times \mu_0 = \text{constant} \times 1.257 \times 10^{-6}$

* For values of skin-effect resistance ratio, see Chapter III of "Principles of Electric Power Transmission," by L. F. Woodruff, and also Chapter III of "Introductory Study of Electrical Characteristics of Power and Telephone Transmission Lines," by Norris and Bingham.

henry per meter. Thus, for the different equations of L and C given in this chapter, the dimensions of L and C are included in the numerical factors.

The inductance per unit length for one wire is one-half of the value given by equation (22–1) or equation (22–1a). Thus,

$$L = 2\times 10^{-7} \ln \frac{D}{r} + \frac{1}{2}\times 10^{-7} \frac{\mu}{\mu_0} \text{ henrys per meter (one wire)} \qquad \textbf{(22–1b)}$$

The term $2\times 10^{-7} \ln \frac{D}{r}$ is due to the current in one conductor linking the flux outside of that one conductor. The term $\frac{1}{2}\times 10^{-7} \frac{\mu}{\mu_0}$ is the inductance due to the internal flux-current linkage or the linkage of flux and current within one conductor. For copper conductors and at high frequencies, the second term approaches zero because of the skin effect (current tends to flow near the surface of the conductor and the flux-current linkage within the conductor is practically zero). At low frequencies, the second term is approximately equal to $\frac{1}{2}\times 10^{-7}$.

The inductive reactance is equal to $2\pi fL$, where f is the frequency. If the internal flux-current linkages can be neglected, then the inductive reactance, in ohms per unit length, for a line of two wires is

$$x = 25.1\times 10^{-7} f \ln \frac{D}{r} \quad \text{ohms per meter (two wires)} \qquad \textbf{(22–2)}$$

or

$$x = 0.00932 f \log_{10} \frac{D}{r} \quad \text{ohms per mile (two wires)} \qquad \textbf{(22–2a)}$$

(*c*) *Capacitance:* When the separation D of the conductors is large compared with the radius r of the conductors, then the capacitance between two wires, per unit length of line, is

$$C = \frac{\frac{k}{k_0}}{35.96\times 10^9 \ln \frac{D}{r}} \quad \text{farads per meter (between two wires)} \qquad \textbf{(22–3)}$$

or

$$C = \frac{0.01944 \frac{k}{k_0}}{\log_{10} \frac{D}{r}} \quad \text{microfarad per mile (between two wires)} \qquad \textbf{(22–3a)}$$

where k_0 = permittivity of free space = $8\ 854\times 10^{-12}$ farad per meter;

$\frac{k}{k_0}$ = relative permittivity, or ratio of permittivity of medium between wires to k_0 for free space.

The capacitive susceptance, in mhos per unit length of line, of two wires is

$$b = 2\pi fC = 1.745 \times 10^{-10} f \frac{\frac{k}{k_0}}{\ln \frac{D}{r}} \quad \text{mhos per meter (between two wires)} \qquad (22\text{–}4)$$

or

$$b = 0.12192 \times 10^{-6} f \frac{\frac{k}{k_0}}{\log_{10} \frac{D}{r}} \quad \text{mhos per mile (between two wires)} \qquad (22\text{–}4a)$$

22–3. Calculation of *R*, *L*, and *C* of a Three-Phase Line.*—In the treatment of a balanced three-phase circuit in the steady-state condition, it is only necessary to consider one phase. This phase, from line to neutral, is regarded as a single-phase circuit. The voltage is the line-to-neutral voltage, and the current is the line current. The power is one-third that of the entire three-phase circuit. The elements are those of one conductor (in the case of a balanced three-phase, four-wire system, the neutral current is zero and the neutral wire may be considered as imaginary). For clearness, the resistance, inductance, and capacitance of one wire in a balanced three-phase, three-wire system will be considered in this article. The shunt leakage conductance will be discussed in the following article.

(*a*) *Resistance:* The resistance of one wire in a balanced three-phase, three-wire system is found from tables in the manner outlined for the case of a two-wire line.

(*b*) *Inductance:* The conductors of three-phase transmission lines are supported on poles or towers in various arrangements, the construction depending on the voltage of the line, the type of supports used, etc. When the conductors are supported in an equilateral arrangement, the three wires are spaced equally and the performances of the three phases will be alike. In this equilateral arrangement, Fig. 22–3(*a*), the resistance, inductance, shunt capacitance, and shunt leakage conductance are said to be the same

* See footnote on page 513.

for all three wires. In practice, it is seldom convenient to space the conductors equally. Expressions for the inductance of one wire will be given for two cases: (*a*) when the conductors are spaced equally, and (*b*) when the conductors are spaced unequally but the line is transposed as shown in Fig. 22–3(*b*).

If the wires are located at the corners of an equilateral triangle, as shown in Fig. 22–3(*a*), and the internal flux-current linkages can be neglected, then the inductance for one wire is

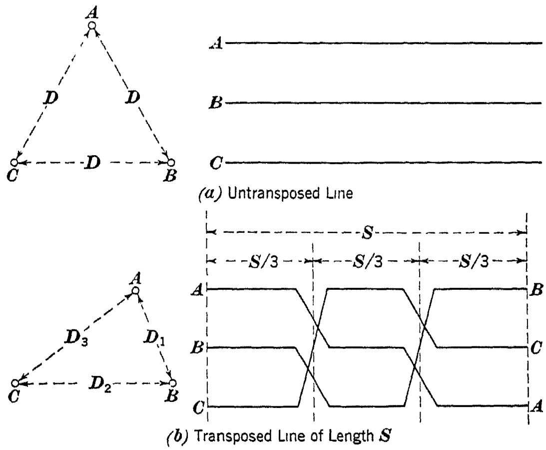


FIG. 22–3

$$L = 2 \times 10^{-7} \ln \frac{D}{r} \qquad \text{henrys per meter (one wire)} \qquad (22\text{–}5)$$

or

$$L = 741.13 \times 10^{-6} \log_{10} \frac{D}{r} \qquad \text{henrys per mile (one wire)} \qquad (22\text{–}5a)$$

When the arrangement is unsymmetrical, it is desirable to transpose* the conductors of a three-phase line to secure like per-

* When a line is transposed, the conductors occupy each position for one-third the length of the line. Transposition will minimize possible inductive interference with nearby lines, particularly with communication circuits. The bad effects of inductive interference are:

1. Introduction of noise in telephone circuits which interferes with speech transmission.

2. High induced voltages which may be hazardous to persons or property. See Chapter 11 on Inductive Coordination of "Communication Engineering," by W. L. Everitt.

formance in the three phases. Thus, if the internal flux-current linkages of the conductors may be neglected and the line is transposed as in Fig. 22–3(*b*), the inductance for one wire is

$$L = 2\times 10^{-7} \ln \frac{\sqrt[3]{D_1D_2D_3}}{r} \quad \text{henrys per meter (one wire)} \qquad (22\text{–}6)$$

or

$$L = 741.13\times 10^{-6} \log_{10} \frac{\sqrt[3]{D_1D_2D_3}}{r} \quad \text{henrys per mile (one wire)} \qquad (22\text{–}6a)$$

(*c*) *Capacitance:* The capacitance of each conductor of a three-phase line is sometimes known as the *capacitance to neutral.* For an equilateral spacing of conductors, as shown in Fig. 22–3(*a*), the capacitance C of one conductor to neutral is

$$C = \frac{\dfrac{k}{k_0}}{17.98\times 10^9 \ln \dfrac{D}{r}} \quad \text{farad per meter (wire to neutral)} \qquad (22\text{–}7)$$

or

$$C = \frac{0.03883\,\dfrac{k}{k_0}}{\log_{10} \dfrac{D}{r}} \quad \text{microfarad per mile (wire to neutral)} \qquad (22\text{–}7a)$$

If the spacing is unequal and the line is transposed, as indicated in Fig. 22–3(*b*), the so-called capacitance to neutral may be calculated by substituting for D in equation (22–7) or equation (22–7*a*) a mean spacing

$$D' = \sqrt[3]{D_1D_2D_3}$$

Example 22–1.—It is desired to find the resistance, inductance, inductive reactance, capacitance, and capacitive susceptance per mile of single conductor of a balanced three-phase, open-wire transmission line. The length of the transmission line is 250 miles. The conductors are of No. 2/0 B & S stranded copper placed in an equilateral arrangement with a spacing of 10 ft. The frequency of the source is 60 cps.

Solution.—From wire tables, the resistance of a No. 2/0 B & S stranded copper wire at approximately 60 F is 0.41 ohm per mile.

The diameter of a No. 2/0 B & S stranded copper conductor is approximately 0.42 in. The spacing D, expressed in the same unit, is 120 in. If the flux-current linkages within the conductors are neglected, equation (22–5a) gives the inductance per mile of one wire. Thus,

$$L = 741.13\times 10^{-6} \log_{10} \frac{120}{0.21} = 0.00204 \text{ henry per mile}$$

The inductive reactance of one wire is

$$x = 2\pi f L = 2\pi \times 60 \times 0.00204 = 0.769 \text{ ohm per mile}$$

The capacitance of one conductor to neutral is given by equation (22–7a). Since the space between the conductors is air, then $\dfrac{k}{k_0} = 1$ and

$$C = \frac{0\ 03883}{\log_{10} \dfrac{120}{0.21}} = 0.01409\ \mu\text{f per mile}$$

The capacitive susceptance of one wire to neutral is

$$b = 2\pi f C = 2\pi \times 60 \times 0.01409 \times 10^{-6} = 5.31 \times 10^{-6} \text{ mho per mile}$$

22–4. Conductance of Shunt Leakage Path.—The conductance of the shunt leakage path per conductor may be a result, particularly at very high transmission voltages, of leakage current, dielectric loss, and corona. It is usually represented by the symbol g or G. Its value depends largely on weather conditions. Line conductance is often neglected in calculations relating to the performance of lines operated at voltages up to about 150 kv. If the insulation of a line is good, the conductance g is practically zero even for high-voltage power lines. It is difficult to calculate the conductance from the dimensions and physical properties of the line materials. Any value given to g in a problem is only an approximation.

22–5. Basic Differential Equations for the Exact Solution of Long Lines.—For convenience, a simple two-parallel-wire transmission line with uniformly distributed elements will be considered. It is assumed that the circuit is linear and operating in the steady state with impressed sinusoidal voltages of a certain frequency. The voltages and currents in every part of the circuit will then be sinusoidal and will have the same frequency. The equations to be derived for the two-wire line apply equally well to one phase of a balanced three-phase, Y-Y circuit if line-to-neutral voltages and elements of one wire are used.

The aim of this article is to set up the basic differential equations. The steady-state solution of these equations and the resulting expressions for voltage and current at any point along the line will be given in the next two articles. The following notation will be used:

r = resistance per meter of line of two wires (ohms)
x = reactance per meter of line of two wires (ohms)
$\dot{z} = r + jx$ = impedance per meter of line of two wires (ohms)
g = conductance from wire to wire per meter (mhos)
b = susceptance from wire to wire per meter (mhos)
$\dot{y} = g + jb$ = admittance from wire to wire per meter (mhos)
$\dot{E}$ = voltage at any point P along the line (volts)
$\dot{I}$ = current in the line conductor at P (amp)
$\dot{E}_r$, $\dot{I}_r$ = voltage, current at load end (volts, amp)
$\dot{E}_s$, $\dot{I}_s$ = voltage, current at sending end (volts, amp)
S = total length of transmission line (meters)

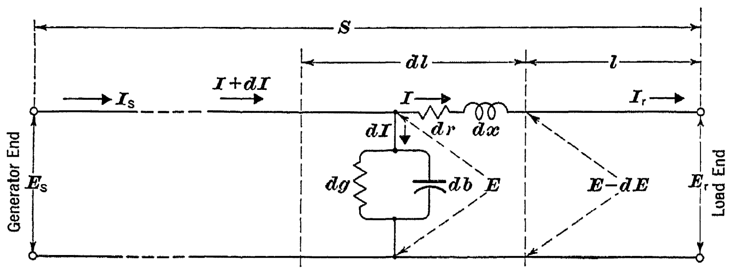


FIG. 22–4

Consider a length dl of line carrying a current $\dot{I}$ at voltage $\dot{E}$ and located at a distance of l meters from the load or receiving end, as shown in Fig. 22–4. Since the voltage drop in any infinitesimal section is equal to the drop per meter multiplied by the length of the infinitesimal section,

$$d\dot{E} = \dot{I}\dot{z}\,dl \qquad (22\text{–}8)$$

Hence, the basic equation for the voltage is

$$\frac{d\dot{E}}{dl} = \dot{I}\dot{z} \qquad (22\text{–}9)$$

Likewise, the current lost in any infinitesimal section of line, due to the shunt admittance from wire to wire, equals the current lost per meter of line multiplied by the length of the infinitesimal section, or

$$d\dot{I} = \dot{E}\dot{y}\,dl \qquad (22\text{–}10)$$

The basic equation for the current is, then,

$$\frac{d\dot{I}}{dl} = \dot{E}\dot{y} \tag{22–11}$$

22–6. Solution of Basic Differential Equations.—To solve for $\dot{E}$ and $\dot{I}$ in the basic differential equations (22–9) and (22–11), differentiate each with respect to l. The results are

$$\frac{d^2\dot{E}}{dl^2} = \dot{z}\frac{d\dot{I}}{dl} \tag{22–12}$$

$$\frac{d^2\dot{I}}{dl^2} = \dot{y}\frac{d\dot{E}}{dl} \tag{22–13}$$

Substituting the value for $\frac{d\dot{I}}{dl}$ from equation (22–11) and the value for $\frac{d\dot{E}}{dl}$ from equation (22–9) gives:

$$\frac{d^2\dot{E}}{dl^2} - \dot{y}\dot{z}\dot{E} = 0 \tag{22–14}$$

$$\frac{d^2\dot{I}}{dl^2} - \dot{y}\dot{z}\dot{I} = 0 \tag{22–15}$$

Equations (22–14) and (22–15) are linear differential equations of the second order with constant coefficients. The solutions are.

$$\dot{E} = A_1 e^{\sqrt{\dot{y}\dot{z}}l} + A_2 e^{-\sqrt{\dot{y}\dot{z}}l} = A_1 e^{\gamma l} + A_2 e^{-\gamma l} \tag{22–16}$$

$$\dot{I} = B_1 e^{\sqrt{\dot{y}\dot{z}}l} + B_2 e^{-\sqrt{\dot{y}\dot{z}}l} = B_1 e^{\gamma l} + B_2 e^{-\gamma l} \tag{22–17}$$

where A_1, A_2, B_1, and B_2 are constants to be determined from the boundary conditions, l is measured from the load end, and

$$\gamma = \sqrt{\dot{y}\dot{z}} = \sqrt{(g+jb)(r+jx)} \tag{22–18}$$

The quantity denoted by γ is called the propagation constant and will be discussed as a complex number $(\alpha + j\beta)$ in Art. 22–11.

(*a*) *Determination of the Constants A_1 and A_2:* Two equations are necessary to find A_1 and A_2. For the boundary conditions at the load end, $l=0$, $\dot{E}=\dot{E}_r$, and $\dot{I}=\dot{I}_r$; and equation (22–16) becomes

$$\dot{E}_r = A_1 + A_2 \tag{22–19}$$

The second equation is obtained by differentiating equation (22–16) with respect to l, making use of equation (22–9), and then applying the boundary conditions at $l=0$. Thus,

$$\frac{d\dot{E}}{dl} = \gamma A_1 e^{\gamma l} - \gamma A_2 e^{-\gamma l} = \dot{I}\dot{z} \tag{22–20}$$

$$\left[\frac{d\dot{E}}{dl}\right]_{l=0} = \dot{I}_r \dot{z} = \gamma A_1 - \gamma A_2 \tag{22–21}$$

Solving equations (22–19) and (22–21) simultaneously gives:

$$\left.\begin{aligned} A_1 &= \frac{\dot{E}_r}{2} + \frac{\dot{I}_r \dot{Z}_0}{2} \\ A_2 &= \frac{\dot{E}_r}{2} - \frac{\dot{I}_r \dot{Z}_0}{2} \end{aligned}\right\} \tag{22–22}$$

where $\dot{Z}_0 = \sqrt{\frac{\dot{z}}{\dot{y}}}$. This quantity $\dot{Z}_0$ is called the characteristic impedance, or surge impedance, of the line. A discussion of the characteristic impedance will be given in Art. 22–13.

(*b*) *Determination of the Constants B_1 and B_2:* For the boundary conditions at the load end where $l=0$, $\dot{E} = \dot{E}_r$, and $\dot{I} = \dot{I}_r$, equation (22–17) becomes

$$\dot{I}_r = B_1 + B_2 \tag{22–23}$$

By differentiating equation (22–17) with respect to l, making use of equation (22–11), and then applying the boundary conditions at $l=0$, we get the second equation for the determination of B_1 and B_2. Thus,

$$\frac{d\dot{I}}{dl} = \gamma B_1 e^{\gamma l} - \gamma B_2 e^{-\gamma l} = \dot{E}\dot{y} \tag{22–24}$$

$$\left[\frac{d\dot{I}}{dl}\right]_{l=0} = \dot{E}_r \dot{y} = \gamma B_1 - \gamma B_2 \tag{22–25}$$

When equations (22–23) and (22–25) are solved simultaneously, the constants are:

$$\left.\begin{aligned} B_1 &= \frac{\dot{I}_r}{2} + \frac{\dot{E}_r \dot{Y}_0}{2} \\ B_2 &= \frac{\dot{I}_r}{2} - \frac{\dot{E}_r \dot{Y}_0}{2} \end{aligned}\right\} \tag{22–26}$$

where $\dot{Y}_0 = \frac{1}{\dot{Z}_0} = \sqrt{\frac{\dot{y}}{\dot{z}}}$. This quantity is called the characteristic admittance, or surge admittance, of the line.

The substitution of the values for A_1, A_2, B_1, and B_2 in equations (22–16) and (22–17) will be considered in the following article.

22–7. General Voltage and Current Equations.—Two sets of steady-state equations for the voltage and current at any point along the transmission line, one for the distance measured from the receiving or load end and another for the distance measured from the sending or generator end, will now be considered.

(*a*) *For l Measured From Receiving End:* This set of equations is convenient when the voltage $\dot{E}_r$ and the current $\dot{I}_r$ at the receiving end are either known or given. When the values of A_1 and A_2 from equation (22–22) are substituted in equation (22–16), the voltage at any point P at a distance of l meters from the load end* is found to be:

$$\dot{E} = \left(\frac{\dot{E}_r}{2} + \frac{\dot{I}_r\dot{Z}_0}{2}\right)e^{\gamma l} + \left(\frac{\dot{E}_r}{2} - \frac{\dot{I}_r\dot{Z}_0}{2}\right)e^{-\gamma l} \tag{22–27}$$

from which

$$\dot{E} = \frac{\dot{E}_r}{2}(e^{\gamma l} + e^{-\gamma l}) + \frac{\dot{I}_r\dot{Z}_0}{2}(e^{\gamma l} - e^{-\gamma l}) \tag{22–27a}$$

or

$$\dot{E} = \dot{E}_r \cosh \gamma l + \dot{I}_r\dot{Z}_0 \sinh \gamma l \tag{22–28}$$

Similarly, by substituting values of B_1 and B_2 from equation (22–26) in equation (22–17), we find that the current at any point P at a distance of l meters from the load end is:

$$\dot{I} = \left(\frac{\dot{I}_r}{2} + \frac{\dot{E}_r\dot{Y}_0}{2}\right)e^{\gamma l} + \left(\frac{\dot{I}_r}{2} - \frac{\dot{E}_r\dot{Y}_0}{2}\right)e^{-\gamma l} \tag{22–29}$$

from which

$$\dot{I} = \frac{\dot{I}_r}{2}(e^{\gamma l} + e^{-\gamma l}) + \frac{\dot{E}_r\dot{Y}_0}{2}(e^{\gamma l} - e^{-\gamma l}) \tag{22–29a}$$

or

$$\dot{I} = \dot{I}_r \cosh \gamma l + \dot{E}_r\dot{Y}_0 \sinh \gamma l \tag{22–30}$$

(*b*) *For l Measured From Supply End:* When the voltage $\dot{E}_s$ and the current $\dot{I}_s$ at the supply end are known, it is convenient to have a set of equations in which $\dot{E}$ and $\dot{I}$ are expressed in terms of these quantities. As indicated in Fig. 22–5, $\dot{E}_s$ is not necessarily the generator voltage. By determining the constants A_1, A_2, B_1, and B_2 of equations (22–16) and (22–17) for the boundary conditions at the sending end, where $l=0$, $\dot{E}=\dot{E}_s$, and $\dot{I}=\dot{I}_s$, we can derive expressions for the voltage and current at any point P at a distance

* See Appendix G on Hyperbolic Functions.

of l meters from the supply end. These expressions are (see Problem 22-9 at the end of this chapter):

$$\dot{E} = \dot{E}_s \cosh \gamma l - \dot{I}_s \dot{Z}_0 \sinh \gamma l \qquad (22\text{–}31)$$

$$\dot{I} = \dot{I}_s \cosh \gamma l - \dot{E}_s \dot{Y}_0 \sinh \gamma l \qquad (22\text{–}32)$$

Equations (22–31) and (22–32) may also be obtained by changing $\dot{E}_r$, $\dot{I}_r$, and l in equations (22–28) and (22–30) into $\dot{E}_s$, $\dot{I}_s$, and $-l$ (l becomes negative because the distance along the line is now measured in the direction of the flow of energy, that is, from the generator toward the receiver). It is to be noted that

$$\cosh(-\gamma l) = \cosh \gamma l$$
$$\sinh(-\gamma l) = -\sinh \gamma l$$

Example 22–2. A 60-cycle, three-phase, three-wire transmission line has the following elements per mile of one wire:

$$r = 0.272 \text{ ohm} \qquad g = 0$$
$$x = 0.762 \text{ ohm} \qquad b = 5.6 \times 10^{-6} \text{ mho}$$

The total length of the transmission line is $S = 100$ miles. The line-to-line voltage at the load end is 66,000 volts. If the balanced three-phase load at the receiving end is 15,000 kva, at 80% power factor lagging, find: (*a*) the propagation constant γ; (*b*) the characteristic impedance Z_0 and the characteristic admittance Y_0; (*c*) the voltage and current at the sending end.

Solution.—(*a*) The line impedance per mile of one wire is

$$\dot{z} = r + jx = 0.272 + j0.762 = 0.809\underline{/70.3^\circ} \text{ ohm}$$

The shunt admittance per mile of one wire to neutral is

$$\dot{y} = g + jb = 0 + j5.6 \times 10^{-6} = 5.6 \times 10^{-6}\underline{/90^\circ} \text{ mho}$$

The propagation constant per mile of one wire is

$$\gamma = \sqrt{\dot{y}\dot{z}} = 0.00213\underline{/80.16^\circ} = 0.000364 + j0.0021 = \alpha + j\beta$$

(*b*) The characteristic impedance and the characteristic admittance are:

$$\dot{Z}_0 = \sqrt{\frac{\dot{z}}{\dot{y}}} = \sqrt{\frac{0.809\underline{/70.3^\circ}}{5.6 \times 10^{-6}\underline{/90^\circ}}} = 380\underline{/-9.83^\circ}$$

$$\dot{Y}_0 = \frac{1}{\dot{Z}_0} = \frac{1}{380\underline{/-9.83^\circ}} = 0.00263\underline{/9.83^\circ}$$

(*c*) Before finding the voltage and current at the supply end, it is convenient to determine first the values of $\cosh \gamma S$ and $\sinh \gamma S$. They are:

$$\begin{aligned}\cosh \gamma S &= \cosh(0.0364 + j0.21)\\ &= (\cosh 0.0364)(\cos 0.21) + j(\sinh 0.0364)(\sin 0.21)\\ &= (1.0007 \times 0.978) + j(0.0364 \times 0.208) = 0.979\underline{/0.48^\circ}\end{aligned}$$

$$\begin{aligned}\sinh \gamma S &= \sinh(0.0364 + j0.21)\\ &= (\sinh 0.0364)(\cos 0.21) + j(\cosh 0.0364)(\sin 0.21)\\ &= 0.211\underline{/80.3^\circ}\end{aligned}$$

With the line-to-neutral voltage at the load end, or $E_r = \frac{66{,}000}{\sqrt{3}} =$ 38,100 volts, being used as the reference vector and with $I_r = \frac{15{,}000{,}000}{\sqrt{3} \times 66{,}000}$ = 131.3 amp lagging 36.8° behind $\dot{E}_r$, the line-to-neutral voltage and the current at the supply end are easily calculated by the use of equations (22–28) and (22–30). Thus,

$$\dot{E}_s = \dot{E}_r \cosh \gamma S + \dot{I}_r \dot{Z}_0 \sinh \gamma S$$
$$= 38{,}000\underline{/0^\circ} \times 0.979\underline{/0\ 45^\circ} + 131.3\underline{/-36.8^\circ} \times 380\underline{/-9\ 83^\circ} \times 0\ 211\underline{/80\ 3^\circ}$$
$$= 37{,}300\underline{/0\ 45^\circ} + 10{,}528\underline{/33\ 67^\circ} = 46{,}700\underline{/7.2^\circ} \text{ volts}$$
$$\dot{I}_s = \dot{I}_r \cosh \gamma S + \dot{E}_r \dot{Y}_0 \sinh \gamma S$$
$$= 131.3\underline{/-36\ 8^\circ} \times 0.979\underline{/0.45^\circ} + 38{,}100 \times 0.00263\underline{/9.83^\circ} \times 0.211\underline{/80.3^\circ}$$
$$= 128.54\underline{/-36.4^\circ} + 21.14\underline{/90\ 1^\circ} = 115.6\underline{/-28^\circ} \text{ amp}$$

It can be seen that the sending-end voltage $\dot{E}_s$ is leading the receiving-end voltage $\dot{E}_r$ by 7.2° in time phase; and the sending-end current $\dot{I}_s$ is lagging 28° behind $\dot{E}_r$ in time phase. The time phase difference between $\dot{E}_s$ and $\dot{I}_s$ is $\theta_s =$ 35 2°.

22–8. Direct-Current Case.—In the direct-current case of a long line with uniformly distributed elements, the numerical calculations are somewhat less tedious than those for the alternating-current case. Since the result in a direct-current line is obtained by setting the frequency equal to zero, the factors $x = 2\pi fL$ and $b = 2\pi fC$ drop out of the equations and the characteristic impedance $\dot{Z}_0$ reduces to $R_0 = \sqrt{\frac{r}{g}}$, the characteristic resistance. Equations (22–28), (22–30), (22–31), and (22–32) may then be written as follows.

(*a*) *For l Measured From Receiving End:*

$$E = E_r \cosh \sqrt{rg}\, l + I_r R_0 \sinh \sqrt{rg}\, l \qquad \textbf{(22–28a)}$$

$$I = I_r \cosh \sqrt{rg}\, l + \frac{E_r}{R_0} \sinh \sqrt{rg}\, l \qquad \textbf{(22–30a)}$$

(*b*) *For l Measured From Sending End:*

$$E = E_s \cosh \sqrt{rg}\, l - I_s R_0 \sinh \sqrt{rg}\, l \qquad \textbf{(22–31a)}$$

$$I = I_s \cosh \sqrt{rg}\, l - \frac{E_s}{R_0} \sinh \sqrt{rg}\, l \qquad \textbf{(22–32a)}$$

22–9. Useful Transmission-Line Equations.—In problems where the generator voltage $\dot{E}_g$, the generator impedance $\dot{Z}_g$, and the load impedance $\dot{Z}_r$ are given, as in Fig. 22–5, the quantities $\dot{I}_r$, $\dot{E}_r$, and $\dot{I}_s$ can be determined from the expressions developed in this article.

If S is the total length of the transmission line, then, from equations (22–28) and (22–30),

$$\dot{E}_s = \dot{E}_r \cosh \gamma S + \dot{I}_r \dot{Z}_0 \sinh \gamma S$$

$$\dot{I}_s = \dot{I}_r \cosh \gamma S + \frac{\dot{E}_r}{\dot{Z}_0} \sinh \gamma S$$

Since $\dot{E}_s = \dot{E}_g - \dot{I}_s \dot{Z}_g$ and $\dot{E}_r = \dot{I}_r \dot{Z}_r$, then by substitution

$$\dot{E}_g - \dot{I}_s \dot{Z}_g = \dot{I}_r \dot{Z}_r \cosh \gamma S + \dot{I}_r \dot{Z}_0 \sinh \gamma S \qquad (22\text{–}33)$$

$$\dot{I}_s = \dot{I}_r \cosh \gamma S + \frac{\dot{I}_r \dot{Z}_r}{\dot{Z}_0} \sinh \gamma S \qquad (22\text{–}34)$$

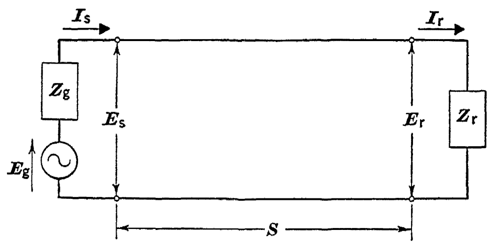


FIG. 22–5

Substituting the value of $\dot{I}_s$ from equation (22–34) in equation (22–33) and solving for the load current, we obtain:

$$\dot{I}_r = \frac{\dot{E}_g \dot{Z}_0}{(\dot{Z}_g + \dot{Z}_r)\dot{Z}_0 \cosh \gamma S + (\dot{Z}_0^2 + \dot{Z}_r \dot{Z}_g) \sinh \gamma S} \qquad (22\text{–}35)$$

The voltage across the load is

$$\dot{E}_r = \dot{I}_r \dot{Z}_r = \frac{\dot{E}_g \dot{Z}_0 \dot{Z}_r}{(\dot{Z}_g + \dot{Z}_r)\dot{Z}_0 \cosh \gamma S + (\dot{Z}_0^2 + \dot{Z}_r \dot{Z}_g) \sinh \gamma S} \qquad (22\text{–}36)$$

Substituting the value of $\dot{I}_r$ from equation (22–35) in equation (22–34) and rearranging, we get:

$$\dot{I}_s = \frac{\dot{E}_g (\dot{Z}_0 \cosh \gamma S + \dot{Z}_r \sinh \gamma S)}{(\dot{Z}_g + \dot{Z}_r)\dot{Z}_0 \cosh \gamma S + (\dot{Z}_0^2 + \dot{Z}_r \dot{Z}_g) \sinh \gamma S} \qquad (22\text{–}37)$$

It should be noted that equations (22–35) to (22–37) have the same denominator.

22–10. Physical Interpretation of Long Line Equations.—The physical significance of the voltage and current equations, where l is measured from the load end, will be more apparent if expressions (22–28) and (22–30) are written in the exponential form, as follows.

$$\dot{E} = \dot{E}_r \cosh \gamma l + \dot{I}_r \dot{Z}_0 \sinh \gamma l$$

$$= \left(\frac{\dot{E}_r}{2} + \frac{\dot{I}_r \dot{Z}_0}{2}\right) e^{\gamma l} + \left(\frac{\dot{E}_r}{2} - \frac{\dot{I}_r \dot{Z}_0}{2}\right) e^{-\gamma l}$$

or

$$\dot{E} = \dot{E}_r^+ e^{\alpha l} e^{j\beta l} + \dot{E}_r^- e^{-\alpha l} e^{-j\beta l} \tag{22–38}$$

$$\dot{I} = \dot{I}_r \cosh \gamma l + \dot{E}_r \dot{Y}_0 \sinh \gamma l$$

$$= \left(\frac{\dot{I}_r}{2} + \frac{\dot{E}_r \dot{Y}_0}{2}\right) e^{\gamma l} + \left(\frac{\dot{I}_r}{2} - \frac{\dot{E}_r \dot{Y}_0}{2}\right) e^{-\gamma l}$$

or

$$\dot{I} = \dot{I}_r^+ e^{\alpha l} e^{j\beta l} + \dot{I}_r^- e^{-\alpha l} e^{-j\beta l} \tag{22–39}$$

where

$$\dot{E}_r^+ = \frac{\dot{E}_r}{2} + \frac{\dot{I}_r \dot{Z}_0}{2}$$

$$\dot{E}_r^- = \frac{\dot{E}_r}{2} - \frac{\dot{I}_r \dot{Z}_0}{2}$$

$$\dot{I}_r^+ = \frac{\dot{I}_r}{2} + \frac{\dot{E}_r \dot{Y}_0}{2}$$

$$\dot{I}_r^- = \frac{\dot{I}_r}{2} - \frac{\dot{E}_r \dot{Y}_0}{2}$$

$$\gamma = \sqrt{\dot{y}\dot{z}} = \alpha + j\beta$$

It can be seen that each term of equation (22–38) is a voltage and that each term of equation (22–39) is a current. At the load end, where $l = 0$, the exponentials equal unity, and the equations become

$$\dot{E}_r = \dot{E}_r^+ + \dot{E}_r^- \tag{22–40}$$

$$\dot{I}_r = \dot{I}_r^+ + \dot{I}_r^- \tag{22–41}$$

Thus, the voltage at any point along the line may be viewed as being equal to the vector sum of two voltage components. Likewise, the current at any point along the line may be considered as consisting of two current components.

The nature of the exponential factors $e^{\gamma l}$ and $e^{-\gamma l}$ may be reviewed to advantage before the voltage and current components are considered as forward and backward waves. By the laws of exponents,

$$e^{\gamma l} = e^{(\alpha + j\beta)l} = e^{\alpha l} e^{j\beta l} \tag{22-42}$$

$$e^{-\gamma l} = e^{-(\alpha + j\beta)l} = e^{-\alpha l} e^{-j\beta l} \tag{22-43}$$

The factor $e^{\alpha l}$ has magnitude only; its angle is zero. At $l = 0$, its value is $e^{\alpha l} = e^0 = 1$. As l increases from the load end, the value of $e^{\alpha l}$ increases for positive values of α and l. Of course, as l increases, $e^{-\alpha l} = \frac{1}{e^{\alpha l}}$ decreases in value. Thus, the multiplication of a voltage or current vector by $e^{\alpha l}$ or $e^{-\alpha l}$ changes the magnitude only.

The factor $e^{j\beta l}$ may be evaluated as follows:

$$e^{j\beta l} = \cos \beta l + j \sin \beta l = 1\underline{/\beta l} \tag{22-44}$$

Thus, this factor has a magnitude of unity but its angle increases with l for positive values of β and l. It may be considered here that the multiplication of a voltage or current vector by $e^{j\beta l}$ advances its phase in the positive or conventional counter-clockwise direction. Similarly,

$$e^{-j\beta l} = \cos \beta l - j \sin \beta l = 1\underline{/-\beta l} \tag{22-45}$$

Hence, multiplication of a voltage or current vector by this factor will mean a dropping behind in phase. Thus, the multiplication of a voltage or current vector by the exponential factor $e^{\gamma l}$ or $e^{-\gamma l}$ will, in general, change the magnitude and phase of the voltage or current vector.

Next the voltage components of equation (22-38) and the current components of equation (22-39) may be interpreted as incident (forward) and reflected (backward) waves. For simplicity, however, only the current components will be considered. It can be seen that, since the equations have the same form, what will be said about the current expression will hold similarly for the voltage equation. As l increases from the load end to the generator end, the first term on the right-hand side of equation (22-39) increases in magnitude because of the effect of $e^{\alpha l}$ and advances in phase because of the effect of $e^{j\beta l}$. This term may also be pictured as a current wave which proceeds from the sending end to the receiving end, decreasing in magnitude by the constant factor e^{α} per unit length and dropping behind in phase by the constant increment β radians per unit length. If it is agreed that, in the direction of travel, a traveling wave is characterized by a retardation of phase and usually also by a decrease in magnitude, then the first term may be called the *incident wave* or the *direct wave*, traveling from the generator end to the load end.

The second term of equation (22–39) decreases in magnitude because of the effect of $e^{-\alpha l}$ and lags behind more and more in phase because of the effect of $e^{-j\beta l}$, as the wave travels from the load end to the generator end. Although this second wave is due to reflection at the load impedance, similar to a wave reflected from a dam in a water canal, it may be viewed as a wave originating at the load end and traveling toward the supply end. This viewpoint satisfies the foregoing characterization of a traveling wave and the wave may be called the *reflected wave.* Thus, the current at any point along a transmission line is the sum of the incident (forward) wave component and the reflected (backward) wave component.

So far it has been indicated that, when a current passes through a unit length of line, it generally changes in magnitude and phase. This change may be interpreted physically. The change in magnitude is caused by the leakage current through the shunt branches, and the change in phase angle is due to the finite amount of time required for the current to be transmitted (propagated) through the unit length. The phase angle β is the angle between two current vectors of the incident (or reflected) wave at two points a unit distance apart, and it should not be confused with the phase angle between a voltage vector and a current vector at a given point on the line.

22–11. Propagation Constant.—The propagation constant of a line with distributed elements is

$$\gamma = \sqrt{yz} = \sqrt{(r+j\omega L)(g+j\omega C)} = \alpha + j\beta \qquad (22\text{–}18)$$

where α is called the attenuation constant and β is the wavelength constant, or phase constant.

It has been shown that the attenuation constant α (in e^{α} or $e^{-\alpha}$) determines, for example, the difference in magnitudes of two forward-wave current vectors at two points a unit distance apart. The phase constant β (in $e^{j\beta l}$ or $e^{-j\beta l}$) measures, for example, the phase difference between two forward-wave current vectors at two points a unit distance apart. Similar changes occur to the backward-wave current vectors with regard to magnitude and phase. Thus, as energy is transferred (propagated) along the line, the propagation constant γ (in e^{γ} or $e^{-\gamma}$) determines the changes in both magnitude and phase angle of the current vector or the voltage vector per unit length along the line.

When the magnitude of the forward-wave current vector and the magnitude of the forward-wave voltage vector remain the same along the entire line, $\alpha = 0$ and the line is said to have no attenuation. When the forward-wave current vectors at the different points along the entire line are in phase and the forward-wave voltage vectors at the different points along the entire line are also in phase, $\beta = 0$ and the line is considered to have no phase shift. Of course, when the attenuation constant is zero for the forward-wave components, it will be zero also for the backward-wave components; and, when the phase constant is zero for the forward-wave components, it will be zero also for the backward-wave components. If z and y are given in ohms and mhos per meter, respectively, then the attenuation constant α is given in nepers* per meter and the wavelength constant β is in radians per meter. In Example 22–2, $\alpha = 0.000364$ neper per mile and $\beta = 0.0021$ radian per mile.

Sometimes, it is desirable to have expressions for α and β in terms of the line elements r, L, g, and C and the frequency f. Then the effects of the line elements and the transmitted frequency on the values of the attenuation constant and the wavelength constant can be determined more easily. Squaring equation (22–18) and rearranging gives

$$\alpha^2 - \beta^2 + j2\alpha\beta = rg - \omega^2 LC + j\omega(gL + rC)$$

Equating the real parts and the j parts, we have:

$$\alpha^2 - \beta^2 = rg - \omega^2 LC$$
$$2\alpha\beta = \omega(gL + rC)$$

When these two equations are solved simultaneously, the results are:

$$\alpha = \sqrt{\tfrac{1}{2}[\sqrt{(r^2 + \omega^2 L^2)(g^2 + \omega^2 C^2)} + (rg - \omega^2 LC)]} \qquad (22\text{–}46)$$

* For the forward wave alone (or the backward wave alone), 1 neper corresponds to a current ratio of 2.7182. Another unit for α is the decibel, which is abbreviated as db. One db corresponds to a current ratio of 1.122. Thus, 1 db = 0.115 neper, and 1 neper = 8.686 db.

$$\beta = \sqrt{\tfrac{1}{2}[\sqrt{(r^2+\omega^2L^2)(g^2+\omega^2C^2)} - (rg - \omega^2LC)]} \qquad (22\text{–}47)$$

Equations (22–46) and (22–47) state that, in general, a variation in frequency will change the values of α and β and, consequently, the value of γ. When some of the elements are negligible, the expressions for α and β become very much simplified, as indicated in Example 22–3.

22–12. Wavelength and Phase Velocity of Propagation.—For simplicity, only the forward component wave of the current will be considered in this article. It can be seen that what will be said about this component wave will hold similarly not only for the backward component wave of the current but also for the two component waves of the voltage.

It will be shown that the wavelength λ is usually expressed in terms of the wavelength constant or phase constant β. If β represents the phase difference between two current vectors of the forward component wave spaced 1 meter apart along the transmission line, then the phase difference between two vectors spaced 2 meters apart will be 2β radians. And for two vectors spaced l meters apart, the phase difference is βl radians. It is conceivable that l could have such a value that $\beta l = 2\pi$ radians. The distance in which the current vector of the forward component wave makes one complete rotation of 360° or 2π radians is called the *wavelength.* It is a length of line on which the current vectors at the opposite ends are exactly 1 cycle or 2π radians apart in time phase. If β is in radians per meter and λ is in meters, then

$$\beta\lambda = 2\pi \text{ radians} \qquad (22\text{–}48)$$

or

$$\lambda = \frac{2\pi}{\beta} \text{ meters} \qquad (22\text{–}49)$$

The velocity with which the forward component wave of the current moves along the line is called the *phase velocity* or the *phase velocity of propagation.* If v is the phase velocity in meters per second and β is the phase rotation of the current vector of the forward component wave in radians per meter, it can be seen that

$$v\beta = \frac{\text{meter}}{\text{second}} \cdot \frac{\text{radians}}{\text{meter}} = \text{radians per second}$$

This relation indicates that, when the forward component wave of the current moves along the line for 1 second, the vector of the forward component wave of the current has rotated through a number of radians equal to $v\beta$. If f is the frequency transmitted, then the number of radians per second is $\omega = 2\pi f$. Thus,

$$v\beta = 2\pi f = \omega \tag{22–50}$$

or

$$v = \frac{2\pi f}{\beta} = \frac{\omega}{\beta} \text{ meters per second} \tag{22–51}$$

It is interesting to note that the phase velocity of propagation is different for each frequency unless β is proportional to the frequency in such a way that the values of f cancel out in equation (22–51) and v is independent of frequency. Also, it should be understood that the phase velocity of propagation is not the velocity at which the current flows along the transmission line. It is the velocity at which the forward component wave moves from the generator end to the load end. The phenomenon is "somewhat" analogous to the application of pressure at one end of a long pipe filled with water. The pressure appears almost immediately at the other end, even though there may be no actual flow of water.

Example 22–3.—In many cable circuits for use at voice frequencies, the inductance and leakage conductance are almost negligible. Because of the small spacing between conductors, the external flux-current linkages are very small and, hence, the inductance is very low. If the cable is well constructed with good insulation, the leakage conductance will be negligible. As an approximation,

$$\dot{z} = r + j\omega L \approx r$$
$$\dot{y} = g + j\omega C \approx j\omega C$$

Find α, β, γ, and v for such a cable. Also, discuss the effect of α and v on the transmission of speech or music.

Solution.—If L and g are assumed to be zero, then equations (22–46) and (22–47) become

$$\alpha = \sqrt{\frac{\omega C r}{2}}$$

$$\beta = \sqrt{\frac{\omega C r}{2}}$$

Hence,

$$\gamma = \alpha + j\beta = \sqrt{\frac{\omega C r}{2}} + j\sqrt{\frac{\omega C r}{2}} = \sqrt{\omega C r}\,\underline{/45^\circ} = \sqrt{j\omega C r}$$

From equation (22–51), the phase velocity of propagation is

$$v = \frac{\omega}{\sqrt{\frac{\omega C r}{2}}} = \sqrt{\frac{2\omega}{Cr}}$$

It can be seen that both α and v are proportional to the square root of the frequency transmitted. In the transmission of speech or music (assume that a voice frequency band from approximately 200 cps to 3,000 cps is being transmitted along a cable) this proportionality of α and v would be undesirable for the following reasons:

(a) The higher frequencies will be attenuated more than the lower ones because α is larger for the higher frequencies.

(b) The higher frequencies will arrive at the receiving end sooner than the lower frequencies because the higher frequencies have greater velocities.

Each of these effects will cause the form of the voice wave at the receiving end to be different from that at the sending end. The result is distortion. For satisfactory transmission, α and v should be independent of frequency as much as possible.

22–13. Characteristic Impedance.—The characteristic, or surge, impedance $\dot{Z}_0$ of a transmission line with distributed elements, as defined in Art. 22–6, is

$$\dot{Z}_0 = \sqrt{\frac{\dot{z}}{\dot{y}}} = \sqrt{\frac{r+jx}{g+jb}} \tag{22–52}$$

This expression states that $\dot{Z}_0$ is independent of the length of the line, since it is proportional to the ratio of z to y. Its value is determined by r, L, g, and C per unit length and by the frequency f. That is, Z_0 may be thought of as some kind of a factor which is characteristic of a given line. For the usual 60-cycle power lines, the numerical value of Z_0 lies between 350 and 400 ohms. The rest of this article will be devoted to the consideration of some of the relations involving Z_0, with the hope that a better picture of the characteristic impedance might be gained from the discussions. The calculation of the characteristic impedance from measurements will be given in Art. 22–14.

The characteristic impedance may be viewed as the impedance offered to the propagation of waves along a transmission line. For example, consider the forward components of equations (22–38) and (22–39). Their ratio is

$$\frac{\dot{E}_r^+}{\dot{I}_r^+} = \frac{\frac{1}{2}(\dot{E}_r + \dot{I}_r\dot{Z}_0)}{\frac{1}{2}(\dot{I}_r + \dot{E}_r\dot{Y}_0)} = \frac{\frac{1}{2}(\dot{E}_r + \dot{I}_r\dot{Z}_0)\dot{Z}_0}{\frac{1}{2}(\dot{I}_r\dot{Z}_0 + \dot{E}_r)} = \dot{Z}_0 \tag{22–53}$$

This equation states that $\dot{Z}_0$ may be interpreted as the impedance which is presented to the forward components of voltage and current at all points of the line. It should be noted that the ratio indicated in equation (22–53) not only is the same at all points of the line but also is independent of the terminal conditions.

More information on the physical significance of the characteristic impedance might be obtained by studying a line which is terminated in a load impedance $\dot{Z}_r = \dot{Z}_0$. Under this condition, $\dot{E}_r = \dot{I}_r\dot{Z}_r = \dot{I}_r\dot{Z}_0$ and equations (22–27) and (22–29) become

$$\dot{E} = \dot{I}_r\dot{Z}_0 e^{\gamma l} \tag{22–54}$$

$$\dot{I} = \dot{I}_r e^{\gamma l} \tag{22–55}$$

The backward or reflected waves are zero, since

$$\frac{\dot{E}_r}{2} - \frac{\dot{I}_r\dot{Z}_0}{2} = \frac{\dot{I}_r\dot{Z}_0}{2} - \frac{\dot{I}_r\dot{Z}_0}{2} = 0$$

$$\frac{\dot{I}_r}{2} - \frac{\dot{E}_r\dot{Y}_0}{2} = \frac{\dot{I}_r}{2} - \frac{\dot{I}_r}{2} = 0$$

The ratio of equation (22–54) to equation (22–55) is

$$\frac{\dot{E}}{\dot{I}} = \dot{Z}_0 \tag{22–56}$$

This relation indicates that the impedance measured at all points of the line will be $\dot{Z}_0$ when the line is terminated in the characteristic impedance $\dot{Z}_0$. That is, the ratio of the actual voltage $\dot{E}$ to the actual current $\dot{I}$ is the same at all points of the line. This information concerning a line terminated in $\dot{Z}_0$ is useful in a more detailed study of the subject of attenuation (not considered in this text).

It has been shown that the termination of a line in its characteristic impedance eliminates the reflected waves. This suggests that a finite line terminated in $\dot{Z}_0$ might be considered to represent an infinitely long line, as far as the distribution of voltage and current in the finite line is concerned. No reflections would occur in an infinite line, just as no reflection would occur in a water canal which is infinitely long. It can be seen that such a terminating condition is very desirable in communication circuits. A telephone line should deliver at the receiving end a wave form which is as nearly like the original input wave as is technically and economically feasible. Any change in the wave form or character of the signal is usually known as distortion. Eliminating the reflected waves means the elimination of distortion due to reflection. For this reason, communication engineers make every effort to terminate each line in $\dot{Z}_0$. Thus, it may be said that a finite line will behave like an infinite line if it is terminated in its characteristic impedance.

22–14. Calculation of Characteristic Impedance From Measurements.—The characteristic impedance $\dot{Z}_0$ of a transmission line may be calculated from measurements of the line impedance at the supply end: (*a*) with load end open; (*b*) with load end short-circuited. The proof gives a good example on the manipulation of the long line equations.

With the load end open, $I_r = 0$ and equations (22–28) and (22–30) become:

$$\dot{E}_s = \dot{E}_r \cosh \gamma S$$
$$\dot{I}_s = \dot{E}_r \dot{Y}_0 \sinh \gamma S$$

where S is the total length of the line.

The impedance of the line at the sending end with the load end open is

$$\dot{Z}_{oc} = \frac{\dot{E}_s}{\dot{I}_s} = \dot{Z}_0 \coth \gamma S \qquad (22\text{–}57)$$

With the load end short-circuited, $E_r = 0$ and

$$\dot{E}_s = \dot{I}_r \dot{Z}_0 \sinh \gamma S$$
$$\dot{I}_s = \dot{I}_r \cosh \gamma S$$

The impedance at the sending end with the load end short-circuited is

$$\dot{Z}_{sc} = \frac{\dot{E}_s}{\dot{I}_s} = \dot{Z}_0 \tanh \gamma S \qquad (22\text{–}58)$$

The product of equations (22–57) and (22–58) is

$$\dot{Z}_0^2 = \dot{Z}_{oc} \dot{Z}_{sc}$$

or

$$\dot{Z}_0 = \sqrt{\dot{Z}_{oc} \dot{Z}_{sc}} \qquad (22\text{–}59)$$

That is, if the sending-end impedance is measured with the load end open-circuited and then with the load end short-circuited, equation (22–59) will give the characteristic, or surge, impedance.

If the series resistance and the shunt leakage conductance are negligible in equation (22–52), the characteristic impedance reduces to

$$\dot{Z}_0 = \sqrt{\frac{L}{C}} \qquad (22\text{–}60)$$

where L and C are the inductance and capacitance per meter and the angle of $\dot{Z}_0$ approaches zero.

PROBLEMS

22–1. An open-wire telephone line consists of two parallel copper wires 0.104 in. in diameter and spaced 12 in. apart. The approximate value of the leakage conductance g from wire to wire is 0.8×10^{-6} mho per mile. Assuming that the internal flux-current linkages can be neglected, find r, L, and C per mile of two wires. Also, change the elements to per meter of two wires.

22–2. An open-wire telephone line, consisting of two parallel copper wires, has the following distributed elements:

$r = 10.4$ ohms per mile of two wires;
$L = 3.67$ millihenrys per mile of two wires;
$C = 0.00835$ μf per mile from wire to wire;
$g = 0.8$ micromho per mile from wire to wire.

The total length of the line is 100 miles. Assume that the frequency is 796 cps, *i.e.*, $\omega = 5000$ (that is, assume that this value of frequency approximates, as far as a single frequency can, the average frequency of the voice wave). Find the inductive reactance and the capacitive susceptance per mile of two wires.

22–3. A 60-cycle, three-phase line consisting of three No. 0000 stranded copper wires is 100 miles long. The conductors are spaced equally 10 ft apart. Assuming that the internal flux-current linkages can be neglected, find: (*a*) r, L, and C per mile of one wire; (*b*) r, L, and C per meter of one wire; (*c*) the inductive reactance and capacitive susceptance per mile of one wire; (*d*) the inductive reactance and capacitive susceptance per meter of one wire.

22–4. A three-phase, three-wire transmission line is constructed of No. 0000 stranded copper conductors. The wires are spaced 6, 8, and 10 ft apart in a triangle. The transmission line is 200 miles long and transposed. Neglecting flux-current linkages within the conductors, find: (*a*) the resistance per mile of one wire; (*b*) the inductance per mile of one wire; (*c*) the capacitance per mile of one wire to neutral.

22–5. An open-wire telephone line, consisting of two parallel copper wires, has the following distributed elements:

$r = 4.14$ ohms per mile of two wires;
$L = 3.64$ millihenrys per mile of two wires;
$C = 0.00845$ μf per mile from wire to wire;
$g = 0.8$ micromho per mile from wire to wire.

The total length of the telephone line is 100 miles. A load impedance $\dot{Z}_r = 3000\underline{/60°}$ at $\omega = 5000$ radians per second is connected across the terminals of the line at the receiving end. If a current of 50 milliamperes at $\omega = 5000$ radians per second flows through the load impedance, what will be the voltage and current at the sending end of the line?

22–6. A balanced three-phase load of 20,000 kva at 80% lagging power factor is connected at the end of a 60-cycle, 100-mile, three-phase transmission line. The line parameters per mile of a single conductor are:

$$r = 0.22 \text{ ohm} \qquad g = 0$$
$$x = 0.75 \text{ ohm} \qquad b = 5.7 \times 10^{-6} \text{ mho}$$

If the line-to-line voltage at the receiving end is 88,000 volts, what must be the voltage and current at the sending end? Use the line-to-neutral voltage $\dot{E}_r$ as reference.

22–7. A 300-mile, 60-cycle, three-phase transmission line has the following distributed elements per mile of one wire:

$$r = 0.15 \text{ ohm} \qquad g = 0$$
$$x = 0.85 \text{ ohm} \qquad b = 5 \times 10^{-6} \text{ mho}$$

The line is supplying power to a balanced three-phase load. If the line-to-neutral voltage at the sending end is $\dot{E}_s = 100{,}000\underline{/0°}$ and the line current at the sending end is $\dot{I}_s = 200\underline{/15°}$, find the vector voltage and the vector current at the load end.

22–8. A 100-mile d-c telegraph line has a resistance of 10 ohms per mile of two wires and a shunt conductance of 10 micromhos per mile from wire to wire. Find the steady-state values of voltage and current at a point 30 miles from the sending end when the receiving end is short-circuited and 32 volts are applied at the sending end.

22–9. Derive equations (22–31) and (22–32) by determining the constants A_1, A_2, B_1, and B_2 of equations (22–16) and (22–17) for the boundary conditions at the sending end. Note carefully that the sign of the term on the right-hand side of equation (22–9) is plus when l is measured from the load end, and the sign must be changed to minus when l is measured from the sending end.

22–10. From equations (22–28) and (22–30) deduce expressions for the voltage distribution and current distribution along the length of the line when the receiver end of the line is: (*a*) short-circuited and (*b*) open-circuited. It is assumed that the sending-end voltage $\dot{E}_s$ is known and the total length of the line is S.

22–11. A 300-mile, 60-cycle, three-phase transmission line has the following constants per mile of one wire:

$$r = 0.15 \text{ ohm} \qquad g = 0$$
$$x = 0.85 \text{ ohm} \qquad b = 5 \times 10^{-6} \text{ mho}$$

If 100,000 volts are maintained between wires at the sending end, plot a curve of no-load voltage against distance as abscissa.

22–12. Plot curves of voltage and current against length of line for the transmission system of Problem 22–11 when the receiver end is short-circuited.

22–13. A generator having a generated voltage of $\dot{E}_g = 100\underline{/0°}$ and an internal impedance $\dot{Z}_g = 1000\underline{/30°}$ is connected to the sending end of the telephone line in Problem 22–5. If a load impedance $\dot{Z}_r = 1000\underline{/30°}$ is connected at the load end, find the current flowing through the load impedance and the current at the sending end. Assume $\omega = 5000$ radians per second.

22–14. An approximate circuit for a short transmission line is shown in Fig. 22–6. It is more convenient to use E_s instead of E_g since an equivalent generator maintaining E_s has no internal impedance. Find the equivalent generator replacing this transmission system to the left of terminals 3 and 4 by Thévenin's Theorem, as far as the load current I_r is concerned

22–15. Repeat Problem 22–14 for a long line where equations (22–28) and (22–30) must be used.

22–16. Find γ, α, β, and v for the telephone line in Problem 22–2, assuming that $\omega = 5000$ radians per second.

22–17. Assume that it is possible to design a line so that

$$\frac{\omega L}{r} = \frac{\omega C}{g}$$

Such a line is called a *distortionless line.* Determine Z_0, γ, α, and β.

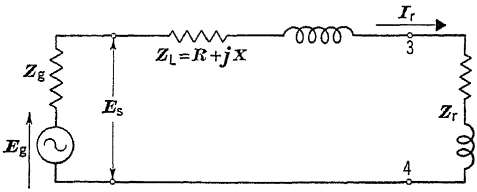


FIG. 22–6

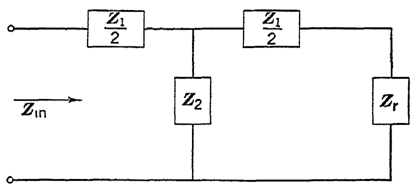


FIG. 22–7

22–18. A length of transmission line, like any other linear bilateral network, may be represented at any single frequency by a T-section as shown in Fig. 22–7. If the terminating impedance $\dot{Z}_r$ is varied, find its value in terms of $\dot{Z}_1$ and $\dot{Z}_2$ such that the input impedance $\dot{Z}_{in}$ will have the same value.

22–19. An equivalent T-section of a telephone line, as indicated in Problem 22–18, has the following pure resistive constants:

$$Z_1 = R_1 = 100 \text{ ohms} \qquad Z_2 = R_2 = 4000 \text{ ohms}$$

(*a*) What is the value of Z_{in}, designated by Z_{sc}, when $Z_r = 0$?
(*b*) What is the value of Z_{in}, designated by Z_{oc}, when $Z_r = \infty$?
(*c*) If a terminating pure resistance Z_r is varied, plot a curve of Z_{in} against Z_r as abscissa.

22–20. An open-wire telephone line has the following distributed parameters per mile of two wires:

$$r = 10.2 \text{ ohms} \qquad g = 5.6 \times 10^{-6} \text{ mho}$$
$$L = 3.5 \text{ millihenrys} \qquad C = 0.008\ \mu\text{f}$$

Find the characteristic impedance if the frequency is: (*a*) 200 cps, (*b*) 1200 cps, and (*c*) 3000 cps. What can you say about the variation of $\dot{Z}_0$ with frequency?

22–21. Consider the 100-mile, open-wire telephone line in Problem 22–5. If $\omega = 5000$ radians per second, find: (*a*) the characteristic impedance $\dot{Z}_0$; (*b*) the characteristic admittance $\dot{Y}_0$; (*c*) the propagation constant γ; (*d*) the attenuation constant α; (*e*) the wavelength constant β; (*f*) the phase velocity of propagation v.

22–22. The following measurements were made at $\omega = 5000$ radians per second on a 10-mile length of a two-wire transmission line:

$$\dot{Z}_{oc} = 2000\underline{/60^\circ}$$

$$\dot{Z}_{sc} = 1200\underline{/-70^\circ}$$

Calculate $\dot{Z}_0$.

APPENDIX A

ALTERNATING-CURRENT INSTRUMENTS

There are many and varied methods of obtaining quantitative data in alternating-current circuits. It is important that the principle upon which the instruments operate should be clearly understood. Only a few of the common instruments will be mentioned here.

A–1. Oscillograph.—The oscillograph consists essentially of a special type of galvanometer and a "rotating mirror" or a "vibrating mirror." A portion of the galvanometer is shown in Fig. A–1.

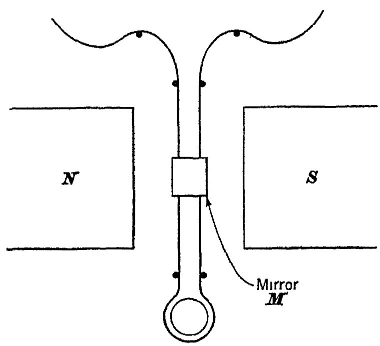


FIG. A–1

The moving element is so designed that it will respond to extremely rapid changes in a voltage or a current. Such an instrument can be used to study unknown alternating-current waves, frequency, time phase of sinusoidal quantities, and transients.

In principle, the galvanometer is similar to the D'Arsonval galvanometer. As shown in Fig. A–1, a single turn of very light phosphor-bronze ribbon is placed between the poles of a powerful magnet. A very small mirror M is cemented to the loop and a definite tension is applied to the loop by means of a spring.

During the actual operation of the oscillograph, a beam of light from an arc or an incandescent lamp is played upon the galvanometer mirror. The effect of passing a current through the loop is to cause the two sides of the loop to move in opposite directions in the magnetic field, thus turning the mirror about a vertical axis. The reflected light from the mirror M is either viewed or recorded photographically. With a constant field, the linear displacement of the spot of light on the viewing screen or the film is proportional to the instantaneous value of the current. To obtain an image of the wave, the motion of the photographic film or the "rotating mirror" must be at right angles to the direction of the movement of the light spot. The motion of the film or the rotation of the "rotating mirror" introduces a time element. The commerical forms of the oscillograph are capable of recording with a high degree of fidelity frequencies as high as 2000 cps and transients which occur in as short an interval as $\frac{1}{1200}$ sec.

A-2. Electrodynamometer Instruments.—In a D'Arsonval type of direct-current meter, a constant field is produced by a permanent magnet. The torque acting on the moving coil at any instant is directly proportional to the current flowing in the coil at that instant. If such a meter is connected to an alternating-current supply, the torque is alternately in opposite directions. For the reason that the average value of a sinusoidal alternating-current wave is zero and the reversals are very rapid, the average torque is zero and there will be no readable deflection. If, however, the field of the permanent magnet is replaced by the field of a stationary coil connected in series with the moving coil, then the fields of the stationary and moving coils will reverse at the same time and the force tending to turn the coil will always be in the same direction. An instrument of this type is called an *electrodynamometer* instrument.

In its simplest form, as shown diagrammatically in Fig. A-2, the electrodynamometer type of instrument consists of two stationary coils S and S' and one moving coil M. The three coils form a series circuit, with the stationary coils so connected that their magnetic fields act in conjunction. The axis of the field Φ_s of the stationary coils is along AB and the axis of the field Φ_m of the moving coil is along CD. The movable coil is mounted on a vertical spindle which turns in jeweled bearings. Two spiral springs oppose the turning of coil M and, at the same time, conduct current to the coil.

Assume that at some instant the directions of the fields are as shown by the arrows. The moving coil M will tend to turn in a clockwise direction so that the number of magnetic linkages in the system is a maximum. But the turning of this coil is retarded by the control springs. The torque is proportional to the product of Φ_s, Φ_m, and sin α, where α is the angle of displacement of the coil M. The magnitude of each of the fields is proportional to the current i, which flows through all three coils since they are in series. Hence, the torque tending to turn coil M at any instant is proportional to the square of the current flowing at that instant. Because of the inertia of the moving element, M

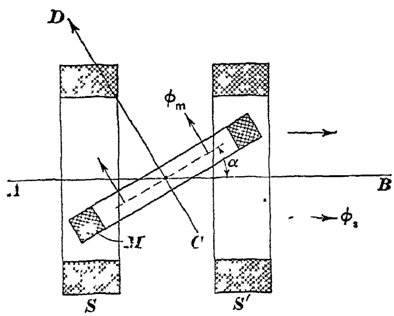


FIG. A-2

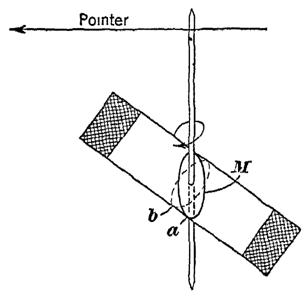


FIG. A-3

cannot move fast enough to follow the rapid fluctuations of the current and, accordingly, it moves to a point corresponding to the average value of the torque. This deflection is a measure of the average value of i^2 and, hence, of the effective value of the current.

A-3. Iron-Vane Instruments.—The iron-vane instrument is based on the principle of magnetic induction. In one form of this type of instrument, shown in Fig. A-3, the moving element consists of one vane M or a small strip of soft iron mounted obliquely on a spindle which is free to turn. This vane takes the place of the moving coil used in the electrodynamometer type of instrument. The stationary coil is inclined at an angle with the spindle.

When the instrument reads zero, the vane lies at an angle to the axis of the coil, as at a, Fig. A-3. When current flows through the stationary coil, a field is established. The vane attempts to place itself parallel to the field, as at b, thus deflecting the needle.

Spiral springs retard the turning moment but they carry no current. The torque acting on the vane is proportional to the product of the field of the stationary coil and the magnetic induction in the iron vane. Each is proportional to the current i and, hence, the average torque is proportional to the average value of i^2 or the square of the effective value of i.

A–4. Ammeters.—Dynamometer ammeters of the portable type are not common, because of the difficulty of leading a current greater than 1 or 2 amp through the spiral springs into the moving coil. Moreover, it is not a simple matter to use a shunt, because the current will not divide between the coil and shunt in the inverse ratio of their resistances on account of the inductance of the coil. When measuring a current wave of irregular shape, the instrument would be in error if it had been calibrated with sine waves at a frequency of 60 cps. For these reasons, alternating-current ammeters of the electrodynamometer type are seldom used.

The iron-vane type of ammeter is so much simpler and less expensive than the type just described that iron-vane ammeters are practically the only type used for commercial alternating-current work. The stationary coil can be wound with a few turns of heavy wire to carry the entire line current. Portable iron-vane ammeters for laboratory use have been constructed to read as high as 200 amp. Since the deflection is proportional to the square of the effective value, the scale is not divided uniformly. The divisions of the scale are somewhat crowded in the part indicating the lowest values, and readings obtained in this range should be regarded as unreliable.

Iron-vane instruments may be used on direct-current circuits. However, their indications are not so accurate as the standard direct-current instruments because of the possible presence of stray fields and the magnetic retentivity of the iron vane. When used in direct-current measurement, two readings—one with and the other without reversal of connections—should be taken, and their average should be considered as the true reading.

A–5. Voltmeters.—Either the electrodynamometer type or the iron-vane type of instrument may be used as a voltmeter by connecting a high resistance in series with the meter element. This resistance is inserted to limit the current when the meter is connected across the voltage to be measured. Since the meter impedance is fixed for a given frequency, the current passing through the meter is proportional to the measured voltage; and the pointer attached to the moving coil or iron vane moves over a scale graduated in volts. As the deflections depend on the square of the voltage, the instrument reads effective values.

When either type of instrument is used to measure the voltage of a direct-current circuit, reversed readings should be taken and their average considered as the true value.

A–6. Thermal Instruments.—The hot-wire type of instrument depends on the average heating effect, in the form of I^2R, of an alternating current. The current to be measured flows through a stretched wire. Since the linear expansion of this wire is pro-

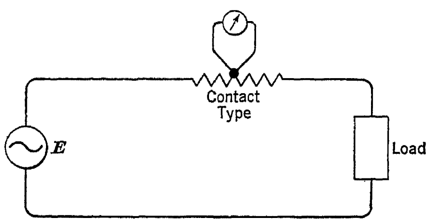


FIG. A–4

portional to the heating or average value of i^2, the deflection is proportional to the square of the effective value of the current. This type of instrument reads equally well on both direct-current and alternating-current circuits. No error is introduced by frequency or wave shape. Its calibration is affected by different room temperatures and, hence, it is not a high precision instrument.

Another thermal type of instrument utilizes a thermocouple and a d-c meter. The contact type, Fig. A–4, consists essentially of two wires of dissimilar metal fastened together at the thermocouple junction. When the junction is heated by the alternating current to be measured, a thermocouple voltage is produced. This voltage is approximately proportional to the heating and this in turn is proportional to I^2R, where I is the effective value of the alternating current and R is the effective resistance of the thermocouple. Thermocouple ammeters are mostly used on high-frequency or radio-frequency circuits.

A–7. Copper-Oxide Instruments.—The copper-oxide or rectifier type of meter consists of a sensitive d-c instrument of the permanent-magnet type connected into the copper-oxide rectifier bridge shown in Fig. A–5. It is adapted for measurements in communication circuits, especially where a high degree of accuracy is not required. When an alternating voltage is impressed across the terminals of the instrument, a direct current will flow through the d-c instrument. The meter can be calibrated to read effective values of alternating current or voltage.

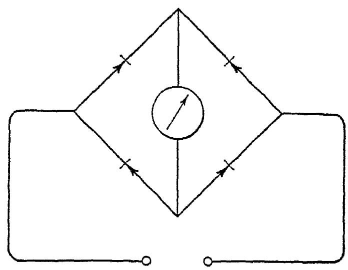


FIG. A–5

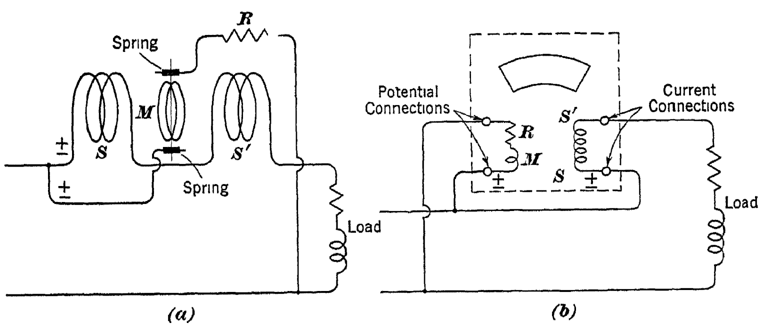


FIG. A–6

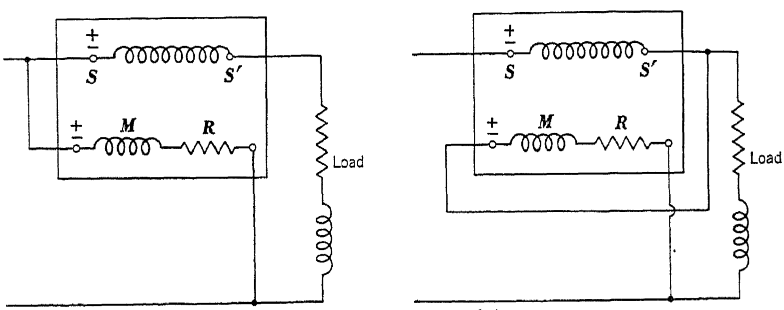
A–8. Single-Phase Wattmeters.—The instantaneous power in an alternating-current circuit is equal to the product of the instantaneous values of current and voltage. The most common type of wattmeter, which reads the average power, operates on the electrodynamometer principle. As shown in Fig. A–6, two stationary coils S and S' are wound with a few turns of heavy wire to carry the load current. They are connected in series with the load, just as an ammeter is. The moving coil M, in series with its high resistance R, is connected across that part of the circuit in which the power is to be measured. The current flowing through the moving coil is approximately proportional to the voltage of the part of the circuit considered.

Because of the inertia of the moving system, the pointer of the meter assumes a steady deflection for constant values of average power. The equation for the average torque is

$$\text{Average torque} = \frac{K_1}{T}\int_0^T \Phi_s i_M \, dt$$

$$= \frac{K_2}{T}\int_0^T ei \, dt = K_2 P$$

where K_1 and K_2 are constants and P is the average power.



(*a*) Current Coil Loss Included (*b*) Potential Coil Loss Included

Fig. A–7

The average power delivered to a load may be measured by a wattmeter. The current and potential terminals are marked by the sign ± in such a way that, when the wattmeter is connected as shown in Fig. A–7, a positive power taken by the load will be indicated by an up-scale reading of the meter.

When the stationary and moving coils are connected as shown in Fig. A–6 (*a*) or Fig. A–7 (*a*), that is, with coil M and coils S and S' connected to the same line, they are practically at the same potential. Most of the line voltage is across the high resistance R. If both the potential-coil connections and the current-coil connections were reversed at the same time, that is, with R and coil $S'S$ connected to the same line, the wattmeter would still read up-scale. However, the potential difference between the stationary coil SS' and the moving coil M would now be equal approximately to the full-line voltage, and electrostatic forces existing between the coils may introduce errors in the readings.

A-9. Frequency Meters.—There are a number of types of commercial frequency meters. The most common form is based on the principle of mechanical resonance. A number of vibrator reeds of different lengths, each having a white index on its end, are mounted in a row on a common mounting. The mass of each reed is so adjusted that the mechanical frequency or vibration of each reed is different. The mounting is caused to vibrate by an electromagnet, the coil of which is connected across the circuit whose frequency it is desired to measure. The reed whose natural frequency is the same as the frequency of the circuit will vibrate with the greatest amplitude. A scale adjacent to the row of reeds is usually calibrated in cycles per second. The reeds are so arranged that there is a reed for every half-cycle over the limited range of the meter.

APPENDIX B

FREQUENCIES FOR MAXIMUM E_C AND E_L IN RLC CIRCUIT

When the frequency is varied in an RLC circuit, the expressions for the frequencies at which the maximum value of E_C and the maximum value of E_L occur may be derived in the following way. At the maximum value, the first derivative of E_C or E_L with respect to f is zero. From this fact, the desired equation for the frequency can be found. The frequency f_C for the maximum value of E_C will be considered first.

The general expression for the current, with an applied emf E_0, is

$$I=\frac{E_0}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}} \tag{B–1}$$

The voltage across the condenser is $E_C=X_CI$, or

$$E_C=\frac{E_0}{\omega C\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}} \tag{B–2}$$

It is desirable to eliminate the radical sign from the denominator. This can be done by using E_C^2; since, when E_C^2 is a maximum, E_C will also be a maximum. Squaring equation (B–2), we obtain:

$$E_C^2=\frac{E_0^2}{\omega^2C^2\left[R^2+\left(\omega L-\frac{1}{\omega C}\right)^2\right]}=\frac{E_0^2}{\omega^2C^2\left[R^2+\frac{(\omega^2LC-1)^2}{\omega^2C^2}\right]}$$

or

$$E_C^2=\frac{E_0^2}{\omega^2C^2R^2+(\omega^2LC-1)^2} \tag{B–3}$$

Differentiating with respect to $\omega=2\pi f$ and setting the result equal to zero, we get:

$$\frac{\partial E_C^2}{\partial\omega}=\frac{-E_0^2\,[2\omega C^2R^2+2(\omega^2LC-1)\;2\omega LC]}{[\omega^2C^2R^2+(\omega^2LC-1)^2]^2}=0 \tag{B–4}$$

from which

$$2\omega^2L^2C+CR^2-2L=0$$

or

$$\omega^2=\frac{1}{LC}-\frac{R^2}{2L^2} \tag{B–5}$$

Also,

$$f_C = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}} \qquad \textbf{(B–6)}$$

The frequency f_L for the maximum value of E_L can be determined in a similar manner. Thus,

$$E_L = X_L I = \frac{E_0 \omega L}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \qquad \textbf{(B–7)}$$

or

$$E_L^2 = \frac{E_0^2 \omega^2 L^2}{R^2 + \frac{(\omega^2 LC - 1)^2}{\omega^2 C^2}} = \frac{E_0^2 \omega^4 L^2 C^2}{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2} \qquad \textbf{(B–8)}$$

By differentiating with respect to $\omega = 2\pi f$ and setting the value of $\frac{\partial E_L^2}{\partial \omega}$ equal to zero, we obtain the following expression:

$$2\omega^2 LC - \omega^2 C^2 R^2 - 2 = 0$$

from which

$$\omega^2 LC \left(2 - \frac{CR^2}{L}\right) = 2$$

Hence,

$$\omega^2 = \frac{1}{LC} \times \frac{2}{2 - \frac{CR^2}{L}} = \frac{1}{LC - \frac{R^2 C^2}{2}} \qquad \textbf{(B–9)}$$

Also,

$$f_L = \frac{1}{2\pi\sqrt{LC - \frac{R^2 C^2}{2}}} \qquad \textbf{(B–10)}$$

APPENDIX C

SELECTIVITY OR SHARPNESS OF RESONANCE IN RLC CIRCUITS

The equation for the determination of sharpness of resonance, *as far as the current flowing in the circuit is concerned*, may be derived in the following way. At the frequencies f_1 and f_2 where the net reactance $X = 2\pi f L - \dfrac{1}{2\pi f C}$ is equal to the resistance R, that is, at the half-power points, the following relations may be written:

$$\omega_2 L - \frac{1}{\omega_2 C} = R \tag{C–1}$$

$$\frac{1}{\omega_1 C} - \omega_1 L = R \tag{C–2}$$

Equating these two equations and rearranging the terms, we obtain

$$\omega_2 L + \omega_1 L = \frac{1}{\omega_1 C} + \frac{1}{\omega_2 C}$$

This gives

$$2\pi L\ (f_2 + f_1) = \frac{1}{2\pi C}\left(\frac{1}{f_1} + \frac{1}{f_2}\right) = \frac{1}{2\pi C}\left(\frac{f_2 + f_1}{f_1 f_2}\right)$$

from which

$$2\pi L = \frac{1}{2\pi f_1 f_2 C} \tag{C–3}$$

or

$$(2\pi)^2\ LC f_1 f_2 = 1 \tag{C–4}$$

In comparing equation (5–53), or $2\pi L = \dfrac{1}{2\pi f_r^2 C}$, with equation (C–3), it may be of interest to note that

$$f_r^2 = f_1 f_2 \tag{C–5}$$

Adding equations (C–1) and (C–2), we get

$$\omega_2 L - \omega_1 L + \frac{1}{\omega_1 C} - \frac{1}{\omega_2 C} = 2R$$

or

$$2\pi L\ (f_2 - f_1) + \frac{1}{2\pi C}\left(\frac{1}{f_1} - \frac{1}{f_2}\right) = 2R$$

from which

$$(f_2 - f_1) + \frac{1}{(2\pi)^2 LC}\left(\frac{f_2 - f_1}{f_1 f_2}\right) = \frac{2R}{2\pi L} \tag{C–6}$$

With the aid of equation (C–4), this becomes

$$(f_2 - f_1) = \frac{R}{2\pi L} \tag{C–7}$$

Dividing both sides by f_r, we have

$$\frac{f_2 - f_1}{f_r} = \frac{R}{2\pi f_r L} = \frac{1}{Q} \tag{C–8}$$

APPENDIX D

EXAMPLE ON THÉVENIN'S THEOREM

The following example on Thévenin's Theorem shows how to determine the sharpness of resonance of a parallel circuit on the basis of the current flowing in the condenser branch.

Example D–1.—A generator with an internal impedance R_g is connected across a parallel combination, as shown in Fig. D–1. (*a*) Find the impedance of the equivalent generator replacing the network to the left of terminals 3 and 4. (*b*) Determine the expression for sharpness of resonance, on the basis of I_C, when the frequency is varied.

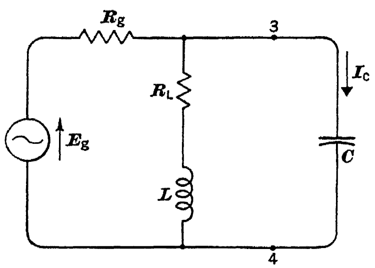


FIG. D–1

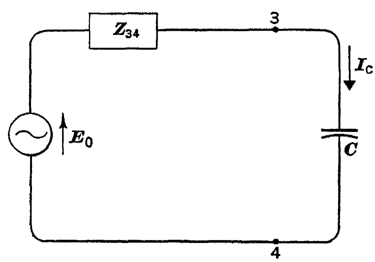


FIG. D–2

Solution.—(*a*) The impedance $\dot{Z}_{34}$ of the equivalent generator is

$$\dot{Z}_{34} = \frac{R_g(R_L + jX_L)}{R_g + R_L + jX_L} \qquad \text{(D–1)}$$

Rationalizing and rearranging terms, we get

$$\dot{Z}_{34} = \frac{R_g R_L(R_g + R_L) + R_g X_L^2}{(R_g + R_L)^2 + X_L^2} + j\frac{R_g^2 X_L}{(R_g + R_L)^2 + X_L^2} = R_{34} + jX_{34} \qquad \text{(D–2)}$$

(*b*) The Q_e of the equivalent series circuit in Fig. D–2 is the ratio of the total inductive reactance to the total resistance. Thus,

$$Q_e = \frac{X_{34}}{R_{34}} = \frac{R_g^2 X_L}{R_g R_L(R_g + R_L) + R_g X_L^2}$$

or

$$Q_e = \frac{R_g X_L}{R_L(R_g + R_L) + X_L^2} \qquad \text{(D–3)}$$

Equation (5–55) gives the degree of sharpness for a series circuit on the basis of the circuit current or the condenser current. This expression is applicable to the series circuit shown in Fig. D–2 when R_g and the $Q = \dfrac{X_L}{R_L}$ of the coil are large. Substituting for Q_e in the equation, we get

$$\frac{(f_2-f_1)}{f_r}=\frac{1}{Q_s}=\frac{R_L}{X_L}+\frac{R_L^2+X_L^2}{R_gX_L}=\frac{1}{Q_{coil}}+\frac{X_L}{R_g} \tag{D-4}$$

where Q_{coil} is the Q of the coil consisting of R_L and L in Fig. D–1.

Equation (D–4) states that the sharpness of resonance can be increased in the following ways:

(*a*) Increase Q of the coil
(*b*) Increase R_g of the generator

As in the case of the series circuit, equation (D–4) can be used for determining the selectivity of the circuit shown in Fig. D–1, on the basis of the condenser voltage E_C, when R_g and the Q of the coil are large.

APPENDIX E

A USEFUL NETWORK THEOREM

A useful theorem* was developed by Jacob Millman for finding the steady-state voltage between two points O and O' of a network. It is only necessary to know the values of the linear bilateral impedances which terminate at the junction point O' and the voltages from the point O to the opposite ends of these impedances. The theory is straightforward and simple to apply in circuit problems. For certain types of problems, such as those involving linear amplifiers, the application of this theorem is a time-saver in comparison with the more conventional solution in which Kirchhoff's Laws are used. It is the purpose here to develop the generalized theorem and to give a few examples showing its application in circuit problems.

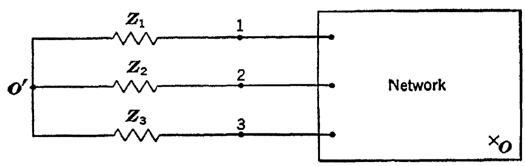


FIG. E-1

Consider the network shown in Fig. E-1. The linear bilateral impedances $\dot{Z}_1$, $\dot{Z}_2$, and $\dot{Z}_3$ are known, and they terminate in a common point O'. The opposite ends of these impedances are indicated as 1, 2, and 3. The point O is any other point in the network. If the voltage drops $\dot{V}_{O1}$, $\dot{V}_{O2}$, and $\dot{V}_{O3}$ from O to the opposite ends of the three impedances are of the same frequency and known, then the steady-state voltage drop from O to O' is given by the relation

$$\dot{V}_{OO'} = \frac{\dot{V}_{O1}\dot{Y}_1 + \dot{V}_{O2}\dot{Y}_2 + \dot{V}_{O3}\dot{Y}_3}{\dot{Y}_1 + \dot{Y}_2 + \dot{Y}_3} \tag{E-1}$$

where $\dot{Y}_1 = \frac{1}{\dot{Z}_1}$, $\dot{Y}_2 = \frac{1}{\dot{Z}_2}$, and $\dot{Y}_3 = \frac{1}{\dot{Z}_3}$.

* "A Useful Network Theorem," by Jacob Millman, Proceedings of the I.R.E., September, 1940, pp. 413–418.

It should be noted that it is not necessary to know the network interconnections between the points O, 1, 2 and 3. Equation (E–1) can also be written as follows:

$$\dot{V}_{OO'} = \frac{\sum_{k=1}^{k=3} \dot{V}_{Ok}\dot{Y}_k}{\sum_{k=1}^{k=3} \dot{Y}_k} \qquad \textbf{(E–2)}$$

Expression (E–1) can be found in the following way. The voltage drop across $\dot{Z}_1$ is

$$\dot{V}_{10'} = \dot{V}_{OO'} - \dot{V}_{O1} \qquad \textbf{(E–3)}$$

and the current flowing through $\dot{Z}_1$ is

$$\dot{I}_{10'} = \frac{\dot{V}_{10'}}{\dot{Z}_1} = \dot{V}_{10'}\dot{Y}_1 = (\dot{V}_{OO'} - \dot{V}_{O1})\dot{Y}_1 \qquad \textbf{(E–4)}$$

Similarly,

$$\dot{V}_{20'} = \dot{V}_{OO'} - \dot{V}_{O2} \qquad \textbf{(E–5)}$$

$$\dot{V}_{30'} = \dot{V}_{OO'} - \dot{V}_{O3} \qquad \textbf{(E–6)}$$

Also,

$$\dot{I}_{20'} = \frac{\dot{V}_{20'}}{\dot{Z}_2} = \dot{V}_{20'}\dot{Y}_2 = (\dot{V}_{OO'} - \dot{V}_{O2})\dot{Y}_2 \qquad \textbf{(E–7)}$$

$$\dot{I}_{30'} = \frac{\dot{V}_{30'}}{\dot{Z}_3} = \dot{V}_{30'}\dot{Y}_3 = (\dot{V}_{OO'} - \dot{V}_{O3})\dot{Y}_3 \qquad \textbf{(E–8)}$$

Applying Kirchhoff's Current Law at point O' gives

$$\dot{I}_{10'} + \dot{I}_{20'} + \dot{I}_{30'} = 0$$

or

$$(\dot{V}_{OO'} - \dot{V}_{O1})\dot{Y}_1 + (\dot{V}_{OO'} - \dot{V}_{O2})\dot{Y}_2 + (\dot{V}_{OO'} - \dot{V}_{O3})\dot{Y}_3 = 0 \qquad \textbf{(E–9)}$$

from which equation (E–1) is obtained.

It can be seen from the proof that the theorem is not restricted to only three impedances as shown in Fig. E–1. The equation for the steady-state voltage drop $\dot{V}_{OO'}$ when n impedances terminate in a common point O' is

$$\dot{V}_{OO'} = \frac{\sum_{k=1}^{k=n} \dot{V}_{Ok}\dot{Y}_k}{\sum_{k=1}^{k=n} \dot{Y}_k} \qquad \textbf{(E–10)}$$

The theorem may be stated: "*If any number of linear bilateral impedances meet in a common junction O′ and the voltages from any other point O in the network to the opposite ends of the impedances are of the same frequency and known, then the steady-state voltage drop from O to O′ is given by equation (E–10).*"

The theorem applies to direct-current circuits and to linear bilateral networks with several sources of different frequencies. In the latter case the problem can be handled quite conveniently by the use of the Superposition Theorem.

Example E–1.—Fig. E–2 shows a T-section with two generators. The impedances $\dot{Z}_1$, $\dot{Z}_2$, and $\dot{Z}_3$ might be the equivalent T impedances of a complicated linear bilateral and passive network (a network which has no sources of emf except those applied externally at the terminals is said to be a passive network). If the voltages $\dot{E}_{g1}$ and $\dot{E}_{g2}$ of the generators are of the same frequency and known, find the currents flowing in the three impedances.

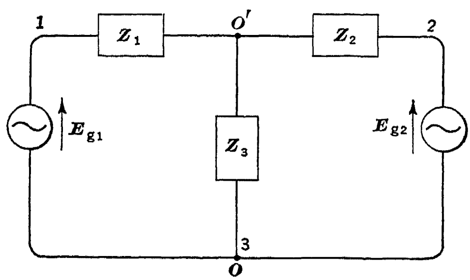


FIG. E–2

Solution.—The point O' is fixed at the junction of the three impedances. For this circuit, it is convenient to choose O at point 3. This choice of O gives $\dot{V}_{O3} = 0$. If $\dot{E}_{g1}$ and $\dot{E}_{g2}$, in the directions of the arrows, are considered to be voltage rises, then the voltage drops across the two generators are $\dot{V}_{O1} = -\dot{E}_{g1}$ and $\dot{V}_{O2} = -\dot{E}_{g2}$. The voltage drop across $\dot{Z}_3$, from equation (E–10), is

$$\dot{V}_{OO'} = \frac{\dot{V}_{O1}\dot{Y}_1 + \dot{V}_{O2}\dot{Y}_2}{\dot{Y}_1 + \dot{Y}_2 + \dot{Y}_3} = \frac{-(\dot{E}_{g1}\dot{Y}_1 + \dot{E}_{g2}\dot{Y}_2)}{\dot{Y}_1 + \dot{Y}_2 + \dot{Y}_3} \tag{E–11}$$

With $\dot{V}_{OO'}$ known, the currents flowing in the different impedances can then be found very easily. Thus,

$$\left.\begin{aligned} \dot{I}_{OO'} &= \frac{\dot{V}_{OO'}}{\dot{Z}_3} = \dot{V}_{OO'}\dot{Y}_3 \\ \dot{I}_{1O'} &= \frac{\dot{V}_{1O'}}{\dot{Z}_1} = (\dot{V}_{OO'} - \dot{V}_{O1})\dot{Y}_1 = (\dot{V}_{OO'} + \dot{E}_{g1})\dot{Y}_1 \\ \dot{I}_{2O'} &= \frac{\dot{V}_{2O'}}{\dot{Z}_2} = (\dot{V}_{OO'} - \dot{V}_{O2})\dot{Y}_2 = (\dot{V}_{OO'} + \dot{E}_{g2})\dot{Y}_2 \end{aligned}\right\} \tag{E–12}$$

Example E–2.—Fig. E–3 shows a network with three linear bilateral impedances connected in wye and three sources of emf of the same frequency also connected in wye (if the three emfs are equal in magnitude and 120° apart, the problem may be considered as one with an unbalanced Y-load connected to a three-phase, Y-connected generator). If the impedances $\dot{Z}_1$, $\dot{Z}_2$, and $\dot{Z}_3$ are given and the generated voltages $\dot{E}_{g1}$, $\dot{E}_{g2}$, and $\dot{E}_{g3}$ are known, determine the current flowing in each of the impedances.

Solution.—For this case, it is convenient to choose O at the junction point of the three sources. If $\dot{E}_{g1}$, $\dot{E}_{g2}$, and $\dot{E}_{g3}$, in the directions of the arrows, are considered to be voltage rises, then the voltage drops across the generators are $\dot{V}_{O1} = -\dot{E}_{g1}$, $\dot{V}_{O2} = -\dot{E}_{g2}$, and $\dot{V}_{O3} = -\dot{E}_{g3}$. The voltage drop from O to O' is

$$\dot{V}_{OO'} = \frac{\dot{V}_{O1}\dot{Y}_1 + \dot{V}_{O2}\dot{Y}_2 + \dot{V}_{O3}\dot{Y}_3}{\dot{Y}_1 + \dot{Y}_2 + \dot{Y}_3} \tag{E–13}$$

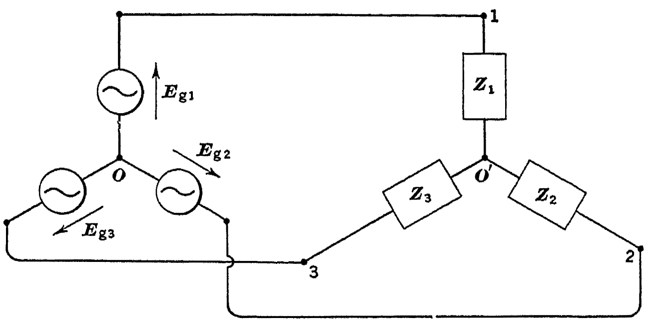


FIG. E–3

The current flowing in the three impedances are given by the following expressions:

$$\left.\begin{aligned} \dot{I}_{1O'} &= \frac{\dot{V}_{1O'}}{\dot{Z}_1} = (\dot{V}_{OO'} - \dot{V}_{O1})\dot{Y}_1 \\ \dot{I}_{2O'} &= \frac{\dot{V}_{2O'}}{\dot{Z}_2} = (\dot{V}_{OO'} - \dot{V}_{O2})\dot{Y}_2 \\ \dot{I}_{3O'} &= \frac{\dot{V}_{3O'}}{\dot{Z}_3} = (\dot{V}_{OO'} - \dot{V}_{O3})\dot{Y}_3 \end{aligned}\right\} \tag{E–14}$$

Example E–3.—Three resistances R_1, R_2 and R_3 are connected in a network containing three sources of emf, as shown in Fig. E–4. Each of the conductors connecting the resistances to the generators has a resistance of 2 ohms. Find the currents in the three line wires.

Solution.—First simplify by grouping the line resistances with the load resistances as follows:

$$Z_1 = 8+2 = 10;\ Y_1 = 0.1$$
$$Z_2 = 3+2 = 5;\ Y_2 = 0.2$$
$$Z_3 = 3+2 = 5;\ Y_3 = 0.2$$

The voltage drops across the generators are: $\dot{V}_{O1} = -\dot{E}_{g1} = -100\underline{/90°} = -j100$; $\dot{V}_{O2} = -\dot{E}_{g2} = -100\underline{/-30°} = -86.6+j50$; $\dot{V}_{O3} = -\dot{E}_{g3} = -100\underline{/-150°} = 86.6+j50$. Then, from equation (E–10),

$$\dot{V}_{OO'} = \frac{-j100\times 0.1 + (-86.6+j50)\times 0.2 + (86\ 6+j50)\times 0.2}{0.1+0.2+0.2}$$

or

$$\dot{V}_{OO'} = \frac{-j10 - 17.32 + j10 + 17.32 + j10}{0.5} = j20$$

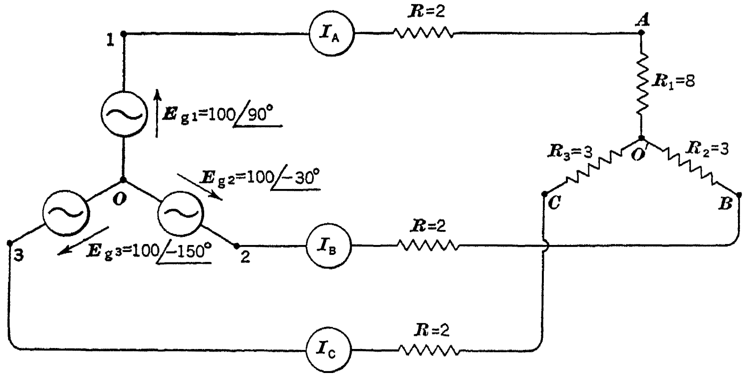


FIG. E–4

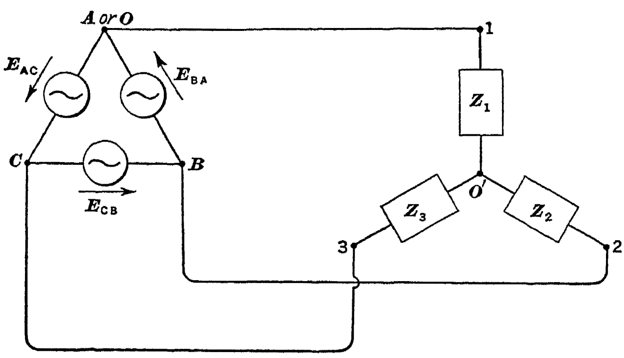


FIG. E–5

Also, from equations (E–14),

$$\dot{I}_A = \dot{I}_{1A} = (j20 + j100)\times 0.1 = j12 = 12\underline{/90°}$$
$$\dot{I}_B = \dot{I}_{2B} = (j20 + 86.6 - j50)\times 0.2 = 17.32 - j6 = 18.32\underline{/-19.1°}$$
$$\dot{I}_C = \dot{I}_{3C} = (j20 - 86.6 - j50)\times 0.2 = -17.32 - j6 = 18.32\underline{/199.1°}$$

Example E–4.—Three generators having the same frequency and having voltages equal in magnitude and 120° apart are connected in delta (three-phase, Δ-connected alternator). If the Y-load in Example E–2 is connected to these generators, as shown in Fig. E–5, find the expression for the current flowing in each of the impedances.

Solution.—Choose the point O to coincide with point A. This gives $\dot{V}_{O1} = 0$. If $\dot{E}_{BA}$ and $\dot{E}_{AC}$ are considered to be voltage rises, then the voltage drops from O to 2 and from O to 3 are $\dot{V}_{O2} = \dot{E}_{BA}$ and $\dot{V}_{O3} = -\dot{E}_{AC}$, respectively. Then, from equation (E–10),

$$\dot{V}_{OO'} = \dot{V}_{1O'} = \frac{\dot{V}_{O2}\dot{Y}_2 + \dot{V}_{O3}\dot{Y}_3}{\dot{Y}_1 + \dot{Y}_2 + \dot{Y}_3} \tag{E–15}$$

Also, the currents flowing in the different impedances are:

$$\left.\begin{aligned} \dot{I}_{1O'} &= \dot{V}_{1O'}\dot{Y}_1 = \dot{V}_{OO'}\dot{Y}_1 \\ \dot{I}_{2O'} &= \dot{V}_{2O'}\dot{Y}_2 = (\dot{V}_{OO'} - \dot{V}_{O2})\dot{Y}_2 \\ \dot{I}_{3O'} &= \dot{V}_{3O'}\dot{Y}_3 = (\dot{V}_{OO'} - \dot{V}_{O3})\dot{Y}_3 \end{aligned}\right\} \tag{E–16}$$

Example E–5.—The equivalent circuit of a simple triode amplifier, Fig. E–6 (*a*), is shown in Fig. E–6 (*b*). The symbols in the figure have the following meaning:

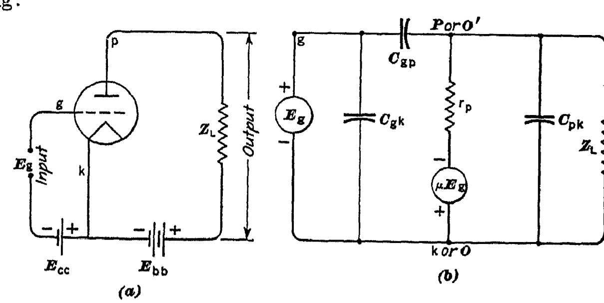


FIG. E–6

g = grid
p = plate
k = cathode (or filament)
C_{gk} = interelectrode capacitance from grid to cathode
C_{pk} = interelectrode capacitance from plate to cathode
C_{gp} = interelectrode capacitance from grid to plate
r_p = tube plate resistance
μ = amplification factor of tube
$\dot{Z}_L$ = load impedance

The problem here is to find the voltage gain of the amplifier. This gain is

$$\text{Gain} = \dot{A} = \frac{\text{Output voltage drop}}{\text{Input voltage drop}}$$

Thus, it is the ratio of the alternating-current output voltage across $\dot{Z}_L$ to the alternating-current input voltage $\dot{E}_g$. Since only alternating-current quantities enter into this problem, the batteries are omitted in the alternating-current equivalent circuit, Fig. E–6(*b*). The voltage of the fictitious generator, which replaces the triode, is 180° out of phase with the input voltage $\dot{E}_g$, and it is often indicated as $-\mu\dot{E}_g$. Determine an expression for the gain $\dot{A}$.

Solution.—Since the output voltage drop across $\dot{Z}_L$ is equal to $\dot{V}_{kp}$, it is convenient, in applying the theorem, to choose O at k and O' at P. The branches containing C_{gp}, r, C_{pk}, and $\dot{Z}_L$ must be considered, since they meet at point P. The voltage drop to the opposite end of r_p is $\mu\dot{E}_g$; the voltage drop to the opposite end of C_{gp} is $-\dot{E}_g$; and the voltage drops to the opposite ends of C_{pk} and $\dot{Z}_L$ are equal to zero. Then, from equation (E–10),

$$\dot{V}_{kp} = \frac{-\dot{E}_g\dot{Y}_{gp} + \mu\dot{E}_g\dot{Y}_p}{\dot{Y}_{gp} + \dot{Y}_p + \dot{Y}_{pk} + \dot{Y}_L} \tag{E–17}$$

where

$$\dot{Y}_{gp} = j\omega C_{gp}$$

$$\dot{Y}_p = \frac{1}{r_p}$$

$$\dot{Y}_{pk} = j\omega C_{pk}$$

$$\dot{Y}_L = \frac{1}{\dot{Z}_L}$$

Thus, the gain is

$$\dot{A} = \frac{\dot{V}_{kp}}{\dot{V}_{kg}} = \frac{\dot{V}_{kp}}{-\dot{E}_g} = \frac{\dot{Y}_{gp} - \mu\dot{Y}_p}{\dot{Y}_{gp} + \dot{Y}_p + \dot{Y}_{pk} + \dot{Y}_L} \tag{E–18}$$

The factor $\dot{A}$ in equation (E–18) is a complex number, although normally only its magnitude is used as the gain of an amplifier.

Equation (E–18) may also be written in terms of the constants of the tube. The transconductance of a triode is defined as follows:

$$g_m = \frac{\mu}{r_p} = \mu Y_p$$

Substituting g_m for μY_p in equation (E–18) gives

$$\dot{A} = \frac{\dot{Y}_{gp} - g_m}{\dot{Y}_{gp} + Y_p + \dot{Y}_{pk} + \dot{Y}_L} \tag{E–19}$$

Example E–6.—A detector amplifier triode RCA-6C5 has the following characteristics:

Grid-plate capacitance	1.8 $\mu\mu$f
Grid-cathode capacitance	4.0 $\mu\mu$f
Plate-cathode capacitance	13.0 $\mu\mu$f
Plate resistance	10,000 ohms
Amplification factor	20

A load resistor of 20,000 ohms is connected at the output terminals of the tube. If a signal of 1000 kc. is impressed at the grid, find the gain of the tube as an amplifier.

Solution.—To find the gain of this triode, it is only necessary to substitute values in equation (E–18). From the data given:

$$\dot{Y}_{gp} = j\omega C_{gp} = j6.28\times 10^6\times 1.8\times 10^{-12} = j11.3\times 10^{-6}$$
$$Y_p = 1/r_p = 1/10{,}000 = 100\times 10^{-6}$$
$$\mu Y_p = 20/10{,}000 = 2000\times 10^{-6}$$
$$\dot{Y}_{pk} = j\omega C_{pk} = j6.28\times 10^6\times 13\times 10^{-12} = j81.6\times 10^{-6}$$
$$Y_L = 1/Z_L = 1/20{,}000 = 50\times 10^{-6}$$

Since the factor 10^{-6} is common in every term of the equation, it may be dropped before making the substitution. Thus,

$$\dot{A} = \frac{j11.3 - 2000}{j11.3 + 100 + j81.6 + 50} = \frac{-2000 + j11.3}{150 + j92.9}$$

$$= \frac{-2000\angle -0.3^\circ}{176\angle 31.8^\circ} = 11.36\angle 147.9^\circ$$

PROBLEMS

E–1. Two 60-cycle generators with negligible internal impedances $\dot{E}_1 = 100\angle 90^\circ$ and $\dot{E}_2 = 100\angle 0^\circ$ are applied to the circuit shown in Fig. E–7. If the

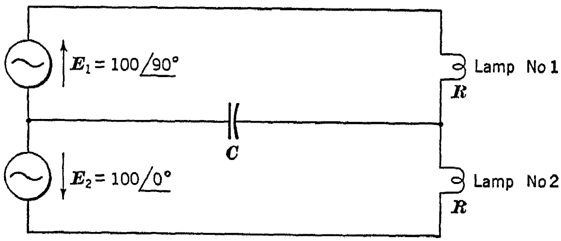


FIG. E–7

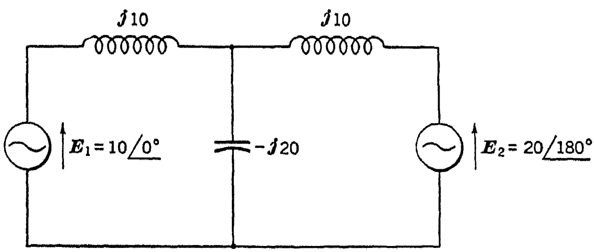


FIG. E–8

two identical incandescent lamps are assumed to have pure and constant resistance $R = 10$ ohms and the condenser C is of negligible resistance and has $X_C = 5$ ohms, which is the brighter lamp of the two?

E–2. A simple amplifier triode has the following characteristics:

$r_p = 2000$ ohms $\qquad C_{gk} =\ \mu\mu f$

$\mu = 4 \qquad C_{pk} = 2\ \mu\mu f$

$C_{gp} = 8\ \mu\mu f$

A load resistor of 10,000 ohms is connected at the output terminals of the tube. If a signal of 100 kc. is impressed at the grid, what is the gain of the tube as an amplifier?

E–3. What is the condenser current in Fig. E–8?

APPENDIX F

UNBALANCED Y-CONNECTED LOAD ON THREE-PHASE SYSTEM

Consider an unbalanced Y-connected load, as indicated in Fig. F–1, which is connected to a three-phase, three-wire system. For the purpose of obtaining the numerical values of the three line currents, it is convenient to derive equations for these quantities in terms of the branch impedances and the line voltage drops.

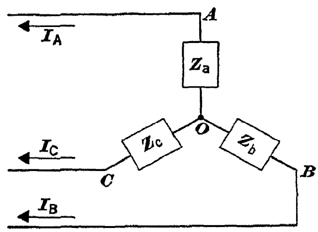


FIG. F–1

If $\dot{E}_{AC}$ and $\dot{E}_{CB}$ are considered as voltage drops, then application of Kirchhoff's Laws gives:

$$\dot{E}_{AC} = -\dot{Z}_a\dot{I}_{OA} + \dot{Z}_c\dot{I}_{OC} \qquad \text{(F–1)}$$

$$\dot{E}_{CB} = -\dot{Z}_c\dot{I}_{OC} + \dot{Z}_b\dot{I}_{OB} \qquad \text{(F–2)}$$

$$0 = \dot{I}_{OA} + \dot{I}_{OB} + \dot{I}_{OC} \qquad \text{(F–3)}$$

Writing these as three simultaneous equations, we have:

$$\dot{E}_{AC} = -\dot{Z}_a\dot{I}_{OA} + 0\ \dot{I}_{OB} + \dot{Z}_c\dot{I}_{OC} \qquad \text{(F–1a)}$$

$$\dot{E}_{CB} = 0\ \dot{I}_{OA} + \dot{Z}_b\dot{I}_{OB} - \dot{Z}_c\dot{I}_{OC} \qquad \text{(F–2a)}$$

$$0 = \dot{I}_{OA} + \dot{I}_{OB} + \dot{I}_{OC} \qquad \text{(F–3a)}$$

Then, using the method of determinants and letting $\Delta = \dot{Z}_a\dot{Z}_b + \dot{Z}_b\dot{Z}_c + \dot{Z}_c\dot{Z}_a$, we obtain:

$$\dot{I}_{OA} = \frac{\dot{E}_{AC}\dot{Z}_b + \dot{E}_{CB}\dot{Z}_c + \dot{E}_{AC}\dot{Z}_c}{-(\dot{Z}_a\dot{Z}_b + \dot{Z}_b\dot{Z}_c + \dot{Z}_c\dot{Z}_a)} = \frac{\dot{E}_{AC}\dot{Z}_b + (\dot{E}_{CB} + \dot{E}_{AC})\ \dot{Z}_c}{-\Delta} \qquad \text{(F–4)}$$

$$\dot{I}_{OB} = \frac{-\dot{E}_{CB}\dot{Z}_a - \dot{E}_{AC}\dot{Z}_c - \dot{E}_{CB}\dot{Z}_c}{-\Delta} = \frac{\dot{E}_{CB}\dot{Z}_a + (\dot{E}_{AC} + \dot{E}_{CB})\ \dot{Z}_c}{\Delta} \qquad \text{(F–5)}$$

$$\dot{I}_{OC} = \frac{-\dot{E}_{AC}\dot{Z}_b + \dot{E}_{CB}\dot{Z}_a}{-\Delta} = \frac{\dot{E}_{AC}\dot{Z}_b - \dot{E}_{CB}\dot{Z}_a}{\Delta} \tag{F–6}$$

Since the voltage drop from A to B is equal to the voltage drop from A to C plus the voltage drop from C to B,

$$\dot{E}_{AB} = \dot{E}_{AC} + \dot{E}_{CB} \tag{F–7}$$

Then equations (F–4) to (F–6) can be written as follows:

$$\dot{I}_{OA} = \frac{\dot{E}_{AC}Z_b + \dot{E}_{AB}\dot{Z}_c}{-\Delta} = \frac{\dot{E}_{CA}\dot{Z}_b + \dot{E}_{BA}\dot{Z}_c}{\Delta} \tag{F–4a}$$

$$\dot{I}_{OB} = \frac{\dot{E}_{CB}\dot{Z}_a + \dot{E}_{AB}\dot{Z}_c}{\Delta} \tag{F–5a}$$

$$\dot{I}_{OC} = \frac{\dot{E}_{AC}\dot{Z}_b + \dot{E}_{BC}\dot{Z}_a}{\Delta} \tag{F–6a}$$

The numerators of the last three equations may be obtained from Fig. F–1 in the following way. For simplicity, let $\dot{Z}_a$ be the "opposite impedance" of $\dot{E}_{CB}$ or $\dot{E}_{BC}$, let $\dot{Z}_b$ be the "opposite impedance" of $\dot{E}_{AC}$ or $\dot{E}_{CA}$, and let $\dot{Z}_c$ be the "opposite impedance" of $\dot{E}_{BA}$ or $\dot{E}_{AB}$. Then the following interpretation may be helpful in determining the numerator of equation (F–4a) for $\dot{I}_A = \dot{I}_{OA}$ in line A:

(*a*) The voltage drops $\dot{E}_{CA}$ and $\dot{E}_{BA}$ from the other two lines to line A must be used.

(*b*) The term $\dot{E}_{CA}\dot{Z}_b$ is the voltage drop $\dot{E}_{CA}$ multiplied by its "opposite impedance" $\dot{Z}_b$; and $\dot{E}_{BA}\dot{Z}_c$ is the voltage drop $\dot{E}_{BA}$ multiplied by its "opposite impedance" $\dot{Z}_c$.

It can be seen that similar observations may be made for the numerators of equations (F–5a) and (F–6a).

APPENDIX G

HYPERBOLIC FUNCTIONS

Hyperbolic functions are useful in many physical problems, such as the cantilever beam, the suspended cable, transients in circuits containing R, L, and C, and voltage and current distribution in long transmission lines. For many electrical engineering problems, it is best to express the electrical quantities in terms of hyperbolic functions or their equivalent exponential functions. Generally this procedure gives shorter formulas than if the quantities are expressed by other relations, and it offers a more accurate solution than the usual approximate methods.

The sine, cosine, tangent, etc. are relations with reference to a unit circle the equation of which is $x^2+y^2=1$. Similarly, the hyperbolic sine, hyperbolic cosine, hyperbolic tangent, etc. are relations with reference to a unit equilateral hyperbola the equation of which is $x^2-y^2=1$. For example, the value of sinh 0.8, which is 0.8881, is obtained from tables of hyperbolic functions; just as the value of sin 0.8, which is 0.7174, is found from tables of trigonometric functions.

It is the purpose here to consider the following: (1) the analogy between the circular functions and the hyperbolic functions; (2) the hyperbolic functions of real numbers; (3) the relationships between circular and hyperbolic functions, such as $\sin jx = j \sinh x$; (4) the hyperbolic functions of complex angles; (5) the angle of hyperbolic functions in the polar form; (6) the inverse hyperbolic functions.

G–1. Analogy of Hyperbolic to Circular Functions.—The hyperbolic functions are so called because they bear relations to the unit hyperbola $x^2-y^2=1$ that are very similar to those borne by the circular functions to the unit circle $x^2+y^2=1$. A source of difficulty in the use of hyperbolic functions is due to the fact that circular functions are very often referred to angles in degrees, such as sin 30°, and there is no analogous unit in hyperbolic functions. The idea of sector area, rather than angles in degrees, is more useful in demonstrating the "similarity" of the hyperbolic and circular functions (that is, we could just as well think of the cosine of an angle as the cosine of a number representing an area). This will now be considered.

Fig. G–1 (*a*) shows a unit circle $x^2+y^2=1$. The circular sector, such as *ONP*, is generated by rotating the radius $r=1$ about the center *O* through a circular angle θ. This angle θ may be defined in terms of the area A_c of the circular sector. By inspection of Fig. G–1 (*a*).,

$$\sin\theta=\frac{y_c}{r} \tag{G–1}$$

$$\cos\theta=\frac{x_c}{r} \tag{G–2}$$

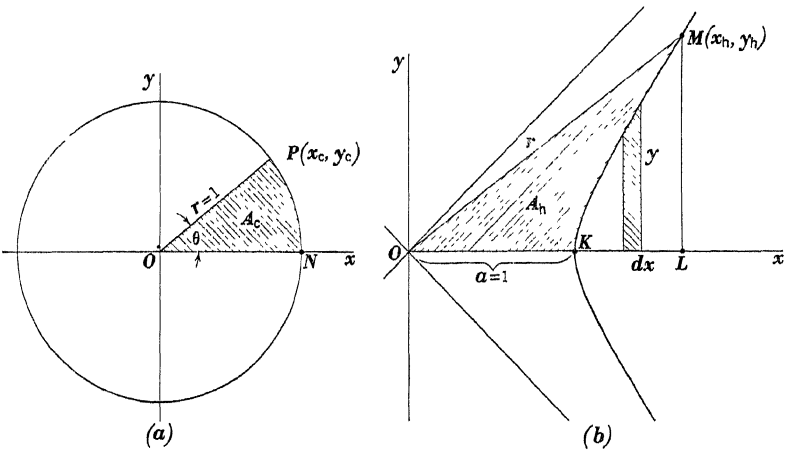


FIG. G–1

The area A_c is proportional to the angle θ expressed in radians. Thus,

$$A_c=\frac{\theta}{2\pi}\times\pi r^2=\frac{\theta r^2}{2} \tag{G–3}$$

or

$$\theta=\frac{2A_c}{r^2} \tag{G–4}$$

Substituting this value for θ and multiplying equation (G–1) by j, and then adding the resultant expression to equation (G–2), we obtain:

$$\cos\frac{2A_c}{r^2}+j\sin\frac{2A_c}{r^2}=\frac{x_c}{r}+j\frac{y_c}{r}=e^{j\frac{2A_c}{r^2}} \tag{G–5}$$

A similar expression may now be derived with reference to a unit equilateral hyperbola $x^2 - y^2 = 1$, as shown in Fig. G–1 (*b*). At any position, such as *OM*, the hyperbolic sector *OKM* may be considered as being swept out by the radius *r*. The area of the triangle *OLM* is $\frac{x_h y_h}{2}$. The area of the cross-hatched hyperbolic section *OKM* is equal to the area of the triangle *OLM* minus the area *KLM*. Hence,

$$A_h = \frac{x_h y_h}{2} - \int_a^{x_h} y\,dx = \frac{x_h y_h}{2} - \int_a^{x_h} \sqrt{x^2 - a^2}\,dx$$

$$= \frac{x_h y_h}{2} - \left\{\frac{1}{2}\left[x\sqrt{x^2 - a^2} - a^2 \log_e \left(x + \sqrt{x^2 - a^2}\right)\right]_a^{x_h}\right\}$$

$$= \frac{x_h y_h}{2} - \frac{1}{2}\left[x_h\sqrt{x_h^2 - a^2} - a^2 \log_e \left(x_h + \sqrt{x_h^2 - a^2}\right) + a^2 \log_e a\right]$$

$$= \frac{x_h y_h}{2} - \frac{1}{2}\left[x_h y_h - a^2 \log_e (x_h + y_h) + a^2 \log_e a\right]$$

or

$$A_h = \frac{a^2}{2} \log_e \frac{x_h + y_h}{a} = \frac{a^2}{2} \log_e \left(\frac{x_h}{a} + \frac{y_h}{a}\right) \tag{G–6}$$

Also,

$$\frac{2A_h}{a^2} = \log_e \left(\frac{x_h}{a} + \frac{y_h}{a}\right)$$

and

$$e^{\frac{2A_h}{a^2}} = \frac{x_h}{a} + \frac{y_h}{a} \tag{G–7}$$

where $\frac{2A_h}{a^2}$ is expressed in radians (or in hyperbolic radians).

Because of the similarity of form with the circular functions given in equation (G–5), the ratios $\frac{x_h}{a}$ and $\frac{y_h}{a}$ are defined as follows:

$$\frac{x_h}{a} = \text{hyperbolic cosine of } \frac{2A_h}{a^2} = \cosh \frac{2A_h}{a^2} \tag{G–8}$$

$$\frac{y_h}{a} = \text{hyperbolic sine of } \frac{2A_h}{a^2} = \sinh \frac{2A_h}{a^2} \tag{G–9}$$

Therefore,

$$e^{\frac{2A_h}{a^2}} = \cosh \frac{2A_h}{a^2} + \sinh \frac{2A_h}{a^2} \tag{G–10}$$

In a similar manner, it can be shown that

$$e^{-\frac{2A_h}{a^2}} = \cosh \frac{2A_h}{a^2} - \sinh \frac{2A_h}{a^2} \tag{G–11}$$

It is known that sin θ and cos θ vary periodically as the point P moves around the circle in Fig. G–1 (a). It can be seen from Fig. G–1 (b) that, as M moves from K along the hyperbola, the ratios $\frac{x_h}{a} = \cosh \frac{2A_h}{a^2}$ and $\frac{y_h}{a} = \sinh \frac{2A_h}{a^2}$ are directly proportional to A_h for a fixed value of $a = 1$. Thus, a difference between these two types of functions is: Circular functions are periodic, whereas hyperbolic functions are not.

G–2. Hyperbolic Functions of Real Numbers.—It is quite simple to use hyperbolic functions of real numbers in the solution of numerical problems. It will be seen in this article that the expressions for sinh x, cosh x, and tanh x are simpler than those for sin x, cos x, and tan x. Tables are available for finding sinh x, cosh x, and tanh x for different values of x. It is reasonable to conclude that hyperbolic functions must satisfy relations similar to those for the circular functions. Some of these relationships will now be considered.

Fundamental relations of circular functions are:

$$e^{jx} = \cos x + j \sin x \tag{G–12}$$

$$e^{-jx} = \cos x - j \sin x \tag{G–13}$$

From these relations, the following functions are obtained:

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \tag{G–14}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \tag{G–15}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{e^{jx} - e^{-jx}}{j\,(e^{jx} + e^{-jx})} \tag{G–16}$$

It is obvious that sin x, for example, is found by subtracting equation (G–13) from equation (G–12) and rearranging the result to give the form (G–14). Also, dividing equation (G–14) by equation (G–15) gives equation (G–16).

By following a similar procedure with the aid of equations (G–10) and (G–11), it can be shown that the hyperbolic sine, the hyperbolic cosine, and the hyperbolic tangent are given by the following expressions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{G–17}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{G–18}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{G–19}$$

By comparison, it can be seen that equations (G–17) to (G–19) are simpler than equations (G–14) to (G–16).

The series expansions for sinh x and cosh x will be given without derivation. They are:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \tag{G–20}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \tag{G–21}$$

Other hyperbolic functions are defined by the following equations:

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \tag{G–22}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \tag{G–23}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \tag{G–24}$$

It is helpful at this point to give a few graphs showing the variations of some of the hyperbolic functions. The hyperbolic functions of real numbers are shown in Fig. G–2. For positive values of x,

sinh x varies from 0 to $+\infty$
cosh x varies from 1 to $+\infty$
tanh x varies from 0 to $+1$

For negative values of x,

sinh x varies from 0 to $-\infty$
cosh x varies from 1 to $+\infty$
tanh x varies from 0 to -1

It can be seen from equations (G–17) and (G–18) that, since e^{-x} is added to e^x in the numerator in cosh x and it is subtracted from e^x in the numerator in sinh x, cosh x will always be greater than sinh x. As x increases to a very large value, the factor e^{-x} becomes negligible and sinh x approaches cosh x as a limit.

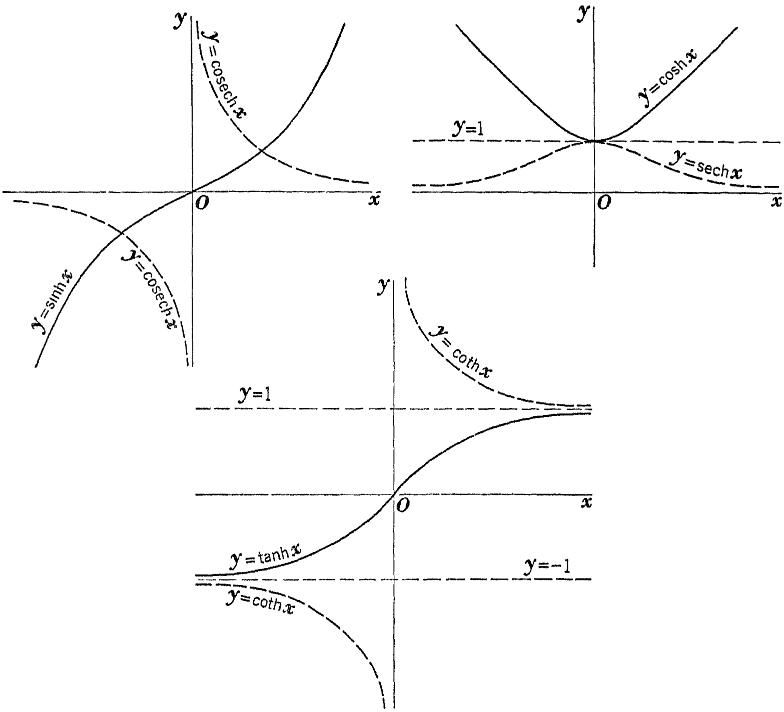


Fig. G–2

The hyperbolic functions satisfy relations similar to those for the trigonometric functions. Some of the relationships follow:

$$\sinh(-x) = -\sinh x \tag{G–25}$$

$$\cosh(-x) = \cosh x \tag{G–26}$$

$$\tanh(-x) = -\tanh x \tag{G–27}$$

$$\frac{d}{dx}\sinh x = \cosh x \tag{G–28}$$

$$\frac{d}{dx}\cosh x = \sinh x \tag{G–29}$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \tag{G-30}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{G-31}$$

$$\sinh 2x = 2 \sinh x \cosh x \tag{G-32}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x \tag{G-33}$$

$$\sinh \frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x - 1)} \tag{G-34}$$

$$\cosh \frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x + 1)} \tag{G-35}$$

$$\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \tag{G-36}$$

$$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \tag{G-37}$$

G–3. Circular and Hyperbolic Functions Related.—It can be shown by the use of the exponential form that the hyperbolic sine and hyperbolic cosine are connected with the trigonometric sine and cosine by the following relations:

$$\left.\begin{array}{ll} \sin jx = j \sinh x & \cos jx = \cosh x \\ \sinh jx = j \sin x & \cosh jx = \cos x \\ \sinh x = -j \sin jx & \cosh x = \cos jx \end{array}\right\} \tag{G-38}$$

Example G–1.—Show that $\sinh jx = j \sin x$.

Solution.—Substituting jx for x in equation (G–17) gives

$$\sinh jx = \frac{e^{jx} - e^{-jx}}{2}$$

From equation (G–14),

$$j \sin x = \frac{e^{jx} - e^{-jx}}{2}$$

Hence,

$$\sinh jx = j \sin x$$

G–4. Hyperbolic Functions of Complex Angles.—The sine and cosine of complex numbers are:

$$\begin{aligned} \sin (x \pm jy) &= \sin x \cos jy \pm \cos x \sin jy \\ &= \sin x \cosh y \pm j \cos x \sinh y \\ \cos (x \pm jy) &= \cos x \cos jy \mp \sin x \sin jy \\ &= \cos x \cosh y \mp j \sin x \sinh y \end{aligned}$$

Similar to these the equivalents of hyperbolic functions of complex numbers (or complex angles) are determined as follows.

(*a*) Equivalent of sinh $(x \pm jy)$:

$$\sinh (x \pm jy) = \frac{e^{(x \pm jy)} - e^{-(x \pm jy)}}{2} = \frac{e^x e^{\pm jy} - e^{-x} e^{\pm jy}}{2}$$

$$= \frac{e^x}{2} (\cos y \pm j \sin y) - \frac{e^{-x}}{2} (\cos y \mp j \sin y)$$

$$= \frac{e^x}{2} \cos y \pm j \frac{e^x}{2} \sin y - \frac{e^{-x}}{2} \cos y \pm j \frac{e^{-x}}{2} \sin y$$

$$= \frac{e^x - e^{-x}}{2} \cos y \pm j \frac{e^x + e^{-x}}{2} \sin y$$

or

$$\sinh (x \pm jy) = \sinh x \cos y \pm j \cosh x \sin y \qquad \textbf{(G–39)}$$

Example G–2.—Find the equivalent of sinh $(0.6 + j0.5)$.

Solution.—By equation (G–39),

$$\begin{aligned} \sinh (0.6 + j0.5) &= \sinh 0.6 \times \cos 0.5 + j \cosh 0.6 \times \sin 0.5 \\ &= \sinh 0.6 \times \cos 28.65° + j \cosh 0.6 \times \sin 28.65° \\ &= 0.6366 \times 0.8776 + j1.1855 \times 0.4794 \\ &= 0.559 + j0.569 = 0.798 \underline{/45.4°} \end{aligned}$$

(*b*) Equivalent of cosh $(x \pm jy)$:

$$\cosh (x \pm jy) = \frac{e^{(x \pm jy)} + e^{-(x \pm jy)}}{2} = \frac{e^x e^{\pm jy} + e^{-x} e^{\mp jy}}{2}$$

$$= \frac{e^x}{2} (\cos y \pm j \sin y) + \frac{e^{-x}}{2} (\cos y \mp j \sin y)$$

$$= \frac{e^x}{2} \cos y \pm j \frac{e^x}{2} \sin y + \frac{e^{-x}}{2} \cos y \mp j \frac{e^{-x}}{2} \sin y$$

$$= \frac{e^x + e^{-x}}{2} \cos y \pm j \frac{e^x - e^{-x}}{2} \sin y$$

or

$$\cosh (x \pm jy) = \cosh x \cos y \pm j \sinh x \sin y \qquad \textbf{(G–40)}$$

Example G–3.—Find the equivalent of cosh $(0.6 + j0.5)$.

Solution.—By equation (G–40),

$$\begin{aligned} \cosh (0.6 + j0.5) &= \cosh 0.6 \times \cos 0.5 + j \sinh 0.6 \times \sin 0.5 \\ &= \cosh 0.6 \times \cos 28.65° + j \sinh 0.6 \times \sin 28.65° \\ &= 1.1855 \times 0.8776 + j0.6366 \times 0.4794 \\ &= 1.04 + j0.3306 = 1.09 \underline{/17.8°} \end{aligned}$$

Similarly,

(*c*)
$$\tanh (x \pm jy) = \frac{\sinh 2x}{\cosh 2x + \cos 2y} \pm j \frac{\sin 2y}{\cosh 2x + \cos 2y} \qquad \textbf{(G–41)}$$

(*d*) $$\coth(x \pm jy) = \frac{\sinh 2x}{\cosh 2x - \cos 2y} \pm j \frac{\sin 2y}{\cosh 2x - \cos 2y} \qquad \text{(G–42)}$$

G–5. Angle of Hyperbolic Functions in Polar Form.—Sometimes it is convenient to express the hyperbolic sine and the hyperbolic cosine in the polar form.

(*a*) $$\sinh(x+jy) = \sinh x \cosh jy + \cosh x \sinh jy$$

or $$\sinh(x+jy) = \sinh x \cos y + j \cosh x \sin y \qquad \text{(G–43)}$$

Also,

$$\sinh(x+jy) = \sqrt{\sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y}\ \underline{/\tan^{-1}\left(\frac{\tan y}{\tanh x}\right)} \qquad \text{(G–44)}$$

When we substitute $(1 - \sin^2 y)$ for $\cos^2 y$ and $(1 + \sinh^2 x)$ for $\cosh^2 x$ and we rearrange terms, equation (G–43) becomes

$$\sinh(x+jy) = \sqrt{\sinh^2 x + \sin^2 y}\ \underline{/\tan^{-1}\left(\frac{\tan y}{\tanh x}\right)} \qquad \text{(G–45)}$$

(*b*) $$\cosh(x+jy) = \cosh x \cosh jy + \sinh x \sinh jy$$

or $$\cosh(x+jy) = \cosh x \cos y + j \sinh x \sin y \qquad \text{(G–46)}$$

Also,

$$\cosh(x+jy) = \sqrt{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y}\ \underline{/\tan^{-1}(\tan y \tanh x)} \qquad \text{(G–47)}$$

When we substitute $(1 + \sinh^2 x)$ for $\cosh^2 x$ and $(1 - \cos^2 y)$ for $\sin^2 y$ and we rearrange terms, equation (G–47) takes the form

$$\cosh(x+jy) = \sqrt{\sinh^2 x + \cos^2 y}\ \underline{/\tan^{-1}(\tan y \tanh x)} \qquad \text{(G–48)}$$

The angular position of the hyperbolic sine will now be considered. In transmission line problems, the complex angle has the form $\alpha + j\beta$, where α is the attenuation constant and β is the wavelength constant. Since, for such cases, α is always defined as a positive real number, the discussion here will be limited to hyperbolic sines with positive values for x. With this assumption for x, it will be shown that the angle of the hyperbolic sine must be in the same quadrant as the imaginary part y of the complex angle $(x+jy)$.

For any given positive value of x, both sinh x and cosh x will be positive and their ratio, or $\tanh x = \frac{\sinh x}{\cosh x}$, may be considered as a positive constant. Equation (G–43) can be written as follows:

$$\sinh (x+jy) = \sinh x \cos y + j \cosh x \sin y = A + jB \qquad \textbf{(G–43a)}$$

Also, the angle of the hyperbolic sine is, from equation (G–44),

$$\theta = \tan^{-1}\left(\frac{\tan y}{\tanh x}\right) = \tan^{-1}\left(\frac{1}{\tanh x}\,\frac{\sin y}{\cos y}\right) = \tan^{-1}\frac{B}{A} \qquad \textbf{(G–49)}$$

Hence, the angle θ depends on the signs of A or cos y and B or sin y.

When the angle y is in the first quadrant, A is positive because cos y is positive and B is positive because sin y is positive. Both A and B being positive, the angle θ of the hyperbolic sine must be in the first quadrant, that is, in the same quadrant as the imaginary part y of the complex angle. When the angle y is in the second quadrant, the sign of A is negative because cos y is negative and the sign of B is positive because sin y is positive. With A negative and B positive, the angle θ must be in the second quadrant, that is, again in the same quadrant as y. This is true also for y in the third quadrant or the fourth quadrant. It can be demonstrated that within a quadrant the angle y does not equal the angle $\theta = \tan^{-1}\left(\frac{\tan y}{\tanh x}\right)$; yet, at $y = 0°$, 90°, 180°, 270°, 360°, etc., the two angles y and θ are equal.

In a similar manner, it can be shown that in every case the angle $\tan^{-1}$ (tan y tanh x) of cosh $(x+jy)$ must be in the same quadrant as the imaginary part y.

Example G–4.—Find the polar form of sinh $(0.6+j0.5)$.

Solution.—By equation (G–45),

$$\sinh (0.6+j0.5) = \sqrt{\sinh^2 0.6 + \sin^2 0.5}\ \Big/ \tan^{-1}\left(\frac{\tan 0.5}{\tanh 0.6}\right)$$

$$= \sqrt{0.6366^2 + 0.4794^2}\ \Big/ \tan^{-1}\frac{0.5476}{0.537} = 0.798\underline{/45.4°}$$

G–6. Inverse Hyperbolic Functions.—The inverse hyperbolic functions are defined in a way similar to that used in defining the inverse circular functions. If $\sinh \alpha = x$, then

$$\alpha = \sinh^{-1} x$$

Also, if $\sinh(\alpha + j\beta) = x + jy$, then

$$\alpha + j\beta = \sinh^{-1}(x + jy)$$

The inverse hyperbolic sine, the inverse hyperbolic cosine, and the inverse hyperbolic tangent are determined as follows.

$$\sinh^{-1}(x \pm jy) = \cosh^{-1}\frac{\sqrt{(1+y)^2+x^2}+\sqrt{(1-y)^2+x^2}}{2} \pm j\sin^{-1}\frac{\sqrt{(1+y)^2+x^2}-\sqrt{(1-y)^2+x^2}}{2} \qquad \text{(G–50)}$$

$$\cosh^{-1}(x \pm jy) = \cosh^{-1}\frac{\sqrt{(1+x)^2+y^2}+\sqrt{(1-x)^2+y^2}}{2} \pm j\cos^{-1}\frac{\sqrt{(1+x)^2+y^2}-\sqrt{(1-x)^2+y^2}}{2} \qquad \text{(G–51)}$$

$$\tanh^{-1}(x \pm jy) = \frac{1}{2}\log_e\sqrt{\frac{(1+x)^2+y^2}{(1-x)^2+y^2}} \pm j\,\frac{\pi - \tan^{-1}\left(\frac{x+1}{\pm y}\right) + \tan^{-1}\left(\frac{x-1}{\pm y}\right)}{2} \qquad \text{(G–52)}$$

In some transmission line problems, the hyperbolic tangent of the propagation constant $\gamma = \alpha + j\beta$ is known. Thus,

$$\tanh\gamma = \tanh(\alpha + j\beta) = A + jB \qquad \text{(G–53)}$$

That is, A and B are known. Then the attenuation constant α and the wavelength constant β can be found from the following equations:

$$\tanh 2\alpha = \frac{2A}{1+A^2+B^2} \qquad \text{(G–54)}$$

$$\tan 2\beta = \frac{2B}{1-(A^2+B^2)} \qquad \text{(G–55)}$$

It should be noted that the solution for β by equation (G–55) is multivalued; that is, if $\tan\beta = -1$, then β may be $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}$, etc.

PROBLEMS

G–1. By the use of tables of hyperbolic and exponential functions, show that

$$\sinh 2 = \frac{e^2 - e^{-2}}{2}$$

G–2. Plot graphs of e^x and e^{-x} for values of $x = 0$ to $x = 3$. Take one-half of the sum of these two curves and compare with a graph obtained by plotting values from a table for $\cosh x$.

G–3. Find sinh 0.5 by the use of the series, and compare the result obtained with the value in the table.

G–4. Repeat Problem G–3 for cosh 0.5.

G–5. Prove that $\cosh^2 x - \sinh^2 x = 1$

G–6. Verify: $\sinh (x+y) = \sinh x \cosh y + \cosh x \sinh y$.

G–7. Show that $\cos jx = \cosh x$.

G–8. Evaluate the following:

$$(a) \quad \sinh (0.8 + j0.4)$$
$$(b) \quad \cosh (1.2 - j0.9)$$

G–9. Determine the value for $\sin^{-1} (3 + j2)$.

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS*

x (Radians)	cosh x	sinh x	tanh x	ϵ^x	ϵ^{-x}	cos x	sin x	tan x	Degrees	x
0.00	1.0000	0.0000	0.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.00°	0.00
0.01	1.0001	0.0100	0.0100	1.0101	0.9900	1.0000	0.0100	0.0100	0.57°	0.01
0.02	1.0002	0.0200	0.0200	1.0202	0.9802	0.9998	0.0200	0.0200	1.15°	0.02
0.03	1.0005	0.0300	0.0300	1.0305	0.9704	0.9996	0.0300	0.0300	1.72°	0.03
0.04	1.0008	0.0400	0.0400	1.0408	0.9608	0.9992	0.0400	0.0400	2.29°	0.04
0.05	1.0013	0.0500	0.0500	1.0513	0.9512	0.9988	0.0500	0.0500	2.86°	0.05
0.06	1.0018	0.0600	0.0599	1.0618	0.9418	0.9982	0.0600	0.0601	3.44°	0.06
0.07	1.0025	0.0701	0.0699	1.0725	0.9324	0.9976	0.0699	0.0701	4.01°	0.07
0.08	1.0032	0.0801	0.0798	1.0833	0.9231	0.9968	0.0799	0.0802	4.58°	0.08
0.09	1.0041	0.0901	0.0898	1.0942	0.9139	0.9960	0.0899	0.0902	5.16°	0.09
0.10	1.0050	0.1002	0.0997	1.1052	0.9048	0.9950	0.0998	0.1003	5.73°	0.10
0.11	1.0061	0.1102	0.1096	1.1163	0.8958	0.9940	0.1098	0.1104	6.30°	0.11
0.12	1.0072	0.1203	0.1194	1.1275	0.8869	0.9928	0.1197	0.1206	6.88°	0.12
0.13	1.0085	0.1304	0.1293	1.1388	0.8781	0.9916	0.1296	0.1307	7.45°	0.13
0.14	1.0098	0.1405	0.1391	1.1503	0.8694	0.9902	0.1395	0.1409	8.02°	0.14
0.15	1.0113	0.1506	0.1489	1.1618	0.8607	0.9888	0.1494	0.1511	8.59°	0.15
0.16	1.0128	0.1607	0.1587	1.1735	0.8521	0.9872	0.1593	0.1614	9.17°	0.16
0.17	1.0145	0.1708	0.1684	1.1853	0.8437	0.9856	0.1692	0.1717	9.74°	0.17
0.18	1.0162	0.1810	0.1781	1.1972	0.8353	0.9838	0.1790	0.1820	10.31°	0.18
0.19	1.0181	0.1912	0.1878	1.2093	0.8270	0.9820	0.1889	0.1923	10.89°	0.19
0.20	1.0201	0.2013	0.1974	1.2214	0.8187	0.9801	0.1987	0.2027	11.46°	0.20
0.21	1.0221	0.2116	0.2070	1.2337	0.8106	0.9780	0.2085	0.2131	12.03°	0.21
0.22	1.0243	0.2218	0.2165	1.2461	0.8025	0.9759	0.2182	0.2236	12.61°	0.22
0.23	1.0266	0.2320	0.2260	1.2586	0.7945	0.9737	0.2280	0.2341	13.18°	0.23
0.24	1.0289	0.2423	0.2355	1.2712	0.7866	0.9713	0.2377	0.2447	13.75°	0.24

0.25	1.0314	0.2526	0.2449	1.2840	0.7788	0.9689	0.2474	0.2553	1.432°	0.25
0.26	1.0340	0.2629	0.2543	1.2969	0.7711	0.9664	0.2571	0.2660	1.490°	0.26
0.27	1.0367	0.2733	0.2636	1.3100	0.7634	0.9638	0.2667	0.2768	15.47°	0.27
0.28	1.0395	0.2837	0.2729	1.3231	0.7558	0.9611	0.2764	0.2876	16.04°	0.28
0.29	1.0424	0.2941	0.2821	1.3364	0.7483	0.9582	0.2860	0.2984	16.62°	0.29
0.30	1.0453	0.3045	0.2913	1.3499	0.7408	0.9553	0.2955	0.3093	17.19°	0.30
0.31	1.0484	0.3150	0.3004	1.3634	0.7334	0.9523	0.3051	0.3203	17.76°	0.31
0.32	1.0516	0.3255	0.3095	1.3771	0.7261	0.9492	0.3146	0.3314	18.33°	0.32
0.33	1.0550	0.3360	0.3185	1.3910	0.7189	0.9460	0.3240	0.3425	18.91°	0.33
0.34	1.0584	0.3466	0.3275	1.4049	0.7118	0.9428	0.3335	0.3537	19.48°	0.34
0.35	1.0619	0.3572	0.3364	1.4191	0.7047	0.9394	0.3429	0.3650	20.05°	0.35
0.36	1.0655	0.3678	0.3452	1.4333	0.6977	0.9359	0.3523	0.3764	20.63°	0.36
0.37	1.0692	0.3785	0.3540	1.4477	0.6907	0.9323	0.3616	0.3879	21.20°	0.37
0.38	1.0731	0.3892	0.3627	1.4623	0.6839	0.9287	0.3709	0.3994	21.77°	0.38
0.39	1.0770	0.4000	0.3714	1.4770	0.6771	0.9249	0.3802	0.4111	22.35°	0.39
0.40	1.0811	0.4108	0.3800	1.4918	0.6703	0.9211	0.3894	0.4228	22.92°	0.40
0.41	1.0852	0.4216	0.3885	1.5068	0.6637	0.9171	0.3986	0.4346	23.49°	0.41
0.42	1.0895	0.4325	0.3969	1.5220	0.6570	0.9131	0.4078	0.4466	24.06°	0.42
0.43	1.0939	0.4434	0.4053	1.5373	0.6505	0.9090	0.4169	0.4586	24.64°	0.43
0.44	1.0984	0.4543	0.4136	1.5527	0.6440	0.9048	0.4259	0.4708	25.21°	0.44
0.45	1.1030	0.4653	0.4219	1.5683	0.6376	0.9005	0.4350	0.4831	25.78°	0.45
0.46	1.1077	0.4764	0.4301	1.5841	0.6313	0.8961	0.4440	0.4954	26.36°	0.46
0.47	1.1125	0.4875	0.4382	1.6000	0.6250	0.8916	0.4529	0.5080	26.93°	0.47
0.48	1.1174	0.4987	0.4462	1.6161	0.6188	0.8870	0.4618	0.5206	27.50°	0.48
0.49	1.1225	0.5098	0.4542	1.6323	0.6126	0.8823	0.4706	0.5334	28.07°	0.49

*This table was originally compiled by F W Norris and L A. Bingham of the University of Nebraska, who made use of the following sources in preparing it.

(1) "Smithsonian Mathematical Tables Hyperbolic Functions" by Becker and Van Orstrand.

(2) "Zehnstellige Logarithmen der Trigonometrischen Funktionen" by J. Peters.

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS—(*Continued*)

x (Radians)	cosh x	sinh x	tanh x	ϵ^x	ϵ^{-x}	cos x	sin x	tan x	Degrees	x
0.50	1.1276	0.5211	0.4621	1.6487	0.6065	0.8776	0.4794	0.5463	28.65°	0.50
0.51	1.1329	0.5324	0.4700	1.6653	0.6005	0.8727	0.4882	0.5594	29.22°	0.51
0.52	1.1383	0.5438	0.4777	1.6820	0.5945	0.8678	0.4969	0.5726	29.79°	0.52
0.53	1.1438	0.5552	0.4854	1.6989	0.5886	0.8628	0.5055	0.5859	30.37°	0.53
0.54	1.1494	0.5666	0.4930	1.7160	0.5827	0.8577	0.5141	0.5994	30.94°	0.54
0.55	1.1551	0.5782	0.5005	1.7333	0.5769	0.8525	0.5227	0.6131	31.51°	0.55
0.56	1.1609	0.5897	0.5080	1.7507	0.5712	0.8473	0.5312	0.6269	32.09°	0.56
0.57	1.1669	0.6014	0.5154	1.7683	0.5655	0.8419	0.5396	0.6410	32.66°	0.57
0.58	1.1730	0.6131	0.5227	1.7860	0.5599	0.8365	0.5480	0.6552	33.23°	0.58
0.59	1.1792	0.6248	0.5299	1.8040	0.5543	0.8309	0.5564	0.6696	33.80°	0.59
0.60	1.1855	0.6367	0.5371	1.8221	0.5488	0.8253	0.5646	0.6841	34.38°	0.60
0.61	1.1919	0.6485	0.5441	1.8404	0.5434	0.8197	0.5729	0.6989	34.95°	0.61
0.62	1.1984	0.6605	0.5511	1.8589	0.5379	0.8139	0.5810	0.7139	35.52°	0.62
0.63	1.2051	0.6725	0.5581	1.8776	0.5326	0.8080	0.5891	0.7291	36.10°	0.63
0.64	1.2119	0.6846	0.5649	1.8965	0.5273	0.8021	0.5972	0.7445	36.67°	0.64
0.65	1.2188	0.6968	0.5717	1.9155	0.5220	0.7961	0.6052	0.7602	37.24°	0.65
0.66	1.2258	0.7090	0.5784	1.9348	0.5169	0.7900	0.6131	0.7761	37.82°	0.66
0.67	1.2330	0.7213	0.5850	1.9542	0.5117	0.7838	0.6210	0.7923	38.39°	0.67
0.68	1.2403	0.7336	0.5915	1.9739	0.5066	0.7776	0.6288	0.8087	38.96°	0.68
0.69	1.2477	0.7461	0.5980	1.9937	0.5016	0.7713	0.6365	0.8253	39.53°	0.69
0.70	1.2552	0.7586	0.6044	2.0138	0.4966	0.7648	0.6442	0.8423	40.11°	0.70
0.71	1.2628	0.7712	0.6107	2.0340	0.4916	0.7584	0.6518	0.8595	40.68°	0.71
0.72	1.2706	0.7838	0.6169	2.0544	0.4868	0.7518	0.6594	0.8771	41.25°	0.72
0.73	1.2785	0.7966	0.6231	2.0751	0.4819	0.7452	0.6669	0.8949	41.83°	0.73
0.74	1.2865	0.8094	0.6292	2.0959	0.4771	0.7385	0.6743	0.9131	42.40°	0.74

0.75	1.2947	0.8223	0.6352	2.1170	0.4724	0.7317	0.6816	0.9316	42.97°	0.75
0.76	1.3030	0.8353	0.6411	2.1383	0.4677	0.7248	0.6889	0.9505	43.54°	0.76
0.77	1.3114	0.8484	0.6469	2.1598	0.4630	0.7179	0.6961	0.9697	44.12°	0.77
0.78	1.3199	0.8615	0.6527	2.1815	0.4584	0.7109	0.7033	0.9893	44.69°	0.78
0.79	1.3286	0.8748	0.6584	2.2034	0.4538	0.7039	0.7104	1.0092	45.26°	0.79
0.80	1.3374	0.8881	0.6640	2.2255	0.4493	0.6967	0.7174	1.0296	45.84°	0.80
0.81	1.3464	0.9015	0.6696	2.2479	0.4449	0.6895	0.7243	1.0505	46.41°	0.81
0.82	1.3555	0.9150	0.6751	2.2705	0.4404	0.6822	0.7312	1.0717	46.98°	0.82
0.83	1.3647	0.9286	0.6805	2.2933	0.4360	0.6749	0.7379	1.0934	47.56°	0.83
0.84	1.3740	0.9423	0.6858	2.3164	0.4317	0.6675	0.7446	1.1156	48.13°	0.84
0.85	1.3835	0.9561	0.6911	2.3396	0.4274	0.6600	0.7513	1.1383	48.70°	0.85
0.86	1.3932	0.9700	0.6963	2.3632	0.4232	0.6524	0.7578	1.1616	49.27°	0.86
0.87	1.4029	0.9840	0.7014	2.3869	0.4190	0.6448	0.7643	1.1853	49.85°	0.87
0.88	1.4128	0.9981	0.7064	2.4109	0.4148	0.6372	0.7707	1.2097	50.42°	0.88
0.89	1.4229	1.0122	0.7114	2.4351	0.4107	0.6294	0.7771	1.2346	50.99°	0.89
0.90	1.4331	1.0265	0.7163	2.4596	0.4066	0.6216	0.7833	1.2602	51.57°	0.90
0.91	1.4434	1.0409	0.7211	2.4843	0.4025	0.6138	0.7895	1.2864	52.14°	0.91
0.92	1.4539	1.0554	0.7259	2.5093	0.3985	0.6058	0.7956	1.3133	52.71°	0.92
0.93	1.4645	1.0700	0.7306	2.5345	0.3946	0.5978	0.8016	1.3409	53.29°	0.93
0.94	1.4753	1.0847	0.7352	2.5600	0.3906	0.5898	0.8076	1.3692	53.86°	0.94
0.95	1.4862	1.0995	0.7398	2.5857	0.3867	0.5817	0.8134	1.3984	54.43°	0.95
0.96	1.4973	1.1144	0.7443	2.6117	0.3829	0.5735	0.8192	1.4284	55.00°	0.96
0.97	1.5085	1.1294	0.7487	2.6379	0.3791	0.5653	0.8249	1.4592	55.58°	0.97
0.98	1.5199	1.1446	0.7531	2.6645	0.3753	0.5570	0.8305	1.4910	56.15°	0.98
0.99	1.5314	1.1598	0.7574	2.6912	0.3716	0.5487	0.8360	1.5237	56.72°	0.99
1.00	1.5431	1.1752	0.7616	2.7183	0.3679	0.5403	0.8415	1.5574	57.30°	1.00
1.01	1.5549	1.1907	0.7658	2.7456	0.3642	0.5319	0.8468	1.5922	57.87°	1.01
1.02	1.5669	1.2063	0.7699	2.7732	0.3606	0.5234	0.8521	1.6281	58.44°	1.02
1.03	1.5790	1.2220	0.7739	2.8011	0.3570	0.5148	0.8573	1.6652	59.01°	1.03
1.04	1.5913	1.2379	0.7779	2.8292	0.3535	0.5062	0.8624	1.7036	59.59°	1.04

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS—(*Continued*)

x (Radians)	cosh x	sinh x	tanh x	ϵ^x	ϵ^{-x}	cos x	sin x	tan x	Degrees	x
1.05	1.6038	1.2539	0.7818	2.8577	0.3499	0.4976	0.8674	1.7433	60.16°	1.05
1.06	1.6164	1.2700	0.7857	2.8864	0.3465	0.4889	0.8724	1.7844	60.73°	1.06
1.07	1.6292	1.2862	0.7895	2.9154	0.3430	0.4801	0.8772	1.8270	61.31°	1.07
1.08	1.6421	1.3025	0.7932	2.9447	0.3396	0.4713	0.8820	1.8712	61.88°	1.08
1.09	1.6553	1.3190	0.7969	2.9743	0.3362	0.4625	0.8866	1.9171	62.45°	1.09
1.10	1.6685	1.3357	0.8005	3.0042	0.3329	0.4536	0.8912	1.9648	63.03°	1.10
1.11	1.6820	1.3524	0.8041	3.0344	0.3296	0.4447	0.8957	2.0143	63.60°	1.11
1.12	1.6956	1.3693	0.8076	3.0649	0.3263	0.4357	0.9001	2.0660	64.17°	1.12
1.13	1.7093	1.3863	0.8110	3.0957	0.3230	0.4267	0.9044	2.1198	64.74°	1.13
1.14	1.7233	1.4035	0.8144	3.1268	0.3198	0.4176	0.9086	2.1759	65.32°	1.14
1.15	1.7374	1.4208	0.8178	3.1582	0.3166	0.4085	0.9128	2.2345	65.89°	1.15
1.16	1.7517	1.4382	0.8210	3.1899	0.3135	0.3993	0.9168	2.2958	66.46°	1.16
1.17	1.7662	1.4558	0.8243	3.2220	0.3104	0.3902	0.9208	2.3600	67.04°	1.17
1.18	1.7808	1.4736	0.8275	3.2544	0.3073	0.3809	0.9246	2.4273	67.61°	1.18
1.19	1.7957	1.4914	0.8306	3.2871	0.3042	0.3717	0.9284	2.4979	68.18°	1.19
1.20	1.8107	1.5095	0.8337	3.3201	0.3012	0.3624	0.9320	2.5722	68.75°	1.20
1.21	1.8258	1.5276	0.8367	3.3535	0.2982	0.3530	0.9356	2.6503	69.33°	1.21
1.22	1.8412	1.5460	0.8397	3.3872	0.2952	0.3437	0.9391	2.7328	69.90°	1.22
1.23	1.8568	1.5645	0.8426	3.4212	0.2923	0.3342	0.9425	2.8198	70.47°	1.23
1.24	1.8725	1.5831	0.8455	3.4556	0.2894	0.3248	0.9458	2.9119	71.05°	1.24
1.25	1.8884	1.6019	0.8483	3.4903	0.2865	0.3153	0.9490	3.0096	71.62°	1.25
1.26	1.9045	1.6209	0.8511	3.5254	0.2837	0.3058	0.9521	3.1133	72.19°	1.26
1.27	1.9208	1.6400	0.8538	3.5609	0.2808	0.2963	0.9551	3.2236	72.77°	1.27
1.28	1.9373	1.6593	0.8565	3.5966	0.2780	0.2867	0.9580	3.3413	73.34°	1.28
1.29	1.9540	1.6788	0.8591	3.6328	0.2753	0.2771	0.9608	3.4672	73.91°	1.29

1.30	1.9709	1.6984	0.8617	3.6693	0.2725	0.2675	0.9636	3.6021	74.48°	1.30
1.31	1.9880	1.7182	0.8643	3.7062	0.2698	0.2579	0.9662	3.7471	75.06°	1.31
1.32	2.0053	1.7381	0.8668	3.7434	0.2671	0.2482	0.9687	3.9033	75.63°	1.32
1.33	2.0228	1.7583	0.8693	3.7810	0.2645	0.2385	0.9712	4.0723	76.20°	1.33
1.34	2.0404	1.7786	0.8717	3.8190	0.2618	0.2288	0.9735	4.2556	76.78°	1.34
1.35	2.0583	1.7991	0.8741	3.8574	0.2592	0.2190	0.9757	4.4552	77.35°	1.35
1.36	2.0764	1.8198	0.8764	3.8962	0.2567	0.2092	0.9779	4.6734	77.92°	1.36
1.37	2.0947	1.8406	0.8787	3.9354	0.2541	0.1995	0.9799	4.9131	78.50°	1.37
1.38	2.1132	1.8617	0.8810	3.9749	0.2516	0.1896	0.9819	5.1774	79.07°	1.38
1.39	2.1320	1.8829	0.8832	4.0149	0.2491	0.1798	0.9837	5.4707	79.64°	1.39
1.40	2.1509	1.9043	0.8854	4.0552	0.2466	0.1700	0.9855	5.7979	80.21°	1.40
1.41	2.1701	1.9259	0.8875	4.0960	0.2441	0.1601	0.9871	6.1654	80.79°	1.41
1.42	2.1894	1.9477	0.8896	4.1371	0.2417	0.1502	0.9887	6.5811	81.36°	1.42
1.43	2.2090	1.9697	0.8917	4.1787	0.2393	0.1403	0.9901	7.0555	81.93°	1.43
1.44	2.2288	1.9919	0.8937	4.2207	0.2369	0.1304	0.9915	7.6018	82.51°	1.44
1.45	2.2488	2.0143	0.8957	4.2631	0.2346	0.1205	0.9927	8.2381	83.08°	1.45
1.46	2.2691	2.0369	0.8977	4.3060	0.2322	0.1106	0.9939	8.9886	83.65°	1.46
1.47	2.2896	2.0597	0.8996	4.3492	0.2299	0.1006	0.9949	9.8874	84.22°	1.47
1.48	2.3103	2.0827	0.9015	4.3929	0.2276	0.0907	0.9959	10.983	84.80°	1.48
1.49	2.3312	2.1059	0.9033	4.4371	0.2254	0.0807	0.9967	12.350	85.37°	1.49
1.50	2.3524	2.1293	0.9052	4.4817	0.2231	0.0707	0.9975	14.101	85.94°	1.50
1.51	2.3738	2.1529	0.9069	4.5267	0.2209	0.0608	0.9982	16.428	86.52°	1.51
1.52	2.3955	2.1768	0.9087	4.5722	0.2187	0.0508	0.9987	19.670	87.09°	1.52
1.53	2.4174	2.2008	0.9104	4.6182	0.2165	0.0408	0.9992	24.498	87.66°	1.53
1.54	2.4395	2.2251	0.9121	4.6646	0.2144	0.0308	0.9995	32.461	88.24°	1.54
1.55	2.4619	2.2496	0.9138	4.7115	0.2122	0.0208	0.9998	48.078	88.81°	1.55
1.56	2.4845	2.2743	0.9154	4.7588	0.2101	0.0108	0.9999	92.621	89.38°	1.56
1.57	2.5074	2.2993	0.9170	4.8066	0.2080	0.0008	1.0000	1255.8	89.95°	1.57
1.58	2.5305	2.3245	0.9186	4.8550	0.2060	−0.0092	1.0000	−108.65	90.53°	1.58
1.59	2.5538	2.3499	0.9202	4.9037	0.2039	−0.0192	0.9998	−52.067	91.10°	1.59

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS—(*Continued*)

x (Radians)	cosh x	sinh x	tanh x	ϵ^x	ϵ^{-x}	cos x	sin x	tan x	Degrees	x
1.60	2.5775	2.3756	0.9217	4.9530	0.2019	−0.0292	0.9996	−34.233	91.67°	1.60
1.61	2.6014	2.4015	0.9232	5.0028	0.1999	−0.0392	0.9992	−25.495	92.25°	1.61
1.62	2.6255	2.4276	0.9246	5.0531	0.1979	−0.0492	0.9988	−20.307	92.82°	1.62
1.63	2.6499	2.4540	0.9261	5.1039	0.1959	−0.0592	0.9982	−16.871	93.39°	1.63
1.64	2.6746	2.4806	0.9275	5.1552	0.1940	−0.0691	0.9976	−14.427	93.97°	1.64
1.65	2.6995	2.5075	0.9289	5.2070	0.1920	−0.0791	0.9969	−12.599	94.54°	1.65
1.66	2.7247	2.5346	0.9302	5.2593	0.1901	−0.0891	0.9960	−11.181	95.11°	1.66
1.67	2.7502	2.5620	0.9316	5.3122	0.1882	−0.0990	0.9951	−10.047	95.68°	1.67
1.68	2.7760	2.5896	0.9329	5.3656	0.1864	−0.1090	0.9940	−9.1208	96.26°	1.68
1.69	2.8020	2.6175	0.9342	5.4195	0.1845	−0.1189	0.9929	−8.3492	96.83°	1.69
1.70	2.8283	2.6456	0.9354	5.4739	0.1827	−0.1288	0.9917	−7.6966	97.40°	1.70
1.71	2.8549	2.6741	0.9367	5.5290	0.1809	−0.1388	0.9903	−7.1373	97.98°	1.71
1.72	2.8818	2.7027	0.9379	5.5845	0.1791	−0.1487	0.9889	−6.6524	98.55°	1.72
1.73	2.9090	2.7317	0.9391	5.6407	0.1773	−0.1585	0.9874	−6.2281	99.12°	1.73
1.74	2.9364	2.7609	0.9402	5.6973	0.1755	−0.1684	0.9857	−5.8534	99.69°	1.74
1.75	2.9642	2.7904	0.9414	5.7546	0.1738	−0.1782	0.9840	−5.5204	100.27°	1.75
1.76	2.9922	2.8202	0.9425	5.8124	0.1720	−0.1881	0.9822	−5.2221	100.84°	1.76
1.77	3.0206	2.8503	0.9436	5.8709	0.1703	−0.1979	0.9802	−4.9534	101.41°	1.77
1.78	3.0493	2.8806	0.9447	5.9299	0.1686	−0.2077	0.9782	−4.7101	101.99°	1.78
1.79	3.0782	2.9113	0.9458	5.9895	0.1670	−0.2175	0.9761	−4.4887	102.56°	1.79
1.80	3.1075	2.9422	0.9468	6.0496	0.1653	−0.2272	0.9738	−4.2863	103.13°	1.80
1.81	3.1371	2.9734	0.9478	6.1104	0.1637	−0.2369	0.9715	−4.1005	103.71°	1.81
1.82	3.1669	3.0049	0.9488	6.1719	0.1620	−0.2466	0.9691	−3.9294	104.28°	1.82
1.83	3.1972	3.0367	0.9498	6.2339	0.1604	−0.2563	0.9666	−3.7712	104.85°	1.83
1.84	3.2277	3.0689	0.9508	6.2965	0.1588	−0.2660	0.9640	−3.6245	105.42°	1.84

1.85	3.2585	3.1013	0.9518	6.3598	0.1572	−0.2756	0.9613	−3.4881	106.00°	1.85
1.86	3.2897	3.1340	0.9527	6.4237	0.1557	−0.2852	0.9585	−3.3609	106.57°	1.86
1.87	3.3212	3.1671	0.9536	6.4883	0.1541	−0.2948	0.9556	−3.2419	107.14°	1.87
1.88	3.3531	3.2005	0.9545	6.5535	0.1526	−0.3043	0.9526	−3.1304	107.72°	1.88
1.89	3.3852	3.2342	0.9554	6.6194	0.1511	−0.3138	0.9495	−3.0257	108.29°	1.89
1.90	3.4177	3.2682	0.9562	6.6859	0.1496	−0.3233	0.9463	−2.9271	108.86°	1.90
1.91	3.4506	3.3025	0.9571	6.7531	0.1481	−0.3327	0.9430	−2.8341	109.43°	1.91
1.92	3.4838	3.3372	0.9579	6.8210	0.1466	−0.3421	0.9396	−2.7463	110.01°	1.92
1.93	3.5173	3.3722	0.9587	6.8895	0.1451	−0.3515	0.9362	−2.6632	110.58°	1.93
1.94	3.5512	3.4075	0.9595	6.9588	0.1437	−0.3609	0.9326	−2.5843	111.15°	1.94
1.95	3.5855	3.4432	0.9603	7.0287	0.1423	−0.3702	0.9290	−2.5095	111.73°	1.95
1.96	3.6201	3.4792	0.9611	7.0993	0.1409	−0.3795	0.9252	−2.4383	112.30°	1.96
1.97	3.6551	3.5156	0.9619	7.1707	0.1395	−0.3887	0.9214	−2.3705	112.87°	1.97
1.98	3.6904	3.5523	0.9626	7.2427	0.1381	−0.3979	0.9174	−2.3058	113.45°	1.98
1.99	3.7261	3.5894	0.9633	7.3155	0.1367	−0.4070	0.9134	−2.2441	114.02°	1.99
2.00	3.7622	3.6269	0.9640	7.3891	0.1353	−0.4161	0.9093	−2.1850	114.59°	2.00
2.01	3.7987	3.6647	0.9647	7.4633	0.1340	−0.4252	0.9051	−2.1285	115.16°	2.01
2.02	3.8355	3.7028	0.9654	7.5383	0.1327	−0.4343	0.9008	−2.0744	115.74°	2.02
2.03	3.8727	3.7414	0.9661	7.6141	0.1313	−0.4432	0.8964	−2.0224	116.31°	2.03
2.04	3.9103	3.7803	0.9668	7.6906	0.1300	−0.4522	0.8919	−1.9725	116.88°	2.04
2.05	3.9483	3.8196	0.9674	7.7679	0.1287	−0.4611	0.8874	−1.9246	117.46°	2.05
2.06	3.9867	3.8593	0.9680	7.8460	0.1275	−0.4699	0.8827	−1.8784	118.03°	2.06
2.07	4.0255	3.8993	0.9687	7.9248	0.1262	−0.4787	0.8780	−1.8340	118.60°	2.07
2.08	4.0647	3.9398	0.9693	8.0045	0.1249	−0.4875	0.8731	−1.7911	119.18°	2.08
2.09	4.1043	3.9806	0.9699	8.0849	0.1237	−0.4962	0.8682	−1.7498	119.75°	2.09
2.10	4.1443	4.0219	0.9705	8.1662	0.1225	−0.5048	0.8632	−1.7099	120.32°	2.10
2.11	4.1847	4.0635	0.9710	8.2482	0.1212	−0.5135	0.8581	−1.6713	120.89°	2.11
2.12	4.2256	4.1056	0.9716	8.3311	0.1200	−0.5220	0.8529	−1.6340	121.47°	2.12
2.13	4.2669	4.1480	0.9722	8.4149	0.1188	−0.5305	0.8477	−1.5979	122.04°	2.13
2.14	4.3086	4.1909	0.9727	8.4994	0.1177	−0.5390	0.8423	−1.5629	122.61°	2.14

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS—(*Continued*)

x (Radians)	cosh x	sinh x	tanh x	ϵ^x	ϵ^{-x}	cos x	sin x	tan x	Degrees	x
2.15	4.3507	4.2342	0.9732	8.5849	0.1165	−0.5474	0.8369	−1.5290	123.19°	2.15
2.16	4.3932	4.2779	0.9738	8.6711	0.1153	−0.5557	0.8314	−1.4961	123.76°	2.16
2.17	4.4362	4.3221	0.9743	8.7583	0.1142	−0.5640	0.8258	−1.4642	124.33°	2.17
2.18	4.4797	4.3666	0.9748	8.8463	0.1130	−0.5722	0.8201	−1.4332	124.90°	2.18
2.19	4.5236	4.4117	0.9753	8.9352	0.1119	−0.5804	0.8143	−1.4031	125.48°	2.19
2.20	4.5679	4.4571	0.9757	9.0250	0.1108	−0.5885	0.8085	−1.3738	126.05°	2.20
2.21	4.6127	4.5030	0.9762	9.1157	0.1097	−0.5966	0.8026	−1.3453	126.62°	2.21
2.22	4.6580	4.5494	0.9767	9.2073	0.1086	−0.6046	0.7966	−1.3176	127.20°	2.22
2.23	4.7037	4.5962	0.9771	9.2999	0.1075	−0.6125	0.7905	−1.2906	127.77°	2.23
2.24	4.7499	4.6434	0.9776	9.3933	0.1065	−0.6204	0.7843	−1.2643	128.34°	2.24
2.25	4.7966	4.6912	0.9780	9.4877	0.1054	−0.6282	0.7781	−1.2386	128.92°	2.25
2.26	4.8437	4.7394	0.9785	9.5831	0.1044	−0.6359	0.7718	−1.2136	129.49°	2.26
2.27	4.8914	4.7880	0.9789	9.6794	0.1033	−0.6436	0.7654	−1.1892	130.06°	2.27
2.28	4.9395	4.8372	0.9793	9.7767	0.1023	−0.6512	0.7589	−1.1653	130.63°	2.28
2.29	4.9881	4.8868	0.9797	9.8749	0.1013	−0.6588	0.7523	−1.1420	131.21°	2.29
2.30	5.0372	4.9370	0.9801	9.9742	0.1003	−0.6663	0.7457	−1.1192	131.78°	2.30
2.31	5.0868	4.9876	0.9805	10.074	0.0993	−0.6737	0.7390	−1.0969	132.35°	2.31
2.32	5.1370	5.0387	0.9809	10.176	0.0983	−0.6811	0.7322	−1.0751	132.93°	2.32
2.33	5.1876	5.0903	0.9812	10.278	0.0973	−0.6883	0.7254	−1.0538	133.50°	2.33
2.34	5.2388	5.1425	0.9816	10.381	0.0963	−0.6956	0.7185	−1.0329	134.07°	2.34
2.35	5.2905	5.1951	0.9820	10.486	0.0954	−0.7027	0.7115	−1.0125	134.65°	2.35
2.36	5.3427	5.2483	0.9823	10.591	0.0944	−0.7098	0.7044	−0.9924	135.22°	2.36
2.37	5.3954	5.3020	0.9827	10.697	0.0935	−0.7168	0.6973	−0.9728	135.79°	2.37
2.38	5.4487	5.3562	0.9830	10.805	0.0926	−0.7237	0.6901	−0.9535	136.36°	2.38
2.39	5.5026	5.4109	0.9834	10.913	0.0916	−0.7306	0.6828	−0.9346	136.94°	2.39

2.40	5.5570	5.4662	0.9837	11.023	0.0907	−0.7374	0.6755	−0 9160	137.51°	2.40
2.41	5.6119	5.5221	0.9840	11.134	0.0898	−0 7441	0.6681	−0.8978	138.08°	2.41
2.42	5 6674	5.5785	0.9843	11.246	0.0889	−0.7508	0.6606	−0 8799	138.66°	2 42
2.43	5.7235	5.6354	0.9846	11.359	0.0880	−0 7573	0.6530	−0.8623	139.23°	2 43
2.44	5 7801	5.6929	0.9849	11.473	0.0872	−0.7638	0.6454	−0.8450	139.80°	2 44
2.45	5.8373	5.7510	0 9852	11.588	0 0863	−0.7702	0 6378	−0.8280	140.37°	2 45
2.46	5.8951	5.8097	0.9855	11.705	0.0854	−0.7766	0.6300	−0.8113	140.95°	2.46
2.47	5.9535	5.8689	0.9858	11.822	0.0846	−0 7828	0.6222	−0.7949	141 52°	2.47
2.48	6.0125	5.9288	0.9861	11.941	0.0837	−0.7890	0.6144	−0.7787	142 09°	2 48
2.49	6.0721	5.9892	0.9864	12.061	0.0829	−0.7951	0.6065	−0.7627	142.67°	2 49
2 50	6 1323	6.0502	0 9866	12.182	0.0821	−0.8011	0.5985	−0.7470	143.24°	2.50
2.51	6.1931	6 1118	0.9869	12.305	0.0813	−0.8071	0.5904	−0.7316	143.81°	2.51
2.52	6.2545	6.1741	0 9871	12.429	0.0805	−0 8130	0.5823	−0.7163	144.39°	2 52
2.53	6.3166	6.2369	0 9874	12.554	0.0797	−0.8187	0.5742	−0.7013	144.96°	2 53
2.54	6.3793	6.3004	0.9876	12.680	0.0789	−0.8244	0.5660	−0.6865	145 53°	2 54
2.55	6.4426	6.3645	0.9879	12.807	0.0781	−0.8301	0 5577	−0.6719	146.10°	2.55
2.56	6.5066	6 4293	0.9881	12 936	0 0773	−0 8356	0.5494	−0 6575	146 68°	2.56
2.57	6.5712	6.4946	0.9884	13 066	0 0765	−0.8410	0 5410	−0.6432	147.25°	2 57
2.58	6 6365	6 5607	0.9886	13.197	0.0758	−0.8464	0.5325	−0.6292	147.82°	2.58
2.59	6.7024	6.6274	0.9888	13.330	0.0750	−0.8517	0.5240	−0 6153	148.40°	2.59
2 60	6.7690	6.6947	0.9890	13 464	0.0743	−0 8569	0 5155	−0.6016	148.97°	2.60
2.61	6.8363	6 7628	0.9892	13.599	0.0735	−0.8620	0.5069	−0 5881	149.54°	2.61
2.62	6 9043	6.8315	0 9895	13.736	0 0728	−0 8670	0.4983	−0.5747	150.11°	2 62
2.63	6 9729	6.9009	0.9897	13.874	0 0721	−0.8720	0.4896	−0.5615	150.69°	2.63
2.64	7 0423	6.9709	0 9899	14.013	0.0714	−0 8768	0.4808	−0 5484	151.26°	2.64
2.65	7.1123	7.0417	0.9901	14 154	0 0707	−0 8816	0.4720	−0.5354	151.83°	2.65
2.66	7.1831	7.1132	0.9903	14.296	0.0699	−0.8863	0.4632	−0.5226	152.41°	2.66
2.67	7.2546	7.1854	0.9905	14.440	0 0693	−0 8909	0.4543	−0 5100	152.98°	2.67
2.68	7.3268	7.2583	0.9906	14.585	0 0686	−0.8953	0 4454	−0.4974	153.55°	2.68
2.69	7.3998	7.3319	0.9908	14.732	0 0679	−0 8998	0.4364	−0.4850	154.13°	2.69

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS—(*Continued*)

x (Radians)	cosh x	sinh x	tanh x	ϵ^x	ϵ^{-x}	cos x	sin x	tan x	Degrees	x
2.70	7.4735	7.4063	0.9910	14.880	0.0672	−0.9041	0.4274	−0.4727	154.70°	2.70
2.71	7.5479	7.4814	0.9912	15.029	0.0665	−0.9083	0.4183	−0.4606	155.27°	2.71
2.72	7.6231	7.5572	0.9914	15.180	0.0659	−0.9124	0.4092	−0.4485	155.84°	2.72
2.73	7.6991	7.6338	0.9915	15.333	0.0652	−0.9165	0.4001	−0.4365	156.42°	2.73
2.74	7.7758	7.7112	0.9917	15.487	0.0646	−0.9204	0.3909	−0.4247	156.99°	2.74
2.75	7.8533	7.7894	0.9919	15.653	0.0639	−0.9243	0.3817	−0.4129	157.56°	2.75
2.76	7.9316	7.8683	0.9920	15.800	0.0633	−0.9281	0.3724	−0.4013	158.14°	2.76
2.77	8.0107	7.9480	0.9922	15.959	0.0627	−0.9318	0.3631	−0.3897	158.71°	2.77
2.78	8.0905	8.0285	0.9923	16.119	0.0620	−0.9353	0.3538	−0.3782	159.28°	2.78
2.79	8.1712	8.1098	0.9925	16.281	0.0614	−0.9388	0.3444	−0.3668	159.86°	2.79
2.80	8.2527	8.1919	0.9926	16.445	0.0608	−0.9422	0.3350	−0.3555	160.43°	2.80
2.81	8.3351	8.2749	0.9928	16.610	0.0602	−0.9455	0.3256	−0.3443	161.00°	2.81
2.82	8.4182	8.3586	0.9929	16.777	0.0596	−0.9487	0.3161	−0.3332	161.57°	2.82
2.83	8.5022	8.4432	0.9931	16.945	0.0590	−0.9519	0.3066	−0.3221	162.15°	2.83
2.84	8.5871	8.5287	0.9932	17.116	0.0584	−0.9549	0.2970	−0.3111	162.72°	2.84
2.85	8.6728	8.6150	0.9933	17.288	0.0578	−0.9578	0.2875	−0.3002	163.29°	2.85
2.86	8.7594	8.7021	0.9935	17.462	0.0573	−0.9606	0.2779	−0.2893	163.87°	2.86
2.87	8.8469	8.7902	0.9936	17.637	0.0567	−0.9633	0.2683	−0.2785	164.44°	2.87
2.88	8.9352	8.8791	0.9937	17.814	0.0561	−0.9660	0.2586	−0.2677	165.01°	2.88
2.89	9.0244	8.9689	0.9938	17.993	0.0556	−0.9685	0.2490	−0.2570	165.58°	2.89
2.90	9.1146	9.0596	0.9940	18.174	0.0550	−0.9710	0.2392	−0.2464	166.16°	2.90
2.91	9.2056	9.1512	0.9941	18.357	0.0545	−0.9733	0.2295	−0.2358	166.73°	2.91
2.92	9.2976	9.2437	0.9942	18.541	0.0539	−0.9756	0.2198	−0.2253	167.30°	2.92
2.93	9.3905	9.3371	0.9943	18.728	0.0534	−0.9777	0.2100	−0.2148	167.88°	2.93
2.94	9.4844	9.4315	0.9944	18.916	0.0529	−0.9798	0.2002	−0.2044	168.45°	2.94

2.95	9.5792	9.5268	0.9945	19.106	0.0523	−0.9817	0.1904	−0.1940	169.02°	2.95
2.96	9.6749	9.6231	0.9946	19.298	0.0518	−0.9836	0.1806	−0.1836	169.60°	2.96
2.97	9.7716	9.7203	0.9948	19.492	0.0513	−0.9853	0.1708	−0.1733	170.17°	2.97
2.98	9.8693	9.8185	0.9949	19.688	0.0508	−0.9870	0.1609	−0.1630	170.74°	2.98
2.99	9.9680	9.9177	0.9950	19.886	0.0503	−0.9885	0.1510	−0.1528	171.31°	2.99
3.0	10.068	10.018	0.9951	20.086	0.0498	−0.9900	0.1411	−0.1426	171.89°	3.0
3.1	11.122	11.077	0.9960	22.198	0.0450	−0.9991	0.0416	−0.0416	177.62°	3.1
3.2	12.287	12.246	0.9967	24.533	0.0408	−0.9983	−0.0584	0.0585	183.35°	3.2
3.3	13.575	13.538	0.9973	27.113	0.0369	−0.9875	−0.1577	0.1598	189.08°	3.3
3.4	14.999	14.965	0.9978	29.964	0.0334	−0.9668	−0.2555	0.2643	194.81°	3.4
3.5	16.573	16.543	0.9982	33.115	0.0302	−0.9365	−0.3508	0.3746	200.54°	3.5
3.6	18.313	18.286	0.9985	36.598	0.0273	−0.8968	−0.4425	0.4935	206.26°	3.6
3.7	20.236	20.211	0.9988	40.447	0.0247	−0.8481	−0.5298	0.6247	211.99°	3.7
3.8	22.362	22.339	0.9990	44.701	0.0224	−0.7910	−0.6119	0.7736	217.72°	3.8
3.9	24.711	24.691	0.9992	49.402	0.0202	−0.7259	−0.6878	0.9474	223.45°	3.9
4.0	27.308	27.290	0.9993	54.598	0.0183	−0.6536	−0.7568	1.1578	229.18°	4.0
4.1	30.178	30.162	0.9995	60.340	0.0166	−0.5748	−0.8183	1.4235	234.91°	4.1
4.2	33.351	33.336	0.9996	66.686	0.0150	−0.4903	−0.8716	1.7778	240.64°	4.2
4.3	36.857	36.843	0.9996	73.700	0.0136	−0.4008	−0.9162	2.2859	246.37°	4.3
4.4	40.732	40.719	0.9997	81.451	0.0123	−0.3073	−0.9516	3.0963	252.10°	4.4
4.5	45.014	45.003	0.9998	90.017	0.0111	−0.2108	−0.9775	4.6373	257.83°	4.5
4.6	49.747	49.737	0.9998	99.484	0.0101	−0.1122	−0.9937	8.8602	263.56°	4.6
4.7	54.978	54.969	0.9998	109.95	0.0091	−0.0124	−0.9999	80.713	269.29°	4.7
4.8	60.759	60.751	0.9999	121.51	0.0082	0.0875	−0.9962	−11.385	275.02°	4.8
4.9	67.149	67.141	0.9999	134.29	0.0074	0.1865	−0.9825	−5.2675	280.75°	4.9
5.0	74.210	74.203	0.9999	148.41	0.0067	0.2837	−0.9589	−3.3805	286.48°	5.0
5.1	82.014	82.008	0.9999	164.02	0.0061	0.3780	−0.9258	−2.4494	292.21°	5.1
5.2	90.639	90.633	0.9999	181.27	0.0055	0.4685	−0.8835	−1.8856	297.94°	5.2
5.3	100.17	100.17	1.0000	200.34	0.0050	0.5544	−0.8323	−1.5013	303.67°	5.3
5.4	110.71	110.70	1.0000	221.41	0.0045	0.6347	−0.7728	−1.2175	309.40°	5.4

APPENDIX H

DEFINITION OF Q (FIGURE OF MERIT)

In Chapter 5 on Series Circuits the Q of an RLC circuit with sinusoidal voltages and current was defined in terms of the frequency band width $(f_2 - f_1)$ which was taken at the half-power points, Fig. H–1. Thus,

$$\frac{f_2 - f_1}{f_r} = \frac{R}{2\pi f_r L} = \frac{R}{\omega_r L} = \frac{1}{Q} \tag{H–1}$$

or

$$Q = \frac{f_r}{f_2 - f_1} = \frac{\omega_r L}{R} \tag{H–1a}$$

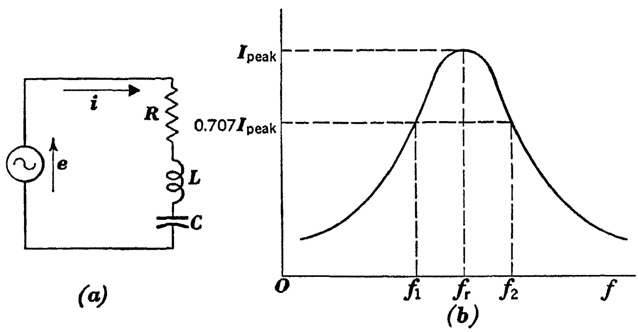


FIG. H–1

The Q of a circuit may also be considered from the standpoint of efficiency with which energy is stored in the circuit. As a measure of such efficiency, Q has been defined as

$$Q = 2\pi \frac{\text{Maximum energy stored per cycle}}{\text{Energy dissipated per cycle}} \tag{H–2}$$

The Q of simple circuits with sinusoidal voltages and sinusoidal currents will be considered in Arts. H–1 to H–4, inclusive. In Art. H–5, it will be shown that, for an RLC circuit, the Q obtained from equation (H–2) will be the same as that given in equation (H–1a).

H–1. *R* and *L* in Series.—Consider a circuit consisting of R and L in series and having sinusoidal voltages and current. The maximum value of the energy stored in the inductance occurs at the maximum value I_m of the current. Thus,

$$\text{Energy Stored} = \frac{1}{2} L I_m^2 \qquad \textbf{(H–3)}$$

The average power dissipated is

$$P = I^2 R = \left(\frac{I_m}{\sqrt{2}}\right)^2 R = \frac{I_m^2 R}{2} \qquad \textbf{(H–4)}$$

The energy dissipated per cycle is equal to the product of the average power dissipated and the period $T = \frac{1}{f}$ (one cycle), or

$$\text{Energy Dissipated} = \frac{I_m^2 R}{2} \times T = \frac{I_m^2 R}{2} \times \frac{1}{f} \qquad \textbf{(H–5)}$$

Then, from equation (H–2),

$$Q = 2\pi \frac{\frac{1}{2} L I_m^2}{\frac{I_m^2 R}{2f}} = \frac{2\pi f L}{R} = \frac{\omega L}{R} \qquad \textbf{(H–6)}$$

H–2. *R* and *C* in Series.—Assume a circuit with R and C in series and with sinusoidal voltages and current. The maximum value of the energy stored in the capacitor C occurs at the maximum value E_{Cm} of the voltage across the capacitor. Hence,

$$\text{Energy Stored} = \frac{1}{2} C E_{Cm}^2 \qquad \textbf{(H–7)}$$

Since $E_{Cm} = X_C I_m = \frac{I_m}{\omega C}$, then

$$\text{Energy Stored} = \frac{1}{2} \frac{I_m^2}{\omega^2 C} \qquad \textbf{(H–8)}$$

The energy dissipated per cycle, as indicated in equation (H–5), is

$$\text{Energy Dissipated} = \frac{I_m^2 R}{2f} \qquad \textbf{(H–9)}$$

Then, from equation (H–2),

$$Q = 2\pi \frac{\frac{1}{2} \frac{I_m^2}{\omega^2 C}}{\frac{I_m^2 R}{2f}} = \frac{1}{\omega C R} \qquad \textbf{(H–10)}$$

H-3. *R* and *L* in Parallel.—In a circuit with R and L connected in parallel, the same voltage is impressed across the two branches. It is assumed that the voltage and currents are sinusoidal. The maximum value of the energy stored in the inductance is

$$\text{Energy Stored} = \tfrac{1}{2} L I_{Lm}^2 \tag{H-11}$$

Since $I_{Lm} = \dfrac{E_m}{X_L} = \dfrac{E_m}{\omega L}$, then

$$\text{Energy Stored} = \frac{1}{2} L \times \frac{E_m^2}{\omega^2 L^2} = \frac{E_m^2}{2\omega^2 L} \tag{H-12}$$

The average power dissipated per cycle is

$$P = I_R^2 R = \left(\frac{E_m}{\sqrt{2}R}\right)^2 R = \frac{E_m^2}{2R} \tag{H-13}$$

The energy dissipated per cycle is

$$\text{Energy Dissipated} = \frac{E_m^2}{2R} \times \frac{1}{f} \tag{H-14}$$

Then, from equation (H-2),

$$Q = 2\pi \frac{\dfrac{E_m^2}{2\omega^2 L}}{\dfrac{E_m^2}{2Rf}} = \frac{R}{\omega L} \tag{H-15}$$

H-4. *R* and *C* in Parallel.—Consider a circuit consisting of R and C in parallel and having sinusoidal voltage and currents. The maximum value of the energy stored in the capacitor C occurs at the maximum value E_m of the voltage across the capacitor. Thus,

$$\text{Energy Stored} = \tfrac{1}{2} C E_m^2 \tag{H-16}$$

The average power dissipated per cycle is

$$P = I_R^2 R = \left(\frac{E_m}{\sqrt{2}R}\right)^2 R = \frac{E_m^2}{2R} \tag{H-17}$$

The energy dissipated per cycle is

$$\text{Energy Dissipated} = \frac{E_m^2}{2Rf} \tag{H-18}$$

Then, from equation (H–2),

$$Q = 2\pi \frac{\frac{1}{2}CE_m^2}{\frac{E_m^2}{2Rf}} = \omega CR \tag{H–19}$$

H–5. *R, L,* and *C* in Series.—Assume a circuit consisting of R, L, and C in series and having sinusoidal voltages and current. If $i = I_m \sin \omega t$, then the maximum energy stored in the circuit at resonance is

$$\text{Energy Stored} = \tfrac{1}{2}Li^2 + \tfrac{1}{2}Ce_C^2 \tag{H–20}$$

Also,

$$e_C = \frac{I_m}{\omega C} \sin (\omega t - 90°) = -\frac{I_m}{\omega C} \cos \omega t$$

and, at resonance,

$$\omega L = \frac{1}{\omega C}$$

Hence,

$$\text{Energy Stored} = \frac{1}{2}LI_m^2 \sin^2 \omega t + \frac{1}{2}\,C\,\frac{I_m^2}{\omega^2 C^2} \cos^2 \omega t$$

$$= \tfrac{1}{2}LI_m^2 \sin^2 \omega t + \tfrac{1}{2}LI_m^2 \cos^2 \omega t$$

or

$$\text{Energy Stored} = \tfrac{1}{2}LI_m^2 (\sin^2 \omega t + \cos^2 \omega t) = \tfrac{1}{2}LI_m^2 \tag{H–21}$$

The energy dissipated per cycle is

$$\text{Energy Dissipated} = I^2RT = \frac{I_m^2 R}{2f_r} \tag{H–22}$$

Then, from equation (H–2),

$$Q = 2\pi \frac{\frac{1}{2}LI_m^2}{\frac{I_m^2 R}{2f_r}} = \frac{\omega_r L}{R} \tag{H–23}$$

which is the same as equation (H–1a).

APPENDIX I

EXAMPLE 1–6 (SEE PAGE 20)

Example 1–6.—A d-c ammeter reads the average value of any periodic wave over a complete cycle. An a-c ammeter reads the value

$$I = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} i^2\, d\alpha}$$

where i is any periodic wave and I^2 is the average value of the i^2-wave.

Find (*a*) the reading of a d-c ammeter and (*b*) the reading of an a-c ammeter connected in a circuit in which the current has the wave form shown in Fig I–1.

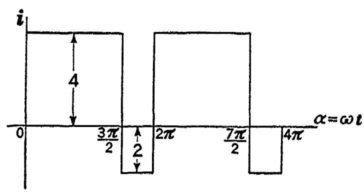


Fig. I–1

Solution.—(*a*) the reading of the d-c ammeter is

$$I_{dc} = \frac{\left(4\times\frac{3\pi}{2}\right) - \left(2\times\frac{\pi}{2}\right)}{2\pi} = 2.5 \text{ amp}$$

(*b*) The reading of the a-c ammeter is found as follows. The i^2-curve will be similar to that of Fig. I–1, but the ordinate 4 becomes $4^2 = 16$ and the negative ordinate 2 becomes a positive ordinate $(-2)^2 = +4$. In this case,

$$I^2 = \frac{\left[4^2\times\frac{3\pi}{2}\right] + \left[(-2)^2\times\frac{\pi}{2}\right]}{2\pi} = 13$$

and

$$I = \sqrt{13} = 3.61 \text{ amp}$$

APPENDIX J

VOLTAGE NOTATION

Since, in Fig. J–1(*a*), the voltage rise e and the voltage drop $e_R = e_{34}$ are alternating voltages, points *2* and *3* will have a (+) polarity during the positive half-cycle and a (−) polarity during the negative half-cycle. Some method of designating the instantaneous (+) polarity or positive sense of a voltage rise and that of a voltage drop is desirable.

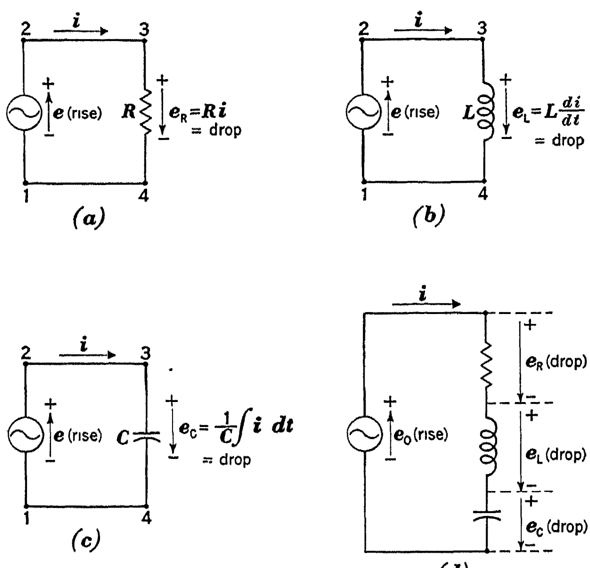


FIG. J–1

When, in tracing the circuit in the direction of the current arrow, the (+) polarity or positive sense of a voltage rise and the (+) polarity or positive sense of a voltage drop are designated by arrows in a circuit, it is very important to make a distinction between the two. As commonly used or implied, the arrowhead is (+) for a voltage rise and the arrowhead is (−) for a voltage drop, as shown for each case in Fig. J–1.

Some authors prefer to omit the arrow entirely and to use only the (+) and (−) signs. Then, in Fig. J-1(*a*), point *3* is (+) and point *4* is (−) when the instantaneous value of $e_R = e_{34}$ is a positive number. And point *3* is (−) and point *4* is (+) when the instantaneous value of $e_R = e_{34}$ is a negative number. Also, point *2* is (+) and point *1* is (−) when the instantaneous value of e is a positive number. It should be noted that, when the (+) and (−) signs are used in the circuit diagram, the terms voltage rise and voltage drop are not necessary.

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